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Egbert Dierker, Hildegard Dierker,
Birgit Grodal

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

# Incomplete Markets and the Firm *

Egbert Dierker †  Hildegard Dierker ‡  Birgit Grodal §

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**Abstract**

In this paper we analyze the welfare properties of the set of Drèze equilibria for economies with incomplete markets and firms. The well known fact that a Drèze equilibrium need not be constrained Pareto optimal is often attributed to a lack of coordination between firms. We show that there are economies with a single firm in which no Drèze equilibrium is constrained Pareto efficient. Even a unique Drèze equilibrium need not be constrained Pareto efficient.

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†Institut für Wirtschaftswissenschaften, Universität Wien, Hohenstaufengasse 9, A-1010 Wien, Austria

‡Institut für Wirtschaftswissenschaften, Universität Wien, Hohenstaufengasse 9, A-1010 Wien, Austria

§Økonomisk Institut and Centre of Industrial Economics, Københavns Universitet, Studiestræde 6, DK-1455 København, Denmark

# 1 Introduction.

In models of perfect or imperfect competition the objective of the firm can be expressed without recourse to shareholders' utility provided markets are assumed to be complete. However, if markets are incomplete, firms obtain an additional role since outputs are risky. As a consequence, the choice of a production plan does not only affect the present wealth of the shareholders, but also their possibilities to ensure themselves against risks.

The risk an individual shareholder faces depends on how his initial endowments vary across the states of nature and, of course, on his preferences that include his attitude towards risk as well as his personal probability assessments concerning future states of nature. In general, there are many assets apart from the shares of a particular firm under consideration that the shareholders of this firm can use to hedge their risk. Incompleteness of markets, however, would become trivialized if the firm could not influence the total span of all assets. If the production decision of a firm does affect the span, then one has to take into account how the firm's production decision influences its shareholders' utilities. In order to disregard this effect one would have to explain why those consumers who can benefit from the insurance possibility offered by the firm do not buy this insurance so that the firm is owned only by those consumers whose benefit can be neglected.

Once it is accepted that not only shareholders' profit motive has to be taken into account, it is apparent that much more information is needed than a profit maximizing firm in an economy with a complete set of markets is usually supposed to have. Within general equilibrium theory with incomplete markets (GEI for short) a literature has emerged that deals with firms on the basis that at least detailed information on the distribution of shareholders' characteristics within each firm is available. This development has been initiated by J.H. Drèze who published a seminal article in 1974, which has been reprinted in Drèze (1987). An extension of Drèze's original contribution can be found in his Yrjö Johansson Lectures (1989). The book by Magill and Quinzii (1996) contains an extensive exposition of the theory of incomplete markets. Production is covered in chapter 6.

Although the informational requirements are highly demanding, it is, in our opinion, desirable to ask how profit maximization as the objective of the firm should be extended to the case of market incompleteness without resorting to ad hoc short cuts. In this paper we want to shed light on the principle underlying the standard solution concepts used in GEI with production. The basic idea there can be summarized as follows. If a firm chooses a production plan with random output, then all shareholders participate in the risk embodied in this production plan in the same proportions, but they value future random income in different ways. Thus, they face a social choice problem [cf. Arrow (1950)].

The method now commonly used to overcome this problem, which goes back to Drèze (1974), requires the production decision of each firm to be such that the firm's shareholders cannot obtain a Pareto improvement through a change in production, if they redistribute their physical wealth at the initial time period $t = 0$ while keeping their shares fixed.

Clearly, it must be ruled out that future endowments or future income can be freely redistributed, since markets are incomplete. A promise of shareholder $A$ to give a (state dependent) sidepayment to $B$ tomorrow in order to get $B$'s concession needed to change the production plan now is not credible in this setting. On the other hand, the credibility issue does neither arise in case of endowments hold at $t = 0$ nor in case of assets hold at $t = 0$. One may ask why $A$ does not use the assets available today to get $B$'s consent. It is tempting to say that it does not matter whether $B$ obtains a certain asset directly from $A$ or whether $A$ gives $B$ the means needed to obtain the asset. This argument, however, takes it for granted that assets can be acquired when sidepayments are made to compensate the losers of a change in production. However, this possibility is ruled out, since shareholdings are supposed to be fixed when sidepayments are made. We shall present examples with the annoying property that every production plan can be improved upon by the choice of another plan accompanied by an exchange of physical wealth and shares at $t = 0$ among the shareholders.

The concept of a stock market equilibrium introduced in Drèze (1974) is defined for economies in which the distribution of individual initial endowments is not specified. However, it is important to observe that in economies with private ownership of endowments, sidepayments should be seen as merely potential. In equilibrium no redistribution is made and the individual decisions leading to an equilibrium are not affected by sidepayments. The role of the sidepayments at $t = 0$ is to eliminate those production decisions which could be improved upon by the shareholders if sidepayments would be possible. We shall later present examples in which sidepayments would allow the economy to move to an allocation that is Pareto superior for all consumers to the equilibrium under consideration. Thus, if sidepayments could be made, they should be carried out, but they are not [1].

There is an important discrepancy between the role of sidepayments in the equilibrium concept just described and the definitions of constrained feasibility and constrained Pareto efficiency. A social planner engaged in creating a constrained optimum can redistribute wealth at date $t = 0$ freely whereas agents never carry out sidepayments.

In traditional general equilibrium theory with a complete system of markets the issue of sidepayments we have just mentioned can be regarded as irrelevant.

[1] These examples do not rely on firms failing to coordinate their production plans efficiently, since only one firm is assumed to exist.

A Walrasian equilibrium, in which no sidepayments are made, is Pareto efficient. Therefore, no social planner is able to use his power to perform a lump sum redistribution so as to make all consumers better off. Efficiency and distribution can be dealt with separately. However, we shall argue in this paper that the discrepancy in the treatment of sidepayments described above becomes crucial when the assumption of market completeness is given up. Indeed, it may well be that the following scenario obtains. The equilibrium concept is designed so as to incorporate the first order condition for constrained Pareto efficiency. Also, constrained Pareto efficient allocations exist. However, none of the equilibria is constrained Pareto efficient.

Readers familiar with Guesnerie's paper (1975) on marginal cost pricing equilibria will notice a striking similarity. The last three sentences in the previous paragraph hold true in the framework examined in Guesnerie (1975), when the appropriate words are substituted [2]. Guesnerie has shown that all marginal cost pricing equilibria can be Pareto inefficient given that wealth is distributed in a predetermined way. Redistribution can be indispensable to obtain efficiency. The phenomenon discovered by Guesnerie also occurs in the present context and it does so for the same reason. Guesnerie considers nonconvex production sets and their nonconvexity entails, in particular, a breakdown of the first fundamental theorem of welfare economics. If markets are incomplete, the constrained feasible set necessarily becomes nonconvex. This global nonconvexity entails that efficiency and distribution become closely entangled.

In the language of chapter 6 of Magill and Quinzii (1996), we are going to analyze partnership economies. In particular, technologies will exhibit constant returns to scale. The solution concept adopted is the partnership equilibrium, which represents a simple case of a Drèze equilibrium. This setting has been chosen, because it lends itself particularly well to a presentation of our ideas without getting involved into additional, distracting issues. Throughout the paper we try to avoid any unnecessary complexity. In Section 2 we introduce a partnership economy with a single firm and discuss the concept of a partnership equilibrium. In Section 3 we give an example of an economy where consumers have quasilinear preferences and where shareholders's social surplus is minimized at a partnership equilibrium. We also show that majority voting can entail a unique outcome which is a surplus minimum. In Section 4 income effects are permitted. We present an example of an economy in which there is a unique partnership equilibrium that is not constrained Pareto efficient. The same result can be obtained using von Neumann-Morgenstern utility functions. Finally we describe an economy with three equilibria, each of which is potentially Pareto dominated by the remaining two equilibria. Section 5 concludes.

[2] The sentences become: Marginal cost pricing equilibria incorporate the first order condition for Pareto efficiency. Also, Pareto efficient allocations exist. However, none of the marginal cost pricing equilibria is Pareto efficient.

## 2 The Partnership Economy with a Single Firm.

The framework of our examples is the partnership economy with a unique firm, which we are now going to introduce and discuss [3]. We consider an economy with two periods $t = 0, 1$ and $S$ possible states of nature at $t = 1$. The states at time $t = 1$ are denoted $s = 1, \cdots, S$ and the unique state at $t = 0$ is included as the state $s = 0$. Thus, there are $S+1$ states in total. We assume that there is a single good in each state. There is a finite number $I$ of consumers. Each consumer $i$ has a consumption set $X^i$ and possesses the initial endowment $e^i \in X^i$. For simplicity, $X^i$ is assumed to be the nonnegative orthant in $\mathbb{R}^{S+1}$. Consumer $i's$ preference relation on $X^i$ is given by a $C^1$ quasiconcave utility function $U^i : X^i \to \mathbb{R}$ which is assumed to be weakly monotone. We assume that $\partial U^i / \partial x_0 > 0$ and that there exists, for any state $s$, a consumer $i$ with $\partial U^i / \partial x_s > 0$.

Furthermore, there is one business venture available to the consumers. The venture determines one production plan $y$ to be chosen from its technology set $Y \subset \mathbb{R}^{S+1}$, which we assume closed, convex, and containing the production plan $y = 0$. In a partnership economy it is assumed that the technology $Y$ exhibits constant returns to scale [cf. Magill and Quinzii (1996), p. 356 f. for a discussion of this assumption]. Production transforms inputs at $t = 0$ into state dependent outputs at $t = 1$. Hence, $y_0 \leq 0$ for all $y = (y_0, \cdots, y_S) \in Y$, whereas $y_s \geq 0$ for $1 \leq s \leq S$. Since there is only one good in each state $s$, a production plan $y = (y_0, y_1, \cdots y_S)$ represents simply an asset transferring wealth across time in a state dependent way.

At $t = 0$ every consumer $i = 1, \cdots, I$ is free to become a partner in the business venture, i.e. to participate in the corresponding asset. As a partner in the firm consumer $i$ invests a positive amount of input at time $t = 0$. In return, he is entitled to the corresponding part of the random output at $t = 1$. Clearly, all partners hold proportions of the same risky asset. Due to differences in initial endowments and preferences different partners will, in general, have different views about what vector $y \in Y$ suits their needs best. However, it is assumed that the venture can only commit itself to realizing one production plan. More explicitly, it is not allowed that $y$ is split into a sum $y = \bar{y} + \hat{y}$ where different partners can hold different proportions of $\bar{y}$ and $\hat{y}$. Asset splitting is ruled out in order to avoid that markets become complete, but the model does not explain why assets cannot be split. It is also assumed that there are no other assets than the one created by the firm. In particular, we do not include bonds into the model, since we want to shed light on the concept of a Drèze equilibrium in the most clear-cut situation.

A partnership equilibrium in which the firm is active can be viewed as follows. The firm proposes a production ray, i.e. a normalized production plan $\tilde{y}_N \in Y$

[3] The definitions in this section follow Magill and Quinzii (1996).

with inputs at $t = 0$ equal to $-1$. Then every consumer $i$ chooses individual production activity levels $\tilde{\alpha}^i$ and instructs the firm to produce $\tilde{\alpha}^i \tilde{y}_N$ for his personal use. Thus, consumer $i$ will finally consume $e^i + \tilde{\alpha}^i \tilde{y}_N$ and he selects $\tilde{\alpha}^i$ so as to maximize his utility given the production ray $\tilde{y}_N$. Clearly, total production of the firm equals $\tilde{y} = \sum_{i=1}^I \tilde{\alpha}^i \tilde{y}_N$ and $\tilde{\vartheta}^i = \tilde{\alpha}^i / (\sum_{i=1}^I \tilde{\alpha}^i)$ can be interpreted as $i$'s share in the firm.

Furthermore, in a partnership equilibrium the following consistency requirement is made in order to generate a link between the interests of the group $\tilde{\mathfrak{I}} = \{i = 1, \cdots, I \mid \tilde{\alpha}^i > 0\}$ of shareholders and the normalized production plan $\tilde{y}_N$. There is no alternative production plan in $y \in Y$ leading to an allocation that is unanimously preferred by all members of $\tilde{\mathfrak{I}}$ after they have redistributed their initial endowment of the input commodity at time $t = 0$ in a suitable way. However, shareholders are, by definition, not allowed to change their shares $\tilde{\vartheta}^i = \tilde{\alpha}^i / (\sum_{i=1}^I \tilde{\alpha}^i)$ in the production activities when the production plan $\tilde{y} = \sum_{i=1}^I \tilde{\alpha}^i \tilde{y}_N$ is replaced by the alternative $y$ even if the risk embodied in $y$ is substantially altered and their wealth is changed. Clearly, this fact makes it harder to achieve a Pareto improvement through a change in production combined with sidepayments among the shareholders.

Observe that the formal definition of a partnership equilibrium given below is phrased in a way that appears to be different from the usual one although the concept itself is not altered. Since the production set has constant returns to scale, no market clearing condition for shares is imposed in a partnership equilibrium. That is to say, the equation $\sum_{i=1}^I \vartheta^i = 1$ should be interpreted as no more than a sometimes convenient normalization. Indeed, there is no market on which shares are bought and sold in a partnership economy.

We use the following notation. Sidepayments are expressed in positive or negative multiples of the commodity vector $e_0 = (1, 0, \cdots, 0)$. In order to include the possibility that the firm is shut down in a partnership equilibrium, the normalized production plan $\tilde{y}_N$ will also be allowed to be equal to 0.

**Definition .** *A system consisting of a normalized production plan $\tilde{y}_N \in Y$ with inputs at $t = 0$ equal to $-1$ or $\tilde{y}_N = 0$, consumption plans $(\tilde{x}^i)_{i=1,\cdots,I} \in \prod_{i=1}^I X^i$, and individual production activity levels $(\tilde{\alpha}^i)_{i=1,\cdots,I} \in \mathbb{R}_+^I$ constitutes a* partnership equilibrium *if*

(i) *for every consumer $i = 1, \cdots, I$, the bundle $\tilde{x}^i = e^i + \tilde{\alpha}^i \tilde{y}_N \in X^i$ maximizes $U^i(e^i + \alpha^i \tilde{y}_N)$ subject to $\alpha^i \geq 0$;*

(ii) *the group of partners $\tilde{\mathfrak{I}} = \{i \mid \tilde{\alpha}^i > 0\} \neq \emptyset$ and it is impossible to find a production plan $y \in Y$ and a system of sidepayments $(\tau^i)_{i \in \tilde{\mathfrak{I}}}$ with $\sum_{i \in \tilde{\mathfrak{I}}} \tau^i = 0$ such that $U^i(e^i + \tau^i e_0 + \tilde{\vartheta}^i y) > U^i(\tilde{x}^i)$ for all $i \in \tilde{\mathfrak{I}}$, where $\tilde{\vartheta}^i = \tilde{\alpha}^i / (\sum_{i=1}^I \tilde{\alpha}^i)$.*

It is apparent that a partnership equilibrium is a fixed point, in which each individual consumer determines his use of the production possibilities taking the normalized production plan $\tilde{y}_N$ as given. On the other hand, when the redistribution of wealth at time $t = 0$ among partners is considered in part (ii) to check the optimality of the production plan, the proportions in which consumers use the production facility are held fixed.

By assumption, firms in a partnership economy have constant returns to scale. The agents only face the social choice problem of determining a production ray. The intensity with which an individual consumer wishes to use a production ray once it is chosen is a purely private matter, since $i$'s choice of any activity level $\alpha^i$ has no consequences for the production possibilities available to $i' \neq i$. In this respect, the use of a ray is analogous to the use of a purely public good with free disposal [4].

In order to illustrate the nature of the assumption that shares are held fixed when production plans are evaluated, consider the following analogy. The inhabitants of town $A$ are planning to connect $A$ to another town $B$ by a freeway. There is no danger of congestion and the construction of the freeway does not cost anything to the inhabitants of $A$. There are only private user costs such as expenses for gasoline, that every inhabitant pays according to his personal use. Also, suppose that individual 1 would like to drive to $A$ twice as often as individual 2. A meeting is called in $A$'s town hall to discuss whether it would not be preferable for the inhabitants of $A$ to build the freeway to $C$ instead of $B$. Clearly, there is no reason to assume that person 1 will drive to $C$ twice as often as person 2. The benefits associated with a freeway towards $C$ cannot be measured using data containing the individual use of a freeway to $B$. Therefore, a comparison based on such an assumption can easily be misleading. For the same reason, when partners meet to discuss a potential change of a production ray to improve their welfare, they should assume that each of them adjusts his personal use of the production facility according to his own taste.

It is well-known that condition (ii) above has an equivalent formulation in terms of profit maximization [cf. Magill and Quinzii (1996), p. 364 f.]. This fact can, of course, also be used to see that only the production ray and not the whole production plan matters in the definition of a partnership equilibrium. Clearly, a production plan $y$ of a firm with constant returns to scale maximizes profits if

[4] If we would deal with a model in which returns to scale are assumed to be strictly decreasing rather than constant, the situation would become asymmetric. The public would appreciate a decrease, but not an increase, in the use of a firm by every consumer $i$, since lowering the activity level entails a reduction in unit costs. The asymmetry between the beneficial effect of a reduction of the intensity with which a person uses a firm with convex costs and the harmful effect caused by an extension is not reflected in any of the equilibrium concepts for economies with incomplete markets and firms that we are aware of.

and only if $\lambda y$ does for any $\lambda > 0$. Define the normalized utility gradient $\pi^i(x)$ of consumer i by [5]

$$\pi^i_s(x) = \frac{\partial_s U^i(x)}{\partial_0 U^i(x)} \quad \text{for } s = 0, \cdots, S\,.$$

Then, assuming $\tilde{x}^i_0 > 0$ for all partners, condition (ii) in the definition of a partnership equilibrium can be reformulated as [6]:

(ii') The group of partners $\tilde{\mathfrak{I}} = \{i \mid \tilde{\alpha}^i > 0\} \neq \emptyset$ and for all plans $y \in Y$

$$(\sum_{i\in\tilde{\mathfrak{I}}} \tilde{\vartheta}^i \pi^i(\tilde{x}^i)) \cdot y \leq (\sum_{i\in\tilde{\mathfrak{I}}} \tilde{\vartheta}^i \pi^i(\tilde{x}^i)) \cdot (\sum_{i\in\tilde{\mathfrak{I}}} \tilde{\alpha}^i \tilde{y}_N)\,,$$

where $\tilde{\vartheta}^i = \tilde{\alpha}^i / (\sum_{i=1}^I \tilde{\alpha}^i)$.

According to equilibrium condition (ii') profits of the firm are maximized with respect to a price system which is the weighted sum of the normalized gradients of the partners of the firm. Hence, the price system does not only depend on the distribution of shares $\tilde{\vartheta}^i$, but also on the consumption plans realized by the shareholders. In partnership economies with several firms, different firms typically maximize profits in the sense of (ii') with respect to different price systems in equilibrium allocations so that prices cannot at all perform the task of coordinating production decisions. This fact is well known and the lack of coordination of production decisions across firms has already been pointed out by Drèze (1974). Hence, we will not address this question in this paper.

Since the optimality requirements expressed in (ii) and (ii') are designed to capture the well being of the partners of a particular firm, we will investigate instead to what extent a partnership equilibrium achieves this goal if coordination failures can be disregarded. For simplicity we assume that there are only two states at $t = 1$, which is the special setting used in the examples in the next sections. The production set of the firm is then given by

$$Y = \{\alpha(-1, \lambda, 1-\lambda) \in \mathbb{R}^3 \mid \alpha \geq 0,\ \lambda_0 \leq \lambda \leq \lambda_1\}$$

for some $\lambda_0 \geq 0$ and $\lambda_0 < \lambda_1 \leq 1$. Moreover, we assume that the initial endowment of every consumer is $e^i = (e^i_0, 0, 0)$ and that $U^i(x_0, \alpha^i\lambda, \alpha^i(1-\lambda))$ is strictly quasiconcave for any given $\lambda \in [\lambda_0, \lambda_1]$.

[5] $\partial_j U^i(x)$ denotes the partial derivative of the function $U^i$ with respect to the coordinate $j$ evaluated at $x$.

[6] The reformulation in Magill and Quinzii (1996) is made under the assumption that consumption plans lie in the interior of $\mathbb{R}^{S+1}_+$. However, in our setting the proof carries over to the case in which $\tilde{x}^i_0 > 0$ for all $i \in \tilde{\mathfrak{I}}$.

Consider a production ray $\lambda$ and a consumer $i$. Let $\alpha^i$ be his optimal production activity level and assume $\alpha^i > 0$, i.e. $i$ is a partner in the firm given the ray $\lambda$. We define partner $i's$ marginal willingness to pay for an infinitesimal change of the production ray $\lambda$ by

$$MW^i(\lambda) = \frac{\partial_\lambda U^i(e^i + \alpha^i(-1, \lambda, 1-\lambda))}{\partial_0 U^i(e^i + \alpha^i(-1, \lambda, 1-\lambda))}.$$

$MW^i(\lambda)$ expresses the marginal willingness to pay for an infinitesimal change in the production ray $\lambda$ in terms of the marginal utility of wealth in the state $s = 0$ at the optimal consumption bundle [7]. In general, this normalization depends, of course, on $\lambda$ and differs among the partners.

The following Remark states that it suffices to focus on production plans $y$ with input $y_0 = -\sum \tilde{\alpha}^i$ in part (ii) of the definition of a partnership equilibrium. The Remark allows us to keep the individual activity levels fixed while checking condition (ii).

**Remark .** *Consider a partnership equilibrium with $\tilde{x}_0^i > 0$ for all partners and $\tilde{y}_N \neq 0$. Then condition* (ii) *holds for all production plans $y \in Y$ iff it holds for all $y \in Y$ such that $-y_0 = \sum \tilde{\alpha}^i$.*

*Proof.* First observe that $y = 0$ together with a system of sidepayments can never lead to an allocation preferred by the partners. Hence, assume that there is a $y \neq 0$ and a system of sidepayments $(\tau^i)_{i \in \tilde{\mathfrak{I}}}$ with $\sum_{i \in \tilde{\mathfrak{I}}} \tau^i = 0$ such that $U^i(e^i + \tau^i e_0 + \tilde{\vartheta}^i y) > U^i(\tilde{x}^i)$. Let $\tilde{y} = (\sum_{i \in \tilde{\mathfrak{I}}} \tilde{\alpha}^i)\tilde{y}_N$. Then $\pi^i(\tilde{x}^i) \cdot (\tau^i e_0 + \tilde{\vartheta}^i(y - \tilde{y})) > 0$ for all partners $i$ by Proposition 31.2 in Magill and Quinzii (1996). Let $\beta > 0$ be such that $\beta y_0 = \tilde{y}_0$. Since the $\tilde{x}^i$ are chosen optimally, we have $\pi^i(\tilde{x}^i) \cdot \tilde{y} = 0$. Hence we obtain $\pi^i(\tilde{x}^i) \cdot (\beta \tau^i e_0 + \tilde{\vartheta}^i(\beta y - \tilde{y})) > 0$ for all $i \in \tilde{\mathfrak{I}}$. Again, according to Proposition 31.2, there exists $\mu > 0$ such that $U^i(\tilde{x}^i + \mu\beta\tau^i e_0 + \tilde{\vartheta}^i \mu(\beta y - \tilde{y})) > U^i(\tilde{x}^i)$ for all $i \in \tilde{\mathfrak{I}}$, since $\tilde{x}_0^i > 0$. Let $\hat{y} = \tilde{y} + \mu(\beta y - \tilde{y})$ and $\hat{\tau}^i = \mu\beta^i\tau^i$. We have $\hat{y} \in Y$ since $Y$ is a convex cone. Moreover, $\hat{y}_0 = \tilde{y}_0 = -\sum_{i \in \tilde{\mathfrak{I}}} \tilde{\alpha}^i$, $\sum_{i \in \tilde{\mathfrak{I}}} \hat{\tau}^i = 0$, and $U^i(e_0^i + \hat{\tau}^i e_0 + \tilde{\vartheta}^i \hat{y}) > U^i(\tilde{x}^i))$ for all $i \in \tilde{\mathfrak{I}}$. □

Now we give a characterization of a partnership equilibrium in terms of the partners' marginal willingnesses to pay. This characterization is useful when we calculate the partnership equilibria in the examples in the Sections 3 and 4.

**Proposition .** *Consider the system $(\tilde{y}_N, (\tilde{x}^i, \tilde{\alpha}^i)_{i=1,\cdots,I})$, where $\tilde{x}_0^i > 0$ for all partners $i$ and where $\tilde{y}_N = (-1, \tilde{\lambda}, 1 - \tilde{\lambda})$ is a normalized production plan with $\tilde{\lambda} \in [\lambda_0, \lambda_1]$. Then $(\tilde{y}_N, (\tilde{x}^i, \tilde{\alpha}^i)_{i=1,\cdots,I})$ is a partnership equilibrium iff*

(i) $\tilde{x}^i = e^i + \tilde{\alpha}^i(-1, \tilde{\lambda}, 1 - \tilde{\lambda})$, *where $\tilde{\alpha}^i$ is $i$'s optimal activity level.*

[7] In the definition of $MW^i(\lambda)$ the optimal production activity level $\alpha^i$ could be allowed to change with $\lambda$ according to Shephard's lemma.

(ii”) *Let* $\tilde{\mathfrak{I}} = \{i = 1, \cdots, I \mid \tilde{\alpha}^i > 0\}$ *be the group of partners. Then* $\tilde{\mathfrak{I}} \neq \emptyset$ *and*

(a) $\sum_{i \in \tilde{\mathfrak{I}}} MW^i(\tilde{\lambda}) = 0$ *and* $\lambda_0 < \tilde{\lambda} < \lambda_1$ *or*

(b) $\sum_{i \in \tilde{\mathfrak{I}}} MW^i(\tilde{\lambda}) \geq 0$ *and* $\tilde{\lambda} = \lambda_1$ *or*

(c) $\sum_{i \in \tilde{\mathfrak{I}}} MW^i(\tilde{\lambda}) \leq 0$ *and* $\tilde{\lambda} = \lambda_0$.

*Proof.* From (ii') and the above Remark we know that $\tilde{\lambda}$ is a partnership equilibrium iff $(\sum_{i \in \tilde{\mathfrak{I}}} \tilde{\vartheta}^i \pi^i(\tilde{x}^i))(\sum_{i \in \tilde{\mathfrak{I}}} \tilde{\alpha}^i)(0, \lambda - \tilde{\lambda}, -(\lambda - \tilde{\lambda})) \leq 0$ for all $\lambda \in [\lambda_0, \lambda_1]$ and hence iff $\sum_{i \in \tilde{\mathfrak{I}}} (\pi_1^i(\tilde{x}^i) - \pi_2^i(\tilde{x}^i))\tilde{\alpha}^i(\lambda - \tilde{\lambda}) \leq 0$ for all $\lambda \in [\lambda_0, \lambda_1]$. Moreover, by definition $MW^i(\tilde{\lambda}) = (\pi_1^i(\tilde{x}^i) - \pi_2^i(\tilde{x}^i))\tilde{\alpha}^i$. Thus $\tilde{\lambda}$ is a partnership equilibrium iff $\sum_{i \in \tilde{\mathfrak{I}}} MW^i(\tilde{\lambda})(\lambda - \tilde{\lambda}) \leq 0$ for all $\lambda \in [\lambda_0, \lambda_1]$. From this statement (a), (b), and (c) follow trivially. □

We are now going to recall the definition of a feasible allocation constrained by market incompleteness and that of constrained Pareto efficiency. These definitions are inspired by the setting in which the first and second welfare theorem in general equilibrium theory with complete markets are formulated. As is usual in the literature, any allocation that can be implemented by a powerful, omniscient planner, who simultaneously determines all production plans, the shares of all consumers and who, moreover, freely redistributes the endowments at time $t = 0$ in a lump sum way, is considered to be constrained feasible.

**Definition .** *An allocation of consumption plans* $(x^i)_{i=1,\cdots,I}$ *is* constrained feasible *if* $x^i \in X^i$ *for all* $i$ *and if there exist a normalized production plan* $y_N$ *and, for all* $i = 1 \cdots, I$*, nonnegative individual production activity levels* $\alpha^i$ *and sidepayments* $\tau^i$ *at time* $t = 0$ *such that*

(a) $y_N \in Y$ *and* $\sum_{i=1}^I \alpha^i > 0$,

(b) $\sum_{i=1}^I \tau^i = 0$,

(c) $x^i = e^i + \tau^i e_0 + \alpha^i y_N$.

In accordance with the above definition of a feasible allocation, constrained efficiency is defined as follows [8].

**Definition .** *An allocation of consumption plans is* constrained Pareto optimal *if it is constrained feasible and if there is no Pareto superior constrained feasible allocation.*

[8] Since we shall later only consider allocations where the consumers have positive wealth at $t = 0$, the strict monotonicity assumption concerning $x_0$ entails that one does not have to distinguish between strict and weak Pareto improvements.

In economies with private ownership of endowments, it is natural to check whether an equilibrium is potentially Pareto dominated.

**Definition .** *The equilibrium corresponding to the ray $\tilde{\lambda}$ is* potentially Pareto dominated *if there are a ray $\lambda$ and sidepayments $\tau^i$ at $t = 0$ with $\sum_{i\in\tilde{\mathfrak{J}}} \tau^i = 0$ such that, after choosing their optimal investments given $\lambda$, all consumers are better off.*

Clearly, a constrained Pareto efficient equilibrium cannot be potentially Pareto dominated. The examples in Section 4 show, however, that it can occur that all Drèze equilibria are potentially Pareto dominated.

Magill and Quinzii (1996), p. 369, state conditions ensuring that a partnership equilibrium satisfies the first order condition for constrained Pareto optimality. From this perspective, a partnership equilibrium seems to be a promising candidate for an appropriate equilibrium concept. However, the following aspects deserve to be discussed.

Partnership equilibria are defined for economies with private ownership. In this respect the concept is similar to a Walrasian equilibrium, but differs from the stock market equilibrium as originally introduced in Drèze (1974), where, as in the usual definition of Pareto efficiency, only aggregate endowments matter. Since no sidepayments are actually carried out, the redistribution mentioned in condition (ii) should be regarded as potential only. A partnership equilibrium is, so to speak, "redistribution proof". Correspondingly, there are obviously no sidepayments involved in conditions (ii') and (ii").

This leads to the question of whether the information necessary to check equilibrium condition (ii) is sufficient to actually carry out sidepayments within the group of partners. The answer is no for the following reason. To check condition (ii) it is sufficient to know the distribution of characteristics of the partners. However, to actually carry a redistribution through, one needs to know more than this distribution. In order to give a sidepayment, that a consumer with a certain characteristic should get, to the right person, one has to identify the person associated with this characteristic. If only the distribution of characteristics is known, individuals remain anonymous and, although sidepayments can be computed, they cannot be paid out. Thus, in terms of informational requirements underlying the concepts of constrained optimality and partnership equilibria, respectively, the planner in the definition of constrained feasibility is assumed to be able to know which consumers have which characteristics, whereas only the distribution of the characteristics matters in the definition of a partnership equilibrium.

# 3 Quasilinear Preferences.

## 3.1 Partnership equilibria.

Since the potential redistribution of initial endowments at $t = 0$ plays a major role in the definition of a partnership equilibrium, it is natural to focus first on a setting in which utility can be transferred by an exchange of endowments at $t = 0$ without affecting the attitude towards risk consumers face at $t = 1$. Thus, we are now going to assume all consumers to have quasilinear preferences that are additively separable in $x_0$. In this case, consumers' surplus is well defined and can be used as a welfare measure. The role of income effects will be examined in Section 4.

The equilibrium characterization in terms of marginal willingnesses to pay as stated in the above Proposition shows that second order conditions are not captured in the concept of a partnership equilibrium. Clearly, the envelope theorem implies that portfolios which have been optimally adjusted to the production decisions can be taken as fixed as long as first order utility changes are regarded only. However, the following example shows that neglecting effects of higher order can have a decisive impact on social welfare. Indeed, the consequences do not only matter locally, but on the whole spectrum of possible production decisions.

In the following example - and most others in this paper - all consumers hold shares for any production ray the only firm might choose. Thus, there cannot be any coordination failure. As a consequence, any failure must be attributed to the fact that, in the definition of a partnership equilibrium, production plans are chosen while portfolios are kept fixed.

We assume that there are two states at $t = 1$ and consider production rays given by $(-1, \lambda, 1-\lambda)$, where $\lambda \in [\,\lambda_0, \lambda_1\,]$. We investigate the symmetric case $\lambda_1 = 1 - \lambda_0$ with $0 < \lambda_0 < 1/2$ [9]. For given $\lambda$, the input $y_0 \leq 0$ at time 0 yields the output $|y_0| \cdot (\lambda, 1-\lambda)$ at time 1.

There are two types of consumers with initial endowments

$$e^1 = (1,\, 0,\, 0)\,, \quad e^2 = (1,\, 0,\, 0)$$

and quasilinear utilities given by

$$U^1(x_0, x_1, x_2) = x_0 + x_1^{0.6},$$
$$U^2(x_0, x_1, x_2) = x_0 + x_2^{0.6},$$

[9] The assumption $\lambda_0 > 0$ entails that every consumer's share $\vartheta^i$ stays away from 0 for all $\lambda$. It is made here only to avoid discussions about changes in the composition of the partners in the firm.

respectively. Ideally one should think of each type of consumers as being represented in the economy by a continuum of mass 1. For convenience, we shall refer to each such continuum of identical consumers as a single consumer denoted $i = 1, 2$, respectively.

For any given $\lambda$ consumer 1 invests $\alpha^1 = 0.6(0.6\lambda)^{1.5}$. Thus, consumer 1, who would like to have $\lambda$ as large as possible, will participate in the partnership even if $\lambda$ is small. For consumer 2, who would like to have $\lambda$ small, the situation is analogous. Moreover, redistribution of endowments at $t = 0$ does not affect the demand for shares. Therefore, the amount of $x_0$ needed to "persuade" a shareholder to agree to changing the production ray from $\lambda$ to some $\lambda'$ is determined in a particularly simple way.

In the situation just described it is not clear on intuitive grounds why the shareholders should not be able to achieve any combination $(\lambda, \alpha^1, \alpha^2)$ for themselves that a benevolent social planner could allocate to them. Economic intuition suggests that shareholders aim at an outcome maximizing consumers' surplus [10].

Before we analyze the set of partnership equilibria we would like to illustrate the nature of the indirect utility functions of the consumers. The *indirect utility* $u^i(x_0, \lambda)$ is the maximal utility consumer $i$ can obtain if his initial endowment is $(x_0, 0, 0)$ and he has free access to the production ray given by $(-1, \lambda, 1 - \lambda)$. Although the preferences are extremely simple, Figure 1 shows that the indirectly preferred sets in the $(x_0, \lambda)$-plane are not convex.

It turns out that the economy under consideration has three partnership equilibria $A, B$ and $C$ corresponding to $\lambda_A = \lambda_0, \lambda_B = 1/2$, and $\lambda_C = \lambda_1$, respectively. It is instructive to investigate the interior equilibrium $B$. For that purpose we first consider the indirect utility $u^1(1, \lambda)$ consumer 1 with endowment $e_0^1 = 1$ obtains if the firm chooses the ray $\lambda$ and if he has his optimal investment $\alpha^1 = 0.6(0.6\lambda)^{1.5}$. Since this utility equals $u^1(1, \lambda) = 1 + 0.4(0.6\lambda)^{1.5}$, the function $u^1(1, \cdot)$ is convex. That is to say, the amount of $x_0$ consumer 1 is willing to give up for an infinitesimal increase in $\lambda$ is growing with $\lambda$. Similarly, the utility level $u^2(1, \lambda)$ consumer 2 obtains if $\lambda$ is chosen equals $u^2(1, \lambda) = u^1(1, 1 - \lambda)$ and is convex in $\lambda$. As a consequence, shareholders' social surplus associated with the ray $\lambda$, $u^1(1, \lambda) + u^2(1, \lambda)$, is convex in $\lambda$. Due to the symmetry between $u^1(1, \lambda)$ and $u^2(1, \lambda)$, the total utility $u^1(1, \lambda) + u^2(1, \lambda)$ has a critical point at $\lambda_B = 1/2$ which must be a global minimum [see Figure 2].

[10] Observe that producer's surplus is identically equal to zero due to the assumption of constant returns to scale.

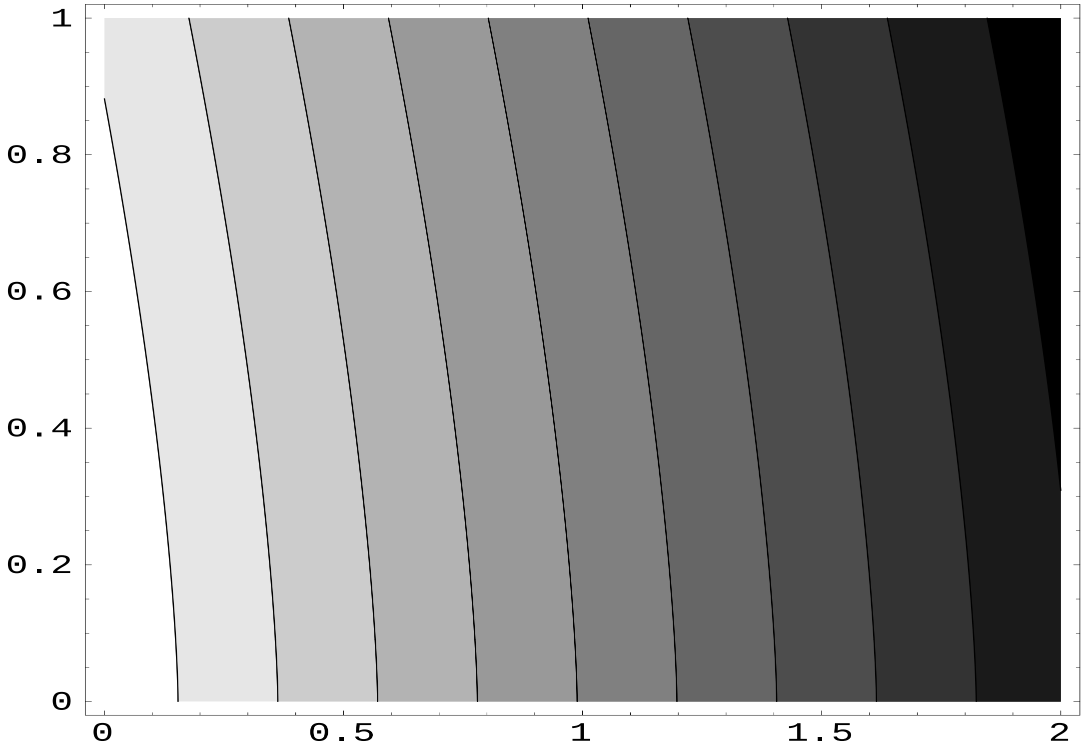


Figure 1: Nonconvex preferred sets of agent 1 in the $(x_0, \lambda)$-plane

Assume now that the firm has proposed to realize $\lambda_B$ and the shareholders meet to discuss this plan. At $\lambda_B$ consumer 1 knows that he is willing to give up as much of $x_0$ for an infinitesimal increase in $\lambda$ as consumer 2 needs to be compensated, since $\partial_\lambda u^1(1,\lambda)|_{\lambda=1/2} + \partial_\lambda u^2(1,\lambda)|_{\lambda=1/2} = 0$ according to the Proposition in Section 2. However, he also knows that he is willing to give up more than $\partial_\lambda u^1(1,\lambda)|_{\lambda=1/2}$ after $\lambda$ has been raised slightly above $\lambda_B = 1/2$, whereas consumer 2 needs less than this amount to be compensated for his additional loss. Due to the convexity of the two indirect utility functions $u^i(1,\lambda)$ with respect to $\lambda$ the compensation process gets additional momentum if the shareholders move further away from $\lambda_B = 1/2$.

Notice that the situation changes drastically if the shareholders are deprived of the possibility to adjust their shares $\vartheta^i$ to variations of $\lambda$. Consider consumer 1 who wants to choose his activity level $\alpha^1$ in proportion to $\lambda^{1.5}$. If $\alpha^1$ is now taken as fixed at its value at $\lambda_B = 1/2$, then the utility reached at ray $\lambda$ equals $\tilde{u}^1(1,\lambda) = c_0 + c_1\lambda^{0.6}$ with a positive constant $c_1$, whereas his indirect utility with share adjustment is a function of the type $u^1(1,\lambda) = c_0' + c_1'\lambda^{1.5}$ with $c_1' > 0$. Thus, by disregarding how consumer 1's individual activity level $\alpha^1$ grows with $\lambda$ the originally convex function $u^1(1,\cdot)$ is turned into a concave function $\tilde{u}^1(1,\cdot)$.

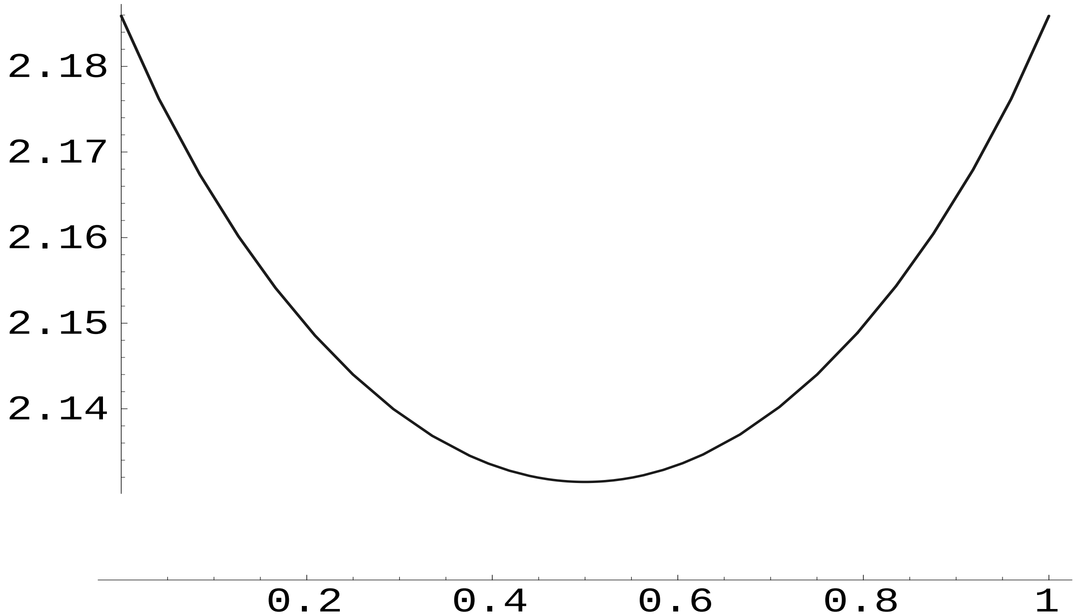


Figure 2: Surplus minimum at the partnership equilibrium $\lambda_B = 1/2$

As a consequence, $\tilde{u}^1(1, \cdot) + \tilde{u}^2(1, \cdot)$ is a concave function and the critical point $\lambda = 1/2$ becomes a maximum.

Suppose now that sidepayments can be made. Then equilibrium $B$ can be improved upon by a sidepayment combined with the choice of another ray. For instance, put $\lambda_0 = 0.01$. Consumer 2, who prefers the boundary equilibrium $\lambda_A = 0.01$ over $\lambda_B$ can offer a sidepayment of about 0.08 to consumer 1 if 1 agrees to switch to $\lambda_A$. If this deal is made, both agents are better off than before. This shows that equilibrium $B$ is potentially Pareto dominated.

In case of quasilinear preferences potential Pareto improvements are possible as long as the social surplus of the partners is not maximized. There is always at least one constrained Pareto efficient Drèze equilibrium, since a social surplus maximum exists. In the present example the maximum is reached at the partnership equilibria $A$ and $C$.

**Remark .** *The definition of a partnership equilibrium is not restrictive enough in order to rule out that shareholders' social surplus is minimized at a partnership equilibrium.*

In the symmetric case considered so far consumers' surplus as a function of $\lambda$ is a parabola taking its minimum at the interior equilibrium $B$. The boundary equilibria $A$ and $C$ are both global surplus maxima. This situation is certainly not robust. Indeed, by introducing asymmetry into the utility functions the interior equilibrium can be moved near to one of the boundary equilibria. As a consequence, we obtain two "bad" and one "good" equilibrium. Consider, for instance, the following setting:

$$U^1(x_0, x_1, x_2) = x_0 + x_1^{0.56}, \qquad e^1 = (1,0,0)$$
$$U^2(x_0, x_1, x_2) = x_0 + x_2^{0.9}, \qquad e^2 = (1,0,0)\,.$$

Using $\lambda_0 = 0.1$ to define bounds on $\lambda$, we get three equilibria corresponding to $\lambda_A = 0.1, \lambda_B \approx 0.104$, and $\lambda_C = 0.9$. Consider the total utility gain brought about by the partnership. The gain reaches its minimum at the interior equilibrium $\lambda_B$ where it equals 0.026221. The boundary equilibrium $\lambda_A$ is not much better, since the gain there amounts to 0.026236. The gain in total utility is maximal at $\lambda_C$ where it is equal to 0.184.

In both, the symmetric as well as the asymmetric example, there is at least one constrained Pareto efficient equilibrium. Since preferences are quasilinear, surplus maximization can be used to obtain such a desirable equilibrium. In a more general setting one would like to "refine" partnership equilibria by ruling out potentially Pareto dominated ones. However, when leaving the quasilinear framework in Section 4 we shall present an example showing that this procedure leads into serious difficulties due to the presence of strong income effects.

## 3.2 Majority voting.

It is often argued that a voting procedure should be used to derive the decisions of firms from the preferences of their shareholders [11]. Clearly, the social choice literature on voting is dominated by paradoxa and impossibility results. Therefore, it hardly lends itself to the development of a general, satisfactory theory of the decisions of firms. Here we want to ask the following question: Assume that the above objection can be ignored and that majority voting leads to a well defined outcome in all cases under consideration. Would it then be desirable on welfare grounds to adopt majority voting as a social decision rule?

The following example illustrates the fact that the outcome of majority voting can very well appear quite unsatisfactory from a normative viewpoint although all shareholders vote sincerely. We consider both, the one person-one vote and the one share-one vote case, and modify the example in the previous subsection slightly in order to always obtain a unique outcome of majority voting. The admissible rays are $\lambda \in [0.4, 0.6]$. Observe that the modified example has again two surplus maximizing partnership equilibria located at the end points of $[0.4, 0.6]$ and a surplus minimizing one at $\lambda = 1/2$.

[11] Majority voting as a decision rule for firms has been studied by several authors, see, e.g., DeMarzo (1993)

There are now three (groups of equal size of) consumers having the utility functions $U^1$ and $U^2$ as defined above and

$$U^3(x_0, x_1, x_2) = x_0 + (x_1 x_2)^{1/3} ,$$

respectively. To be specific, let $e^1 = e^2 = e^3 = (1, 0, 0)$. Clearly, consumer 3 likes $\lambda$ to be as close to $1/2$ as possible, whereas the other two agents want $\lambda$ to be at one of the endpoints of the interval $[0.4, 0.6]$ of admissible production rays. As a consequence, consumer 3 is a median voter. His most preferred choice $\lambda = 1/2$ defeats any other potential $\lambda$ by a majority of two thirds, if each shareholder has one vote. For any $\lambda$ all consumers want to participate in the firm and a straightforward calculation shows that their social surplus reaches its minimum at $\lambda = 1/2$. The social surplus exhibits the same qualitative shape as in Figure 2.

The construction of the above example illuminates the fact that majority voting and welfare considerations can be totally unrelated. We modified the symmetric example in subsection 3.1, in which majority voting always ends in a draw, by adding a median voter. Obviously, one can move the median voter's best choice freely around so that the outcome of majority voting and thereby the resulting value of social welfare becomes totally arbitrary. Observe that the example is robust except for the fact that a slight change of the median voter's utility function $U^3$ moves the outcome of majority voting to a $\lambda$ near the surplus minimizing one instead of the minimizer itself. In particular, the relative sizes of the three groups of consumers can be made unequal without any change of the result.

Now we assume that majority voting occurs under the one share-one vote rule. The number of shares depends, of course, on the $\lambda$ under consideration. At $\lambda = 1/2$ the shares of the three consumers are given by $(0.363, 0.363, 0.273)$. Therefore, $\lambda = 1/2$ passes the test against any other $\lambda$ with a majority of 58 percent. Furthermore, consider any admissible $\tilde{\lambda} \neq 1/2$. We show that $\tilde{\lambda}$ fails to pass a similar test. Obviously, in any comparison between $\tilde{\lambda}$ and $1/2$ consumer 3 always votes together with one of the other consumers against the remaining consumer. The shares consumer 1 or 2 can have for any $\lambda \in [0.4, 0.6]$ are below 48 percent. Thus, consumer 3 is again a median voter and defeats $\tilde{\lambda}$.

**Remark .** *Majority voting can entail a unique outcome at which shareholders' welfare expressed in terms of social surplus reaches its minimum. Thus, since the surplus minimum is necessarily a partnership equilibrium, majority voting can select the "worst" partnership equilibrium. The conclusion holds independently of whether each person or each share possesses one vote.*

### 3.3 Mismatch of firm and owner.

We are now going to discuss shortly another phenomenon that can occur if we allow the firm to be owned by part of the consumers only. For that purpose, consider the asymmetric version of the example in subsection 3.1 and let $\lambda$ vary over the whole interval $[0, 1]$. We obtain a partnership equilibrium at $\lambda = 0$ if the firm is owned by agent 2. Remember that $\lambda = 0$ is optimal for agent 2 and that agent 1 is not interested in becoming a partner if $\lambda = 0$. Another partnership equilibrium with agent 1 as sole proprietor is obtained at $\lambda = 1$. At $\lambda = 0$ consumers get utilities $u^1(1, 0) = 1, u^2(1, 0) = 1.03874$, whereas utilities at $\lambda = 1$ are $u^1(1, 1) = 1.21036, u^2(1, 1) = 1$, respectively.

Consider the socially less desirable equilibrium corresponding to $\lambda = 0$. If sidepayments are permitted, it can easily be overcome as follows. If agent 1 pays, say, 5 percent of his initial endowment at $t = 0$ as a sidepayment to agent 2 in order to get full control over the firm, both agents become better off.

Observe that this Pareto improvement relies on the following facts. First, the sidepayment is actually carried out. Second, the deal can only be done if shares are not fixed. Third, to obtain a Pareto improvement it is necessary that owners and nonowners get together. However, the definition of a partnership equilibrium does not take any of these points into account. Moreover, this example shows the close connection between issues of distribution and Pareto efficiency clearly. Also, it illustrates the point that the social planner in the definition of constrained Pareto efficiency is more powerful than the agents are in the definition of a partnership equilibrium.

To conclude this section we want to point out an extremely simple, instructive type of mismatch between firm and owner. Assume there is a consumer $j$ such that $\pi^j(e^j) \cdot y \leq 0$ for all $y \in Y$. Then a partnership equilibrium is obtained irrespective of the needs of all persons different from $j$ if the firm is closed down, i.e. $\tilde{y} = 0$, and assigned to $j$ as the sole proprietor, i.e. $\tilde{\vartheta}^i = 0$ for $i \neq j$ and $\tilde{\vartheta}^j = 1$. Clearly, a partnership should, in principle, be available to the whole society. However, as this example shows, the definition of a partnership equilibrium is not sufficiently welfare oriented. Again, the reason is that part (ii) of the definition is based on the principle that shares and thereby the membership in the club of partners is given.

# 4 Sidepayments, Income Effects, and Pareto Comparisons.

## 4.1 A unique, but potentially Pareto dominated partnership equilibrium.

In the previous section quasilinearity has been assumed in order to obtain a setting in which the redistribution of wealth at time $t = 0$ does not affect any consumer's evaluation of an asset. Although the existence of socially undesirable partnership equilibria such as surplus minima could be shown in this particular framework, one would expect additional, disturbing phenomena to occur if quasilinearity is not required. Clearly, if consumers' surplus is well defined, a surplus maximum exists. As a consequence, there is at least one partnership equilibrium that can be considered a desirable outcome. We are now going to ask whether a similar conclusion obtains if income effects are permitted.

In the following example there are two types of consumers. The first one exhibits strong income effects while the second one has the quasilinear utility function

$$U^2(x_0, x_1, x_2) = x_0 + {x_2}^{1/2}$$

and the initial endowment $(1, 0, 0)$. We take the power of $x_2$ to equal $1/2$ for the following reason. In this particular case, the marginal willingness to pay for a change of $\lambda$ is constant. More precisely, since type 2's endowment is taken to be $(1, 0, 0)$, his indirect utility becomes $u^2(1, \lambda) = 1 + (1 - \lambda)/4$. As a consequence, the sidepayment that is just sufficient to compensate type 2 for a change $\Delta\lambda$ in production is of the linear form $\Delta\lambda/4$. For the same reason, the indirectly preferred set of agent 2 in the $(x_0, \lambda)$-plane is convex, but raising the power $1/2$ slightly would destroy this property.

Types 1 and 2 are complementary to each other in the following sense. While 2's investment decision at $t = 0$ depends on the ray $\lambda$, but not on any sidepayment, 1's decision will strongly depend on sidepayments, but not on $\lambda$. Therefore, the effects in the example can be clearly attributed to the individual types.

As in the previous section, consumer 1 is only interested in state 1 while a consumer of type 2 only cares about state 2. To embody strong income effects into consumer 1's preferences we proceed as follows. Consider the CES-indifference curve $x_0^{0.9} + x_1^{0.9} = 1$ and take its image under the affine mapping $(x_0, x_1) \mapsto (x_0, \gamma x_1)$ for $0 \le \gamma < 1$. The resulting indifference pattern defines the preferences of consumer 1 in the part that is relevant for the example. We will calibrate the example such that the consumption of agent 1 stays well below the curve

$x_0^{0.9}+x_1^{0.9}=1$. In particular, 1's consumption of good 0 will have an upper bound of 0.73 that is far below 1. Therefore, the "turning point" $(1,0)$ associated with the above construction is outside the domain of interest in the example. There is no need to extend the preferences to the whole nonnegative orthant, but such an extension can easily be made. The endowment of consumer 1 is (0.95, 0, 0). In the relevant range his preferences are given by

$$U^1(x_0,x_1,x_2)=\frac{x_1}{(1-x_0^{9/10})^{10/9}}\,.$$

A computation yields that maximization of $U^1$ leads to an investment into production that is independent of the proposed ray $\lambda$ and that is of size $(x_0-x_0^{10})$ if consumer 1 possesses the amount $x_0<1$ at $t=0$ (after sidepayments). Therefore, he will consume $x_0^{10}$ at $t=0$. In a partnership equilibrium no sidepayments occur by definition and consumer 1 invests $0.95-0.95^{10}\approx 0.35$ so that approximately 0.6 units of his endowment remain for consumption at $t=0$. Clearly, these numbers vary if sidepayments are made. The pattern is such that consumer 1 invests less in the production activity if he is made richer. On the other hand, if consumer 1 gives some of his wealth to the consumers of type 2 in order to get a more favorable ray, i.e. a larger $\lambda$, consumer 1 invests more and more into production.

Assume that there are one consumer of type 1 and two agents of type 2 and let the technology consist of all rays given by the normalized production plans $(-1,\lambda,1-\lambda)$ where $\lambda\in[2/3,\ 0.99]$. The upper bound $0.99<1$ is chosen in order to avoid an autarky solution. A numerical computation shows that there is a unique partnership equilibrium at $\tilde{\lambda}\approx 0.7$ in this example [12].

We want to show that the equilibrium is potentially Pareto dominated by every other $\lambda\in[2/3,\ 0.99]$, that is to say, it is dominated by every non-equilibrium choice of $\lambda$, if appropriate sidepayments are made and agents choose their investment as they please [13].

[12] If we would add a third or fourth agent of type 2, uniqueness would still obtain, since the marginal willingness $MW^1(\lambda)$ of consumer 1 is decreasing and $MW^2(\lambda)$ is identically equal to the constant -1/4. More weight on agents of type 2 entails that the equilibrium value of $\lambda$ moves further downwards.

[13] We could also have taken the left endpoint of the interval smaller than 2/3. However, we want to make clear that the phenomenon of a unique, interior equilibrium that is everywhere potentially Pareto dominated over the whole range of feasible production rays depends on a sufficiently strong income effect, but not on the particular way in which the preferences of consumer 1 have been constructed. Therefore, the interval of feasible rays is selected here in such a manner that 1's consumption at $t=0$ after any sidepayment considered in the argument lies far below 1.

For that purpose, fix the utility level of both consumers of type 2 at its equilibrium value. Thus, put $\Delta\lambda = \lambda - \tilde{\lambda}$ and let each consumer of type 2 get the sidepayment $\Delta\lambda/4$. Accordingly, consumer 1 gets the sidepayment $-\Delta\lambda/2$. For $\Delta\lambda > 0$ consumer 1 gets a negative sidepayment. As a consequence, he consumes less at $t = 0$, but he invests more in production. A computation shows that 1's utility is monotonically increasing in $\Delta\lambda > 0$. Similarly, for $\Delta\lambda < 0$ the sidepayment to be given to 1 becomes positive. Hence, the wealth and also the consumption at $t = 0$ of consumer 1 increase. However, due to the restriction $\lambda \geq 2/3$ his consumption never exceeds 0.73. That is to say, over the whole range of rays under consideration 1's consumption at $t = 0$ stays well below the critical value 1. Again, the utility of consumer 1 increases if $\lambda$ moves further away from its equilibrium value. Thus, if the sidepayment given to both consumers of type 2 is just sufficient to make them indifferent to their position in the partnership equilibrium, consumer 1 does best if the economy moves far away from the equilibrium ray $\tilde{\lambda}$. The corresponding utility increase of consumer 1 is shown in Figure 3.

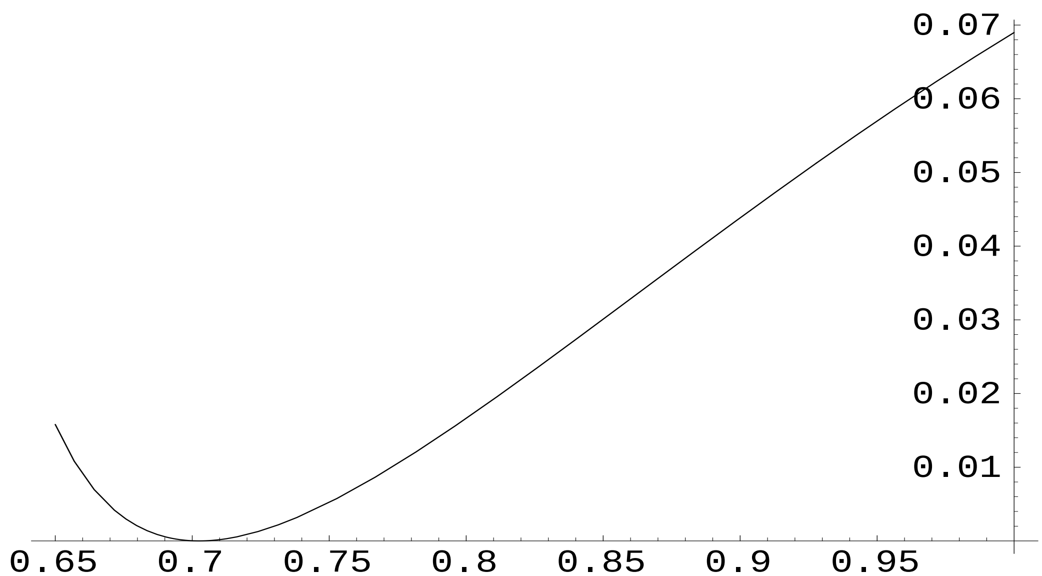


Figure 3: Utility increase of agent 1 after the sidepayment

**Remark .** *There exist robust examples of economies with a unique partnership equilibrium that is potentially Pareto dominated by any other possible choice of a production ray $\lambda$.*

If redistribution would be possible and sidepayments could actually be made, Pareto improvements could easily be achieved. One possibility would then be that consumer 1 gives an amount at least equal to $\Delta\lambda/4$ to type 2 in order to get

2's consent to a change of size $\Delta\lambda > 0$. For instance, if consumer 1 gives each of his partners the amount 0.1 at $t = 0$, then all agents become better off if $\lambda = 0.99$. Alternatively, the consumers of type 2 could give 0.01 at $t = 0$ to agent 1 so that 1 becomes so rich that he cares sufficiently little about the ray $\lambda$. In this case type 2 can obtain the $\lambda$ he likes most, i.e. $\lambda = 2/3$. Obviously, if sidepayments are possible, there are many ways to obtain Pareto improvements that affect shareholders quite differently. Therefore, if sidepayments were permitted, one would need a theory selecting among a large set of possible outcomes. Apparently, this fact constitutes a major reason why sidepayments are, in general, not carried out in the theory of incomplete markets with individually specified ownership of initial endowments.

## 4.2 Von Neumann-Morgenstern utilities.

One might think that the last example is ruled out if agents maximize expected utility. Therefore, we address this question now. Suppose there is a lottery giving an agent the bundle $(x_0, x_1)$ with probability $p$ and the bundle $(x_0, x_2)$ with probability $(1 - p)$. The first component of these two bundles represents consumption at date $t = 0$ and is independent of the outcome of the lottery. The second stands for the consumption at $t = 1$ and is random. It is assumed that $p$ is the same for every consumer, since $p$ is the publicly known probability that the world will be in state 1 tomorrow. Moreover, each consumer has a utility function over consumption today and tomorrow that is independent of which state will obtain at $t = 1$.

More precisely, each lottery is characterized by the triple $(x_0, x_1, x_2)$ and the number $p$. Preferences over the lotteries are defined by

$$U^1(x_0, x_1, x_2) = p\log\left(\frac{x_1}{(1 - x_0^{0.9})^{1/0.9}}\right) + (1 - p)\log\left(\frac{x_2}{(1 - x_0^{0.9})^{1/0.9}}\right)$$
$$U^2(x_0, x_1, x_2) = p\frac{x_1}{(1 - x_0^{0.99})^{1/0.99}} + (1 - p)\frac{x_2}{(1 - x_0^{0.99})^{1/0.99}}\,.$$

Each consumer has an endowment of 0.98 at $t = 0$ and no endowment at $t = 1$.

In this example we consider one agent of type 1 and 3 agents of type 2. We choose $p = 0.45$ and assume that the firm has production rays in the interval $\lambda \in [0.1, 0.2]$ at its disposal. As in case of consumer 1 in the previous section, the investment of both consumers considered here is independent of the choice of $\lambda$ and depends on their wealth only. In particular, the investment is independent of the probability $p$.

For every $\lambda \in ]0, 1[$ each agent has a monotonically declining marginal willingness to pay for a slight increase of $\lambda$. Hence the partnership equilibrium must be unique. It turns out to be at $\tilde{\lambda} \approx 0.13$. The equilibrium consumption of both agents is about 0.82, which is clearly below the critical level of 1.

In order to avoid numerical methods to calculate the compensation a consumer needs to stay at his equilibrium utility level when $\lambda$ is varied, we argue as follows. The example has been calibrated such that for each consumer the indirectly preferred set is the complement of a convex set (in the interval of rays $\lambda$ under consideration) [14]. Therefore, we take the linear compensation scheme determined by the marginal willingnesses $MW^1(\tilde{\lambda}) = -MW^2(\tilde{\lambda})$.

More precisely, consumer 1's marginal willingness to pay for an infinitesimal change of $\lambda$ is approximately 0.47 at the equilibrium ray $\tilde{\lambda} \approx 0.13$ and the marginal willingness of each consumer of type 2 is $MW^2(\tilde{\lambda}) \approx -0.47/3$. Therefore, at the ray $\lambda$ consumer 1 receives a sidepayment of about $-0.47 \cdot (\lambda - \tilde{\lambda})$ and each consumer of type 2 receives $0.47/3 \cdot (\lambda - \tilde{\lambda})$. Computing the values of the utility functions for each of the consumers after this linear compensation scheme is carried out one observes the following result: For each of the consumers utility reaches its minimal value at the equilibrium ray $\tilde{\lambda}$.

**Remark .** *Even if all consumers have von Neumann-Morgenstern utility functions, there can be a unique partnership equilibrium that is potentially Pareto dominated by any other possible choice of a production ray $\lambda$.*

## 4.3 Sidepayments and mutual potential Pareto domination of partnership equilibria.

In all the examples discussed so far there is at least one partnership equilibrium that is not potentially Pareto dominated by another one. Thus, one might ask whether such must always be the case. At first glance, the elimination of partnership equilibria that are dominated by another partnership equilibrium seems to be a rather modest selection criterion. However, the example presented below demonstrates that it may very well be the case that no partnership equilibrium survives this procedure. Since the example is nondegenerate, it shows that one cannot even show the generic existence of a partnership equilibrium that is not potentially Pareto dominated by another one. Indeed, in the following example there will be three partnership equilibria each of which is potentially Pareto dominated by every other one.

Clearly, the example is due to the fact that distributional issues can very well affect efficiency if markets are incomplete. As pointed out in Section 2, a similar phenomenon has first been discovered by Guesnerie (1975) in the context

[14] Qualitatively, the situation is as in Figure 1.

of general equilibrium theory with a nonconvex aggregate production set (and a complete system of markets). Guesnerie has presented an example of an economy with a given distribution of wealth in which none of the marginal cost pricing equilibria is Pareto efficient. To obtain an efficient marginal cost pricing equilibrium it is necessary to redistribute wealth among consumers. The example we are now going to describe rests on the same principle as Guesnerie's.

There are two types of consumers. The preferences of the first one are obtained by starting out with the CES-indifference curve $x_0^{0.9} + x_1^{0.9} = 5$ and by taking its image under the affine mapping $(x_0, x_1) \mapsto (x_0, \gamma x_1)$ for $0 \leq \gamma < 1$. Thus, the preferences are the same as those of consumer 1 in the previous examples in this section with the exception that the consumption of good 0 must now have an upper bound below 5 rather than 1. In the area below the curve $x_0^{0.9} + x_1^{0.9} = 5$ the preferences can be represented by the utility function

$$U^1(x_0, x_1, x_2) = \frac{x_1}{(5^{9/10} - x_0^{9/10})^{10/9}} .$$

The initial endowment of type 1 is $(4.6, 0, 0)$. A consumer of type 2 has the utility function

$$U^2(x_0, x_1, x_2) = 0.7 x_0^{0.7} + x_2$$

and the initial endowment $(1.7, 0, 0)$.

In the economy under consideration there are one consumer of type 1 and three consumers of type 2. The firm can choose any ray generated by $(-1, \lambda, 1 - \lambda)$ with $\lambda \in [0.01\,, 0.56]$. To determine the set of partnership equilibria, observe that the sum of the marginal willingnesses of all consumers is positive before it hits 0 at $\lambda_A \approx 0.41$, and then negative until it becomes 0 again at $\lambda_B \approx 0.48$, and that it stays positive for $\lambda > \lambda_B$. Thus, there are three partnership equilibria corresponding to $\lambda_A \approx 0.41, \lambda_B \approx 0.48$ and $\lambda_C = 0.56$. At these equilibria the consumer of type 1 reaches the utility levels

$$U_A^1 \approx 0.407, \qquad U_B^1 \approx 0.476, \qquad U_C^1 \approx 0.553,$$

respectively. A consumer of type 2, who prefers low values of $\lambda$, obtains

$$U_A^2 \approx 1.137, \qquad U_B^2 \approx 1.065, \qquad U_C^2 \approx 1.018,$$

respectively.

**Remark .** *In the economy just described there are three partnership equilibria. Each of the equilibria is potentially Pareto dominated by the remaining two equilibria. This feature is robust with respect to small parameter changes.*

First, we claim that the equilibrium $A$ corresponding to $\lambda_A$ is potentially Pareto dominated by both other equilibria, $B$ and $C$. This can be seen as follows. Since consumer 1 prefers $\lambda_B$ over $\lambda_A$, but his partners do not, consumer 1 has to check whether the amount he needs to give them so that their utility level does not fall below $U_A^2$ is not so large that he is not compensated for the loss by obtaining the preferred ray $\lambda_B$. Suppose now that consumer 1 gives to each of his partners 0.14 units of wealth at $t = 0$. Then his initial endowment at $t = 0$ becomes $4.6 - 0.42 = 4.18$. A computation shows that his partners are better off at $\lambda_B$, if they receive the sidepayment 0.14 so that they have no reason to turn the offer down. Also, consumer 1 gains, since his utility rises from 0.407 to 0.412. Therefore, equilibrium $A$ is potentially Pareto dominated by $B$.

To see that $A$ is also potentially Pareto dominated by $C$, assume that consumer 1 now distributes 0.84 units of wealth at $t = 0$ evenly among his partners. If $\lambda_C$ is chosen, consumer 1 reaches the utility level 0.425 and his partners obtain the utility 1.141. Again, this represents a potential Pareto improvement.

Next, we claim that equilibrium $B$ is potentially Pareto dominated by $A$ and also by $B$. If each of the consumers of type 2 gives 0.1 to consumer 1 and they switch to $\lambda_A$, then 1's utility rises from 0.476 to 0.492 and the utility of each of the consumers of type 1 goes up from 1.065 to 1.078. Similarly, if consumer 1 gives each of the consumers of type 2 the amount 0.13 and $\lambda_C$ is chosen, then all consumers are better off.

Finally, we claim that $C$ is potentially Pareto dominated by $A$ and by $B$. If 1 obtains a sidepayment of size 0.37 and production switches to $\lambda_A$, then 1's utility goes up from 0.553 to 0.567, whereas his partners reach the utility level $1.065 > 1.018$. In order to go to the ray $\lambda_B$, consider a sidepayment to consumer 1 of size $3 \cdot 0.09 = 0.27$ in order to get a potential Pareto improvement.

The situation can best be summarized in a drawing. For that purpose, consider first the partnership equilibrium $A$ and fix the utility level $U_A^2$ each of the consumers of type 2 obtains at $A$. There is a well defined change $\Delta x_0^A(\lambda)$ of the initial wealth of each consumer of type 2 such that he reaches precisely the utility level $U_A^2$, if his initial endowment is $(e_0^2 + \Delta x_0^A(\lambda),\, 0\,, 0)$ and the ray $\lambda$ is chosen. Since there are 3 consumers of type 2, the amount $-3 \cdot \Delta x_0^A(\lambda)$ can be made available to consumer 1. The continuous line touching the horizontal axis at $\lambda_A \approx 0.41$ shows the utility increase over $U_A^1$ agent 1 can obtain at the ray $\lambda$ if he receives the sidepayment $-3 \cdot \Delta x_0^A(\lambda)$. Clearly, Figure 4 shows that agent 1 can be made better off at $B$ as well as at $C$ than he is at the reference equilibrium $A$ without any other agent being worse off. A similar statement also holds true for the rays in between the equilibria.

The dashed line that is tangent to the horizontal axis at $\lambda_B \approx 0.48$ shows the utility increase consumer 1 can get over the utility level $U_B^1$ if the consumers of type 2 get sidepayments that keep them precisely at $U_B^2$. Similarly, the dashed-dotted line cutting the horizontal axis from below at $\lambda_C = 0.56$ indicates consumer 1's utility increase corresponding to the third equilibrium $C$.

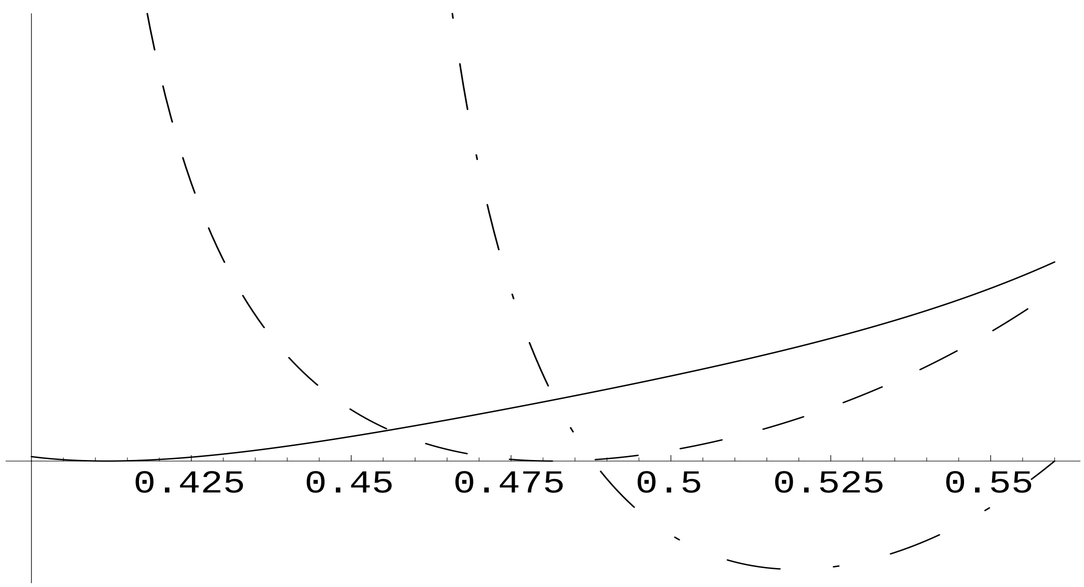


Figure 4: Mutual domination after sidepayments

We could have chosen $\lambda_B$ as upper bound for $\lambda$ in order to get an economy with two equilibria each of them being potentially dominated by the other one. However, this example would not have been robust, since extending the technology to rays arbitrarily little above $\lambda_B$ would have given rise to a third equilibrium at the upper bound. The bound 0.56 has been selected in order to make the additional boundary equilibrium $C$ potentially Pareto dominated by both other equilibria.

# 5 Conclusion

We have examined the welfare properties of Drèze equilibria in the particularly simple framework of a partnership economy with a single firm in order to rule out the well-known coordination failures across firms. These equilibria can be characterized in terms of the gradients of the utility functions of the partners and can be computed by finding the zeros of the sum of the marginal willingnesses to pay for an infinitesimal change of the production plan aggregated over all partners. This characterization captures the first order condition of constrained Pareto efficiency.

Examples show that majority voting can result in a uniquely determined outcome at which shareholders' social surplus is near to its minimum, because this is the outcome preferred by the median voter. Thus, we are led to conclude that majority voting does not constitute a sound basis for a welfare oriented theory of the behavior of a firm when markets are incomplete, even if strategic misrepresentation of voters' preferences can be disregarded. A major difference between outcomes of majority voting and Drèze equilibria can be expressed as follows: The firm's production decision is governed by the utility gradient of the median shareholder in case of voting whereas it depends on the weighted sum of the utility gradients of all shareholders in case of Drèze equilibria and is thereby more closely related to shareholders' social surplus.

However, even if consumers have quasilinear preferences the set of Drèze equilibria can have undesirable properties. In particular, at a Drèze equilibrium shareholders' social surplus can reach its minimum. This is due to the fact that the definition of a Drèze equilibrium only takes welfare changes of first order into account. Thus, no distinction is made between an interior maximum and any other critical point.

More importantly, if the framework of quasilinear preferences is left, robust examples of economies with a unique, but constrained Pareto inefficient equilibrium can be given. Therefore, the inclusion of higher order welfare changes would entail a refinement of the equilibrium concept that can lead to nonexistence. Moreover, restricting attention to expected utility maximization does not rule out situations in which the unique Drèze equilibrium is socially undesirable.

A constrained Pareto efficient equilibrium cannot be potentially Pareto dominated [see Section 2 for a precise definition]. However, there exist nondegenerate examples with multiple Drèze equilibria each of which is potentially Pareto dominated by every other one.

These disturbing phenomena can be explained as follows. It has already been emphasized by Drèze (1974) that the set of feasible consumption allocations is necessarily nonconvex due to the fact that shares $\vartheta^i$ and production plans $y$ enter the definition of feasibility in a multiplicative way. Clearly, whenever one of the factors $\vartheta^i$ and $y$ is kept constant while the other one is varied, the product becomes linear and the nonconvexity remains ineffective. The very structure of the definition of a partnership equilibrium as a fixed point with two properties referring to consumption and production, respectively, builds upon such a separation of changes in shares $\vartheta^i$ and changes in production $y$. For the same reason, this definition does not capture welfare effects of second or higher order. If one takes the welfare effects fully into account and imposes the condition that equilibria must not be potentially Pareto dominated, the nonconvexity of the feasible set may very well lead to the nonexistence of potentially undominated equilibria.

Our examples reflect the presence of a severe conflict between issues of distribution and constrained Pareto optimality if the set of feasible states is nonconvex, that is absent in the theory of perfect competition with complete markets. According to the first and second welfare theorem lump sum redistributions and efficiency do not interfere with each other. That is to say, lump sum redistribution is neither needed to obtain Pareto efficiency in a Walrasian equilibrium nor does it prevent equilibria from being efficient if it is performed. However, Guesnerie discovered that the intuition conveyed by the first and second welfare theorem hinges upon the assumption that global nonconvexities are absent. More precisely, Guesnerie (1975) investigated the efficiency of marginal cost pricing equilibria when the technology is nonconvex and pointed out that the above intuition becomes false due to the presence of the nonconvexity. As shown in this paper, the phenomenon discovered by Guesnerie has important consequences for the theory of incomplete markets.

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