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# Individual Randomness in Economic Models with a Continuum of Agents

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**Abstract**

This paper studies measurable processes modeling a continuum of random variables such that individual uncertainty cancels out exactly at the aggregate level (in the sense of a strong law of large numbers). These processes provide an analytically tractable framework for the analysis of stochastic mass phenomena in economics, without departing from the usual measure theory techniques. The paper shows the abundance of such processes and studies the implications on independence and correlation among the individual random variables. The approach is based on modeling the aggregate level first and derive the properties of the individual one afterwards. The main difference with other approaches is that exact (rather than approximate) results are provided in standard Borel spaces. This is important if the set of agents affected in a specific way by a random element or shock needs to be studied, as happens e.g. in many dynamical models.

**Keywords:** Large economies, Law of large numbers, Aggregate risk.

**JEL Classification Nos.:** C6, C7, D8

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# 1 Introduction

Models with a continuum of random variables are extensively used in economics. Such random variables represent, for example, individual risk of agents in a continuum population (an interval). The motivation for the use of the continuum in the first place is the intuitive idea that if agents in a large population face individual uncertainty, this uncertainty vanishes upon aggregation.

The list of models relying on such a construction is simply too large to quote. We review here some important examples.

Prescott and Townsend [17] put forward a general framework for adverse selection and moral hazard problems. In their model, there are finitely many types of agents, and a continuum of agents of each type. Then a law of large numbers is used to establish that there is no aggregate uncertainty within each type. For instance, in a risk insurance model, where type $i$ agents have a probability of accident $\theta_i$, they state that "of people of type $i$, $\theta_i$ is also the fraction that will suffer a loss" (see Prescott and Townsend [17, p.24]).

Lucas [13] studies a monetary economy with a continuum of individuals, where each of them receives an independent preferences shock in such a way that the fraction of individuals suffering a given shock is identified with the individual probability of suffering that shock. That is, he implicitly uses a law of large numbers to state that "with a continuum of agents, there is no aggregative uncertainty" (see Lucas [13, p.206]).

Grossman and Helpman [9], in their well-known quality ladders model, postulate a continuum of products, whose quality is improved with a certain (independent) probability $\iota dt$ in a time interval of length $dt$. They state that "by the law of large numbers, a fraction $\iota$ of the products are continually being improved" (see Grossman and Helpman [9, p.49]).

Other examples include large (non-atomic) games with strategic uncertainty, as in Schmeidler [18], Mas Colell [14] or Pascoa [16], where, in principle, a continuum of players choosing mixed strategies randomize independently and their randomizations must be aggregated to obtain the distribution of actions in the population that each individual faces.

In Harrington [11], a continuum of individuals observe different realizations of an environmental stochastic element. Then a law of large numbers is implicitly used to state that the fraction of agents observing a certain realization is exactly equal to the probability of this realization.

A related problem is the one posed by models with a continuum of randomly matched agents, with examples including many evolutionary models and the above mentioned paper by Harrington [11]. That framework was analyzed in a related paper (see Alós-Ferrer [5]).

These models, and many others, postulate an interval of agents, endowed with identically distributed random variables, and informally invoke a strong law of large numbers to identify the sample average across the population with the mean of the individual random variables.

Mathematically, this is formalized by a process $x(i, \omega)$, with $\omega$ and element in a sample space $\Omega$ and $i$ an agent in a population $I$, or, more generally, a parameter in an appropriate space, typically an interval. The functions $x(i, \cdot)$ are the individual random variables, which are assumed be identically distributed with a fixed random variable $X$ of distribution $f$. The law of large numbers would then say that, for any $\omega$, the sample function $x(\cdot, \omega)$ should also have distribution $f$. A less demanding formulation, which is the one used in the examples above, would establish the equality between the mean of $X$ and the sample average of $x(\cdot, \omega)$, for any $\omega$. For example, if a continuum of agents are tossing coins ($X$ being "toss a coin"), half of them should always obtain heads.

Judd [12] and Feldman and Gilles [7] proved that, if the individual uncertainty is idiosyncratic, such a law of large numbers does not exist, because the sets of agents obtaining a certain realization (say, heads) may not be measurable, and, even if they are, they need not have the appropriate measure (i.e., numerically equal to the probability of the considered realization, say, one half). The mathematical problem is that the process $x(i, \omega)$ is not (jointly) measurable, which prevents Fubini-type results.

Since the establishment of this impossibility result, the literature has tried to find a way around the problem, relaxing or changing assumptions to obtain more sophisticated constructions verifying the properties assumed by economic models. Judd [12] showed that there exist extensions of the Kolmogorov probability space (the basic space on which $x(i, \omega)$ is defined) such that the sample average (for the whole population) equals the mean, with probability one. He notes, though, that these extensions are arbitrary. We can find analogous, equally reasonable extensions such that the "law of large numbers" does not hold. In fact, Feldman and Gilles showed that it is impossible to have an extension satisfying the same property for all Borel subsets of agents (a problem named *absence of homogeneity*), i.e. we will always have some counterintuitive behavior. We will refer to this result as FG in the sequel.

The absence of homogeneity is specially worrying, since, if the population is represented by an interval, one would expect any subinterval to have analogous properties to those of the whole population. Bringing more sophisticated weapons to the battle, Green [8] constructs a process which fulfills homogeneity[1]. This does not contradict FG because the space of agents is itself constructed in order to guarantee the property, and it is not a Borel space, but an abstract probability space endowed with a $\sigma$-algebra which is not countably generated, and thus cannot be easily interpreted. Moreover, Green finds again the freedom of choice encountered by Judd [12], noticing that we could have different constructions satisfying different numerical equalities.

The problem of the lack of a law of large numbers, and its importance in Economics, has received a great deal of attention in recent years. Of course there is no way around the basic impossibility result. A continuum of independent, identically distributed random variables will *not* satisfy the Strong Law of Large Numbers. The only possible solution is to relax some part of this sentence.

One possible approach is to renounce to the strong law of large numbers, asking instead for weaker results. Uhlig [21] proposed to change the integral used to compute the sample averages, hence redefining the problem. Al-Najjar [1, 3] elaborates on this approach, which is of special interest with respect to the (highly demanding) problems which appear in the framework of large games. There, the problem is one of definition, and the crucial question is how does each individual agent perceive the aggregate level. Hence, it might be reasonable to say that the individual agent perceives the population aggregate behavior as if certain integral concept were at work, even if no statement can be made about

[1] This process is analogous to the construction illustrated by Anderson [6], based on nonstandard analysis)

the set of agents who choose a certain action after randomization[2].

The approach of Uhlig and Al-Najjar consists essentially in renouncing to the exact cancellation of individual risk that a strong law of large numbers would provide. The results obtained are actually enough for some static frameworks (as long as one accepts to redefine the problem), specially if the only important thing is how does each individual perceive the whole population. The approach is, however, problematic because it does not solve the measurability problem. The sets of agents obtaining a particular realization remain potentially non-measurable. Also, the *exact* cancellation of individual risk is crucial for many macroeconomic and finance applications (see e.g. the references in [12, 7]).

Consider a population dynamical system with an explicit time component. In such dynamic models, the results prescribed by a law of large numbers at time $t$ (i.e. the sets of agents which have obtained a given realization) have a very precise interpretation which is to be used to generate the results for $t+1$. Those sets have to be measurable. If not, the dynamical system cannot even be defined. Moreover, very often, the link between the population model and the (deterministic) dynamic equations describing its evolution along time is an exact claim which, if formalized, must be put in the form of a strong law of large numbers, e.g. exactly how many agents have obtained a certain realization. Examples include most of evolutionary game theory (see e.g. [19, 22]), random matching models (see [5] for a discussion), and in general any dynamic model where it is important to keep track of the sets of agents that have experienced a specific realization (e.g. [11, 15]). Consider, for instance, evolutionary game theory. While in large games the crucial question is how does each individual perceive the aggregate level, in an evolutionary setting the individual is irrelevant, and all that matters is the measure of the set of individuals of its same type. For such models, the measurability problem cannot be bypassed; it must be solved.

Sun [20] presents a complete exposition of the possibilities of changing the usual continuum framework to hyperfinite models based on non-standard analysis. This approach, of enormous mathematical possibilities, delivers exact results (which can be called a strong law of large numbers) at the cost of renouncing to the "classical" spaces of agents, as e.g. the unit interval, in favor of more abstract entities. While ultimately this might be the only way out for specially demanding models, the question arises whether it is really necessary to fully abandon the continuum model.

In contrast to previous work, this paper focuses on the standard space of agents, i.e. a Borel space over an interval, and tries to answer the question of what the implications of exact aggregate results (Strong Law of Large Numbers) are, always without changing the usual framework and without leaving the realm of standard measure theory. The primary aim of the paper is to study and characterize the family of standard (as opposed to non-standard or hyperfinite) processes satisfying the exact (strong) law of large numbers.

The main idea is to change the approach to the problem and concentrate attention in the converse implication of that studied in the literature. Suppose that we have a process such that individual uncertainty cancels out exactly at the aggregate level. This process could be the one underlying the economic

[2] For a precise discussion of the relationship between the subjective individual level and the law of large numbers, see [4]

models cited above. The question is then what can be said about such a process. Does it exist? Is it unique? Can it be characterized?

The vital requirement for tractability is, of course, joint measurability. As a consequence, the claim for full independence among the individual random variables is lost. The idea of sacrificing full independence in favor of exact results is not new. Feldman and Gilles [7] already point to it, citing an example where individual uncertainty and exact aggregate results are obtained at the cost of losing independence. The difference with the present work is that here the converse question is analyzed. We turn to the analysis and tentative characterization of the full family of processes where individual uncertainty and deterministic aggregate outcomes are obtained.

Renouncing to full independence is of course a restriction for some situations, but not always. Close examination of many economic models reveal that, frequently, independence is added to the hypothesis of the model because it is thought to be necessary for the application of an unspecified law of large numbers. In many of them, though, independence is not needed. In random matching models, independence is actually excluded, because if one agent is matched to another one, the second has to be matched to the first. Also, as Al-Najjar [1] points out, "strong distributional assumptions should be avoided in the study of large games where correlation due to sunspots, correlated types, or correlated randomization devices is quite natural." To cite another extreme example, if a subject crashes his car against his neighbor's, it is hard to understand how could their probabilities of accident have been independent.

In models where independence is crucial, the approach here will yield only approximate results, as we will detail later on.

Section 2 introduces notation and defines the object of study, which is named *population extension* of a random variable. Section 3 shows the existence and non-uniqueness of population extensions. Section 4 tackles the issue of homogeneity, showing that *ex ante*, each measurable subset of agents satisfies the law of large numbers. Section 5 formalizes in which sense are the individual random variables not independent. Section 6 introduces the concept of *randomness basis*, which are detailed, maximal "maps" describing the correlation relations among individual variables within a population extension. The main result of the paper is that any population extension can be described through a randomness basis. Section 7 shows that randomness bases serve both as a classifying device and to pin down how many independent random variables can we have in a specific population extension. Section 8 studies a type of particularly well-behaved population extensions. Section 9 sketches some selected applications.

# 2 Notation and definitions

Consider a random variable

$$X : (A, \mathcal{A}, \lambda) \to S$$

defined on a probability space $(A, \mathcal{A}, \lambda)$ and taking values on a (finite[3]) set $S = \{1, 2, ..., N\}$. Such a variable could be a model for the sources of random-

[3]We focus on simple random variables for concreteness. However, with the appropriate reformalization, the main results are still true for arbitrary real random variables.

ness that a given, fixed individual faces. By taking a large population of such individuals, the researcher expects to "wash away" the uncertainty and obtain a deterministic aggregate model. E.g., if $X$ models a coin toss, one expects to be able to say that, in a large population, half the agents obtain heads.

Let $I = (0, 1]$ be the set of agents. The triple formed by this set, the $\sigma$-algebra of its Borel subsets $\mathcal{B}$, and the Lebesgue measure $\mu$, is the space of agents.

Denote $\lambda_s = \Pr(X = s) := \lambda(\{a \in A \ / \ X(a) = s\})$ for each $s \in S$.

## 2.1 Traditional approach

To model idiosyncratic risk on this space of agents, the first approach is to endow each agent with a copy of the random variable $X$. This creates a continuum of i.i.d. random variables, $\{x_i\}_{i \in I}$ and, by Kolmogorov's Extension Theorem, there exists a probability space to represent them, i.e. we have

$$x : (\Omega, \mathcal{F}, P) \to S^I$$

The result we are interested in is that, for (almost) each realization of the randomness, and for each (measurable) subset of agents, the fraction of agents who obtain a particular realization $s \in S$ equals the probability $\Pr(X = s)$. Formally, this means that, for all $s \in S$, for almost all $\omega \in \Omega$, and for any measurable subset of agents of positive measure, $B \in \mathcal{B} \ / \ \mu(B) > 0$,

$$\mu(\{i \in B \ / \ x_i(\omega) = s\}) = \lambda_s \cdot \mu(B)$$

Feldman and Gilles [7] showed that this "Law of Large Numbers" is in general false. The first problem is that the set $\{i \in B \ / \ x_i(\omega) = s\}$ needs not be measurable. The second, that even if it is, the equality needs not hold.

Following Judd [12], it is possible to see that the situation is even more curious. Given a fixed set $B$, the set of realizations $\omega$ such that $\{i \in B \ / \ x_i(\omega) = s\}$ is measurable is itself non-measurable. It has outer measure 1 and inner measure 0, a fact which can be used to construct extensions of the probability space $(\Omega, \mathcal{F}, P)$ such that the above equality (for $B = I$) holds with any arbitrary probability $0 \le p \le 1$, a disturbing freedom of choice. Moreover, because of FG, a different extension will be needed for each different Borel set, i.e. there exists no extension satisfying simultaneously the above equation for all measurable sets of agents.

## 2.2 An alternative approach

The traditional approach is based on the implicit choice of modeling firstly at the individual level and aggregating afterwards. The impossibility result FG could arguably be interpreted as casting doubt on the aptness of this modeling decision.

In economics, when continuum models are called for, the most important level is the aggregate one. Exactly as insurance companies estimate the probabilities of accident from cross-panel data and then infer an individual measure of risk, in these cases it would be more reasonable to start with a model for the population as a whole and then infer the properties of individual risk. This

"Copernican" turn would allow the researcher to concentrate attention on the relevant characteristics of a population model.

The aim of this paper is to study the object which the researchers need and characterize it. This object would be a random process at the population level, displaying aggregate stability (and hence analytical tractability) but individual uncertainty. This uncertainty at the microscopic level can be studied because the population process will induce individual random variables.

It should be kept in mind that this population process is a modeling tool. The lack of uncertainty at the aggregate level should not be viewed as a result, but as a condition which reflects some observed result. The tool is built to model economic situations displaying aggregation effects, and hence should exhibit this characteristic.

It should also display enough mathematical regularity to be useful as a tool. The extensions found by Judd [12] or Green [8] are problematic, not only because of the above-mentioned "disturbing freedom of choice," but also because the associated probability spaces are analytically untractable. For instance, the population process can be considered as a random function from the product of the sample space and the set of agents. Both in Judd [12] and in Green [8], it can be seen that their processes cannot be jointly measurable in their two arguments. However, joint measurability is precisely the kind of mathematical regularity condition which would make the process tractable.

Let us start defining our object of study. Our population process will randomly specify a (measurable) partition of the population, i.e. the sets of agents obtaining each of the possible realizations. Thus, let $\mathcal{P}(I)$ be the set of partitions of $I$ into $|S|$ subsets, and let $\mathcal{R}(I)$ be the set of measurable partitions of $I$ into $|S|$ subsets.

$$\begin{aligned} \mathcal{P}(I) &= \{R \in (2^I)^S \ / \ \cup\{R_s \ / \ s \in S\} = I \ \wedge \ R_s \cap R_{s'} = \emptyset \ \forall \ s, s' \in S\} \\ \mathcal{R}(I) &= \mathcal{P}(I) \cap \mathcal{B}^S \end{aligned}$$

Note that the notation $R_s$ refers to the $s$-coordinate of the element $R \in (2^I)^S$, that is, $R_s \in 2^I$.

In order to transform $X$ into a suitable population-wide random variable, we should use a new mapping which directly specifies the partition of the population in measurable subsets, such that the agents of a specific subset are those which have obtained a specific realization. That is, we are looking for a mapping $X^I$ taking values on $\mathcal{R}(I)$ such that, with probability one, $\mu(X^I_s) = \Pr(X = s) \ \forall \ s \in S$.

More formally, we want to have a mapping $X^I$ taking values on $\mathcal{R}(I)$ and defined on a probability space $(\Omega, \mathcal{F}, P)$ (which here does not need to be Kolmogorov's product space) such that

$$P(\omega \in \Omega \ / \ \mu(X^I_s(\omega)) = \lambda_s) = 1 \quad \forall \ s \in S$$

Again, note that $X^I_s(\omega)$ refers to the $s$-coordinate of the element $X^I(\omega) \in \mathcal{P}(I)$. If $X^I(\omega) \in \mathcal{R}(I)$, then $X^I_s(\omega) \in \mathcal{B}$, which is implicitly required by the last equation.

Such a mapping could always be re-interpreted as a continuum of random variables $\{x^I_i\}_{i \in I}$, taking $x^I_i(\omega) = s$ iff $i \in X^I_s(\omega)$. The difference with the previous approach is that the set of agents which obtain a certain realization is measurable by definition. This is not "assuming away" the problem because the question is whether such "population random variables" exist or not and which properties need they have. Let us now formally define the object.

**Definition 2.1.** Given the set of agents $I$ and a random variable $X : (A, \mathcal{A}, \lambda) \to S$, a mapping $X^I : (\Omega, \mathcal{F}, P) \to \mathcal{P}(I)$ is said to be a *population extension of* $X$ *if it verifies*

1. *Joint measurability: the mapping* $x : I \times \Omega \to S$ *defined by* $x(i, \omega) = s$ *iff* $i \in X_s^I(\omega)$, *is a random variable.*
2. *Individual uncertainty:* $P(\{\omega \in \Omega \ / \ i \in X_s^I(\omega)\}) = \lambda_s \ \forall \ s \in S, i \in I$
3. *Aggregate stability:* $P(\{\omega \in \Omega \ / \ \mu(X_s^I(\omega)) = \lambda_s\}) = 1 \ \forall \ s \in S$

*where* $\lambda_s = \lambda(\{a \in A \ / \ X(a) = s\}) \ \forall \ s \in S$.

The first condition is simply a joint measurability requirement. It implies that the variables $\{x_i^I\}_{i \in I}$ defined by $x_i^I(\omega) = s$ iff $i \in X_s^I(\omega)$, and the mappings given by $\mu(X_s^I(\omega)) \ \forall \ s \in S$, are random variables, making the following conditions meaningful. It also allows us to treat the population extension as a random variable, since it actually takes values in the measurable partitions $\mathcal{R}(I)$. See Appendix B for a "population extension" which does not fulfill joint measurability.

The second condition says that, from the individuals point of view, uncertainty is indeed captured by the original random variable $X$. The third condition establishes that uncertainty disappears at the aggregate level.

In summary, a population extension is simply a mathematical model, analogous to a random variable (joint measurability condition) displaying uncertainty at the microscopic level (individual uncertainty condition) but such that this uncertainty disappears upon aggregation (aggregate stability condition).

# 3 Existence and Non-Uniqueness

In order to show existence of non-trivial extensions, it is enough to give one example. Let $S, I$, and $X$ be given. We construct now a population extension of $X$.

*Example* 3.1. The Wheel Extension.

The half-open interval of agents $(0, 1]$ can be interpreted as a circumference. Suppose that this circumference is randomly rolled (calling for a random variable uniformly distributed on $(0, 1]$) and then placed over another circumference with an arc labeled "$s$", of length $\lambda_s$, for each possible realization $s$ of $X$. Then, we say that agents placed over the arc "$s$" have obtained an "$s$". This seems to be the simplest procedure to obtain a population extension of $X$.

Formally, the population extension is given by $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, $P = \mu$, and

$$X_s^I(\omega) = \begin{cases} (\Gamma_{s-1} - \omega, \Gamma_s - \omega] & \text{if } \omega \leq \Gamma_{s-1} \\ (0, \Gamma_s - \omega] \cup (\Gamma_{s-1} + 1 - \omega, 1] & \text{if } \Gamma_{s-1} < \omega \leq \Gamma_s \\ (\Gamma_{s-1} + 1 - \omega, \Gamma_s + 1 - \omega] & \text{if } \omega > \Gamma_s \end{cases}$$

where $\Gamma_s = \sum_{r=1}^{s} \lambda_r, \Gamma_0 = 0$. This extension can be generalized to any subinterval.

To show non-uniqueness, it is enough to construct a different example. In fact, we can do better than that, by constructing a whole family of extensions.

*Example* 3.2. The Z-Extensions.

Let $Z : I \to S$ be any measurable function such that $\mu(Z^{-1}(s)) = \lambda_s \ \forall \ s \in S$. We shall see later that such functions exist.

The population extension $X^I(Z)$ of $X$ is defined through $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, $P = \mu$, and

$X^I_s(\omega) = [(Z^{-1}(s) - \omega) \cup (Z^{-1}(s) + 1 - \omega)] \cap (0, 1]$

This example is non-vacuous, because every $R \in \mathcal{R}(I)$ such that $\mu(R_s) = \lambda_s \ \forall \ s \in S$ specifies a suitable function $Z$. If $R_s = (\sum_{r=1}^{s-1} \lambda_r, \sum_{r=1}^{s} \lambda_r] \ \forall \ s \in S$, we obtain the Wheel Extension. A different example is given by

$R_s = (\frac{1}{2} - \frac{1}{2}\sum_{r=1}^{s} \lambda_r, \frac{1}{2} + \frac{1}{2}\sum_{r=1}^{s} \lambda_r] \setminus R_{s-1} \ \forall \ s \in S$, with $R_0 = \emptyset$.

As a preliminary result, we have (somewhat trivially) showed the abundance of processes which qualify as population extensions. Our aim is now to study their general properties.

*Remark* 3.3. The population extension takes values on $\mathcal{R}(I) \subset \mathcal{B}^S$. This set has cardinality $\aleph_1$. Given a family $\{x_i\}_{i \in I}$, one could have tried to define the "population extension" taking values on

$$\mathcal{R}^*(I) = \{R \in \mathcal{P}(I)^S \ / \ \cup \{R_s \ / \ s \in S\} = I \ \wedge \ R_s \cap R_{s'} = \emptyset \ \forall \ s, s' \in S\}$$

but, because of the FG impossibility result, these partitions are not measurable in general. In fact, this set has cardinality $\aleph_2$. This cardinality difference is at the very core of the original problem.

# 4 Homogeneity

Feldmann and Gilles [7] show that the "law of large numbers" fails with a continuum of independent random variables. This failure takes the form of a realization of the underlying probability space and a Borel subset of agents (even an interval) such that the proportion of agents of that subset obtaining a given realization $s$ is not equal to the probability of that realization according to the variable $X$. In order to study this failure, we will call such behavior a "pathology" and formally define it in our framework.

**Definition 4.1.** Given a population extension $X^I$ of a random variable $X$, a *pathology is a pair* $(\omega, B) \in \Omega \times \mathcal{B}$ *such that* $\mu(X^I_s(\omega) \cap B) \neq \mu(B) \cdot \lambda_s$ *for some* $s \in S$.

Pathologies are problematic from the economic point of view. In an economy with a large number of traders and risk, it would be expected that any large coalition of traders could be able to form a risk-pooling coalition, at least in expected terms. See [8] for a discussion. The bad news is that pathologies are bound to exist. The good news is that, in expected terms, the law of large numbers is still true for any positive-measure subset of agents.

**Proposition 4.2.** *Given any population extension* $X^I$ *of a random variable* $X$*, then for all* $\omega \in \Omega$ *there exists a set* $B \in \mathcal{B}$ *such that* $(\omega, B)$ *is a pathology.*

*Proof.* Let $\omega \in \Omega$, and suppose that, for all $B \in \mathcal{B}$, $(\omega, B)$ is not a pathology. Given $s \in S$, this means that $\mu(X_s^I(\omega) \cap B) = \mu(B) \cdot \lambda_s$ for all $B \in \mathcal{B}$. Call $z_\omega^s(i) = 1$ iff $i \in X_s^I(\omega)$, 0 otherwise. Then, $z_\omega(\cdot)$ is a measurable function such that $\int_B z_\omega^s(i) d\mu(i) = \mu(B) \cdot \lambda_s$ for all $B \in \mathcal{B}$. The Radon-Nykodym theorem implies then that $z_\omega(i) = \lambda_s$ almost everywhere, a contradiction. ■

**Proposition 4.3.** *Given any population extension $X^I$ of a random variable $X$, pathologies cancel out in the aggregate, i.e. for all $B \in \mathcal{B}$ and for all $s \in S$,*

$$\int_\Omega \mu(X_s^I(\omega) \cap B) dP(\omega) = \mu(B) \cdot \lambda_s$$

*Proof.* Given $s \in S$, call $z^s(\omega, i) = 1$ iff $i \in X_s^I(\omega)$, 0 otherwise. Then, $z^s(\cdot, \cdot)$ is a jointly measurable function. Moreover,

$\int_\Omega \mu(X_s^I(\omega) \cap B) dP(\omega) = \int_\Omega \int_B z^s(\omega, i) d\mu(i) dP(\omega) =$
$= \int_B \int_\Omega z^s(\omega, i) dP(\omega) d\mu(i) = \int_B P(\{\omega \in \Omega \ / \ x(\omega, i) = s\}) d\mu(i) =$
$= \int_B \lambda_s d\mu(i) = \mu(B) \cdot \lambda_s$

where the second equality holds by Fubini's theorem. ■

The first proposition means that what has been taken as a law of large numbers fails. The proof is inspired in Feldman and Gilles [7], although they state the result only for families of i.i.d. random variables, whereas here we see that it also applies when independence is not assumed (the fact that the individual variables come from a population extension is also irrelevant for the proof). The second proposition, though, shows us that the joint measurability built into the definition of a population extension allows us, if not to get rid of the pathologies, at least to conclude with a statement that certainly has some flavour of a law of large numbers: pathologies disappear in the aggregate. For example, in an economy with a large number of traders and risk, any large coalition would expect risk to cancel out for its members (see Section 9).

Since a pathology is the failure of the sought equality for a specific realization of the underlying probability space, and the previous result just shows that integrating over all such realizations must give us precisely that equality, it might be tempting to suggest that the problem with the law of large numbers could now be reinterpreted as the lack of one further integration step. What is actually at work, from the purely mathematical point of view, is that the joint measurability requirement allows for the use of Fubini's theorem and hence to transfer the regularities at the aggregate level in the agents space to the aggregate level in the sample space.

# 5 Independence

A law of large numbers establishes an aggregate result for a set of independent random variables.[4] When we try to model situations which call for such an aggregation result in the continuum framework, it is obviously counterintuitive to have the aggregate result at the whole population level, and to observe its

[4]Notice that the random variables of an arbitrary family are independent if and only if the variables of each finite subfamily are independent. This is stronger that the notion of pairwise independence. However, the whole analysis could be carried for this weaker notion, and the results in this and the following section would hold true.

failure at a given subinterval (which should be, in some sense, isomorphic to the whole population). And yet this “absence of homogeneity” is what the impossibility result of Feldman and Gilles shows. In the framework of a population extension, we have disposed of the independence assumption, and decided that the aggregate level should be the first one to be modeled. In the previous section, we have even restored homogeneity up to a point. We can now establish the precise relationship with the independence - or lack of independence - at the individual level. The following theorem and its proof are closely related to Green [8, Theorem 4].

**Theorem 5.1.** *Given any population extension $X^I$ of a random variable $X$, there exists no set $J \subset I$ such that $J$ is dense in a Borel set of strictly positive measure and the induced random variables $\{x_j\}_{j \in J}$ are independent.*

*Proof.* Let $J$ be dense in $A \in \mathcal{B}$, $\mu(A) > 0$, and suppose that $\{x_j\}_{j \in J}$ are independent.

Consider any $s \in S$ such that $0 < \lambda_s < 1$. Then, $x^{-1}(s) \in \mathcal{B} \times \mathcal{F}$.

By Lemma A.1 (see Appendix A), for all $\varepsilon > 0$ there exist $N \in \mathbb{N}, B_1, ..., B_N \in \mathcal{B}, G_1, ..., G_n \in \mathcal{F}$ such that the sets $\{B_n\}_{n=1}^N$ are disjoint and

$$(\mu \times P)(((A \times \Omega) \cap x^{-1}(s))\Delta \cup_{n=1}^N (B_n \times G_n)) < \varepsilon.$$

By restricting to the sub-$\sigma$-algebra $\mathcal{B} \cap A$, it can be assumed that $B_1, ..., B_N \subset A$.

Then, by Fubini's Theorem,

$$\begin{gathered}(\mu \times P)(((A \times \Omega) \cap x^{-1}(s))\Delta \cup_{n=1}^N (B_n \times G_n)) = \int P(\omega \ / \ (i, \omega) \in \\ (((A \times \Omega) \cap x^{-1}(s))\Delta \cup_{n=1}^N (B_n \times F_n)))d\mu(i) = \\ = \int_A P(x_i^{-1}(s)\Delta\{G_n \ / \ i \in B_n\})d\mu(i)\end{gathered}$$

which implies that there exists a Borel set $C \subset A$, $\mu(C) > 0$, such that for all $i \in C$, $P(x_i^{-1}(s)\Delta\{G_n \ / \ i \in B_n\}) < \varepsilon$.

Take a fixed $B_m$ such that $\mu(B_m \cap C) > 0$. This exists because, if not, since $C$ is infinite, there exists $i \in C \backslash \cup_{n=1}^N B_n$, implying that $\lambda_s = P(x_i^{-1}(s)) < \varepsilon$, a contradiction for $\varepsilon$ small enough.

Since $J$ is dense in $B$, there exist two different agents, $j, k \in J \cap B_m \cap C$. Then, $P(x_j^{-1}(s)\Delta G_m) < \varepsilon$ and $P(x_k^{-1}(s)\Delta G_m) < \varepsilon$.

Since $P(x_j^{-1}(s)) = P(x_k^{-1}(s)) = \lambda_s$, this implies $P(x_j^{-1}(s) \cap x_k^{-1}(s) \cap G_m) = P(x_j^{-1}(s) \cup x_k^{-1}(s) \cup G_m) - P((x_j^{-1}(s)\Delta G_m) \cup (x_k^{-1}(s)\Delta G_m)) > \lambda_s - 2\varepsilon > \lambda_s^2$ for $\varepsilon$ small enough. But, since $j, k \in J$, we have that $P(x_j^{-1}(s) \cap x_k^{-1}(s)) = \lambda_s^2$, a contradiction. ■

**Corollary 5.2.** *Given any population extension $X^I$ of a random variable $X$, there cannot exist a subset of positive measure $J \subset I$ such that the random variables $\{x_j^I\}_{j \in J}$ are independent.*

This last, hardly-surprising, result establishes the extent to which we have renounced to the full independence requirement. Basically, once we accept that the properties we want exclude a continuum of independent random variables, there would not be such a family of random variables, even as a subset. What is important to notice is that the regularity properties of a population extension are extended to subsets, as we saw in the previous section and as we will see below. Hence, what this last negative result is telling us is that a population extension has coherent properties, with no real conceptual contradiction between the whole family of random variables and a given subset.

# 6 Randomness Basis

As we have just seen, the family $\{x_i^I\}_{i\in I}$ cannot contain a subset of independent variables of positive measure. In the examples above, the maximal subsets of independent variables in this family have cardinality 1. In this sense, those examples are "minimal". The question which immediately arises is how would "maximal" examples be, i.e. to what extent can we get families of i.i.d. random variables within a population extension.

This question turns more interesting when we realize that the answer could be a mean to find a systematic way to describe population extensions of random variables. This is the objective of this section. We shall build upon "maximal" sets of independent and non-independent variables to understand the common features of population extensions and the possible variety that we might encounter.

*Remark* 6.1. All the results in this section, including the main theorem, hold true when applied to an arbitrary family of random variables, not coming from a population extension. The proofs remain the same.

**Definition 6.2.** Given a population extension $X^I$, a set $d \in \mathcal{B}$ is called a *cluster for $X^I$ if for all $i, j \in d$, the random variables $x_i^I, x_j^I$ are* not *independent.*

**Definition 6.3.** A *random system $D$ for a population extension $X^I$ is a family of disjoint clusters for $X^I$ such that, for every set of agents $\{i(d) \ / \ d \in D\}$, the random variables$\{x_{i(d)}^I \ / \ d \in D\}$ are independent.*

**Definition 6.4.** Given two Borel sets $d, d' \in \mathcal{B}$, we say that $d$ is *almost included in $d'$ if $\mu(d \backslash d') = 0$. Denote it by $d \subset d'$ a.e.*

*Remark* 6.5. If $d \subset d'$ a.e., then $\mu(d) = \mu(d \cap d') + \mu(d \backslash d') = \mu(d \cap d') \leq \mu(d')$.

The next result shows that it is possible to obtain maximal clusters with respect to almost inclusion.[5]

**Lemma 6.6.** *Let $C$ be a chain in $\mathcal{B}$ according to the relation "to be almost included in". Then,*

1. *There exists a countable subchain $\{d_n\}_{n=1}^{\infty} \subset C$ such that $d_\infty := \liminf d_n$ is an upper bound for the chain $C$ in $\mathcal{B}$.*

2. *Consider a population extension $X^I$. If all the elements of the chain $C$ are clusters for $X^I$, then $d_\infty$ is also a cluster for $X^I$.*

*Proof.* Let $C$ be such a chain,[6] and let $r$ be the supremum of the set of real numbers $\{\mu(d) \ / \ d \in C\}$. If $r = 0$, then the assertion is trivial. Thus, suppose

[5] Of course, it is not possible to do the same with respect to set inclusion. The introduction of the almost inclusion relation allows us to concentrate in positive measure sets, avoiding the conceptual problems associated to null-sets.

[6] Note that the natural candidate for an upper bound, $\cup\{d \ / \ d \in C\}$, is the union of a maybe uncountable sequence of Borel-measurable subsets of the interval $I$, and thus may be non-measurable.

that $r > 0$. For each $n \in \mathbb{N}\backslash\{0\}$, let $d_n \in C$ such that $\mu(d_n) \geq r - \frac{1}{n}$. These sets can be chosen in such a way that $d_n \subset d_{n+1}$ a.e., for all $n$.

Define[7] $d_\infty := \liminf d_n = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} d_m$. This set is obviously measurable. Moreover, $d_n \subset d_\infty$ a.e. for all $n$.

$$\mu(d_n \backslash \cap_{m=n}^{\infty} d_m) = \mu(\cup_{m=n}^{\infty} d_n \backslash d_m) \leq \sum_{m=n}^{\infty} \mu(d_n \backslash d_m) = 0 \rightarrow \mu(d_n \backslash d_\infty) = 0$$

and this in turn implies that $\mu(d_\infty) = r$.

Let $d \in C$. We have to prove that $d \subset d_\infty$ a.e.

Given $n$, then either $d_n \subset d$ a.e. or $d \subset d_n$ a.e. If $d \subset d_n$ a.e. for some $n$, then $d \subset d_\infty$ a.e. and we are done. If $d$ is not almost included in $d_n$ for any $n$, then, $d_n \subset d$ a.e. for all $n$. Thus, $\mu(d) \geq \mu(d_n) \geq r - \frac{1}{n}\ \forall\, n$, implying that $\mu(d) = r$.

Consider the intersection $d \cap d_\infty$. If $\mu(d \cap d_\infty) < r$, then exists $n$ such that $\mu(d \cap d_\infty) < r - \frac{1}{n}$. But, since $d \cap d_n \subset d \cap d_\infty$ a.e., we have that $r - \frac{1}{n} \leq \mu(d_n) = \mu(d \cap d_n) + \mu(d_n \backslash d) = \mu(d \cap d_n) \leq \mu(d \cap d_\infty) < r - \frac{1}{n}$, a contradiction. Thus, $\mu(d \cap d_\infty) = r$, implying that $\mu(d \backslash d_\infty) = \mu(d) - \mu(d \cap d_\infty) = r - r = 0$, and, hence, that $d \subset d_\infty$ a.e.

In summary, $d_\infty$ is an upper bound for $C$.

To prove the second part, take $i, j \in d_\infty$. Then, there exist $n, n'$ such that $i \in \cap_{m=n}^{\infty} d_m$ and $j \in \cap_{m=n'}^{\infty} d_m$. Let $n'' = \max\{n, n'\}$. Then, $i, j \in d_{n''}$ and, since $d_{n''}$ is a cluster, the random variables $x_i^I, x_j^I$ are not independent. ■

The following well-known property will be used several times in the sequel.

**Lemma 6.7.** *Let $B$ be a family of disjoint Borel sets of $I$ of strictly positive measure. Then, $B$ is countable.*

*Proof.* Obviously, $B_n = \{b \in B \ / \ \mu(b) > \frac{1}{n}\}$ is finite for all $n \in \mathbb{N}\backslash\{0\}$. Since $B = \cup_{n=1}^{\infty} B_n$, it follows that $B$ is countable. ■

The following result is an immediate consequence of the lemma.

**Proposition 6.8.** *Given a random system $D$ for a population extension, define $m(D) = \{d \in D \ / \ \mu(d) > 0\}$. Then, $m(D)$ is countable.*

**Definition 6.9.** Given two random systems $D, D'$ for a population extension $X^I$, we say that *$D'$ is broader than $D$ if $\forall\, d \in D\ \exists\, d' \in D'$ such that $d \subset d'$ a.e. Denote this relation by $\succ$. A randomness basis is a maximal random system with respect to $\succ$.*

A randomness basis is, thus, a precise (up to null sets) way to explain the structure of a population extension in terms of independence and correlation. It can be conceived as a map which tells us which sets of agents are affected by correlated shocks (the clusters). Across these sets, the sources of randomness affecting agents are independent.

The next result shows that randomness basis always exist. The proof, although lengthy, essentially relies on a Zorn's Lemma argument.

**Theorem 6.10.** *Any population extension $X^I$ has at least a randomness basis.*

[7] Note that $\cup_{n=1}^{\infty} d_n$ is a measurable set of measure $r$. In fact, this set verifies the first part of the statement, but not necessarily the second.

*Proof.* Consider any subset $M \subset I$, such that the variables $\{x_i^I\}_{i \in M}$ are independent. Such sets exist trivially (e.g. a singleton) and can be regarded as random systems. So, the set of random systems for $X^I$ is nonempty. This set is preordered by the binary relation "to be broader than".

We want to apply Zorn's Lemma to this set. Thus, let $\mathcal{D}$ be a chain of random systems. We have to find an upper bound for this chain.

Define $m(\mathcal{D}) = \cup\{m(D) \ / \ D \in \mathcal{D}\}$.

**Step 1** Define the following binary relation: if $d, d' \in m(\mathcal{D})$, we say $dRd'$ whenever $d \subset d'$ a.e. or $d' \subset d$ a.e. Then, $R$ is an equivalence relation.

$R$ is obviously reflexive and symmetric. It is also transitive: if $dRd'$ and $d'Rd''$, we have the following possibilities. If $d \subset d'$ a.e. and $d' \subset d''$ a.e., then $d \subset d''$ a.e. and $dRd''$. If $d' \subset d$ a.e. and $d'' \subset d'$ a.e., then $d'' \subset d$ a.e. and $dRd''$.

The remaining two cases are: first, $d \subset d'$ a.e. and $d'' \subset d'$ a.e.; second, $d' \subset d$ a.e. and $d' \subset d''$ a.e. We consider them simultaneously.

Let $D, D'' \in \mathcal{D}$ such that $d \in D$, $d'' \in D''$. Without loss of generality, and since $\mathcal{D}$ is a chain, we can assume that $D'' \succ D$. Thus, there exists $d^* \in D''$ such that $d \subset d^*$ a.e. If $d^* = d''$, then $dRd''$. Otherwise, take $i \in (d \cap d') \cap d^*$ and $j \in d'' \cap d'$. These elements exist because $\mu(d), \mu(d'), \mu(d'') > 0$. Then, $x_i^I$ and $x_j^I$ are not independent, because $i, j \in d'$ and $d'$ is a cluster. But $i \in d^*, j \in d''$ and $d^*, d''$ are different clusters in the same random system $D''$, a contradiction.

Thus, $R$ is an equivalence relation.

**Step 2** Consider the quotient set $m(\mathcal{D})/R$. Each of the classes $C$ of this quotient set is a chain of clusters for $X^I$ for the almost inclusion relation. By Lemma 6.6, for each chain $C$ there exists a countable subchain $\{d_n(C)\}_{n=1}^\infty \subset C$ such that $d_\infty(C) = \liminf d_n(C)$ is a cluster for $X^P$ and an upper bound for $C$. Define $D^* = \{d_\infty(C) \ / \ C \in m(\mathcal{D})/R\}$. This is a set of disjoint measurable sets of $I$ (see below), of positive measure, and hence $m(\mathcal{D})/R$ must be countable by Lemma 6.7.

For each $n \in \mathbb{N}\backslash\{0\}$ and for each $C \in m(\mathcal{D})/R$, choose $D_n(C) \in \mathcal{D}$ such that $d_n(C) \in D_n(C)$. To see that the sets $d_\infty(C)$ are disjoint, consider two different equivalence classes $C, C' \in m(\mathcal{D})/R$, and let $i \in d_\infty(C) \cap d_\infty(C')$. Then there exist $d_n(C), d_m(C')$ such that $i \in d_n(C) \cap d_m(C')$. Without loss of generality, assume $D_n(C) \succ D_m(C')$. Then, there exists $d^* \in D_m(C'), d^* \neq d_m(C')$, such that $d_n(C) \subset d^*$ a.e. Let $j \in d_n(C) \cap d^*$ (this set is nonempty because $\mu(d_n(C)) > 0$). Since $i, j \in d_n(C)$, we have that $x_i^I, x_j^I$ are not independent. But, since $i \in d_m(C'), j \in d^*$, we have that $x_i^I, x_j^I$ are independent, a contradiction.

$D^*$ is the prime candidate for an upper bound of the chain of random systems. However, it is not necessarily so, because $D^*$ needs not be a random system unless we establish some relation between the sequences $\{d_n(C)\}_{n=1}^\infty$.

**Step 3** There exists a sequence of random systems $\{E_n\}_{n=1}^\infty$ such that, for all $C \in m(\mathcal{D})/R$, the sequence of clusters $\{e_n(C)\}_{n=1}^\infty = \{C \cap E_n\}_{n=1}^\infty$ has the same properties as $\{d_n(C)\}_{n=1}^\infty$, i.e. $\liminf e_n(C)$ is a cluster for $X^I$ and an upper bound for $C$.

Consider a given $C$ and its associated sequence $\{d_n(C)\}_{n=1}^{\infty}$. Denote $r(C) = \mu(d_\infty(C))$. Now, consider another $C'$ and its associated sequence $\{d_n(C')\}_{n=1}^{\infty}$. This latter sequence induces another sequence in $C$, given by $\{C \cap D_n(C')\}_{n=1}^{\infty}$. Let $b_n$ be the (unique) element in $C \cap D_n(C')$, if it exists, and $b_n = \emptyset$ otherwise (we will still write $b_n \in C \cap D_n(C')$ in this case). If there exists $b_n$ such that $\mu(b_n) = r(C)$, we say that $\{d_n(C')\}_{n=1}^{\infty}$ covers $\{d_n(C)\}_{n=1}^{\infty}$. Now, distinguish two cases:

**Case 1** There exists $\{d_n(C)\}_{n=1}^{\infty}$ such that no other $\{d_n(C')\}_{n=1}^{\infty}$ covers it.

Fix this $C$, and define $e_n(C) = d_n(C)$ for all $n$, $E_n = D_n(C)$.

Now consider any other $C' \in m(\mathcal{D})/R$, and define $b_n$ as above. Consider the nontrivial subcase $\sup\{\mu(b_n)\} > 0$. We will show that $\sup\{\mu(e_n(C'))\} = r(C')$. In order to see it, consider any $\varepsilon > 0$. There exists $d_n(C')$ such that $\mu(d_n(C')) > r(C') - \varepsilon$. Then, consider $b_n$, which can be assumed to have strictly positive measure in this subcase. As $\mu(b_n) < r(C)$, there exists $d_m(C), m > n$, such that $\mu(d_m(C)) > \mu(b_n)$, implying both that $b_n \subset d_m(C)$ a.e. and that the converse is not true. Thus, since $b_n \in D_n(C')$ and $d_m(C) \in D_m(C)$, we must have that $D_m(C) \succ D_n(C')$, which in turn implies that $d_n(C')$, which is the element in $D_n(C') \cap C'$, must be almost included in the element in $D_m(C) \cap C'$, i.e. $e_m(C')$. Thus, $\mu(e_m(C')) > r(C') - \varepsilon$. In summary, for every $\varepsilon > 0$ there exists $m$ such that $\mu(e_m(C')) > r(C') - \varepsilon$, implying $\sup\{\mu(e_n(C'))\} = r(C')$. But this last fact is enough to establish that $\{e_n(C')\}_{n=1}^{\infty}$has the same properties as $\{d_n(C')\}_{n=1}^{\infty}$, analogously to the proof of Lemma 6.6.

If $\sup\{\mu(b_n)\} = 0$, then any $D_m(C)$ must be such that $D_m(C) \succ D_n(C')$ for all $n$ and the same conclusion follows trivially.

**Case 2** For all $\{d_n(C)\}_{n=1}^{\infty}$, there exists $\{d_n(C')\}_{n=1}^{\infty}$ which covers it.

In this case, for all $C$, there exists $e(C) \in C$ such that $\mu(e(C)) = r(C)$. Since $m(\mathcal{D})/R$ is countable, we can enumerate it. Hence, let $m(\mathcal{D})/R = \{C_n\}_{n=1}^{\infty}$. We construct now the sequence $\{E_n\}_{n=1}^{\infty}$ by induction.

Let $E_1 \in \mathcal{D}$ be such that $e(C_1) \in E_1$. Then, for any $n > 1$, let $E'_n \in \mathcal{D}$ be such that $e(C_n) \in E'_n$. If $E'_n \succ E_{n-1}$, define $E_n = E'_n$. If $E_{n-1} \succ E'_n$, define $E_n = E_{n-1}$. Obviously, this sequence verifies the desired properties.

Given the claim just proved, we can rename our clusters and simply state $d_n(C) = e_n(C)$ for all $n$ and $C$. Now, the sequences $\{d_n(C)\}_{n=1}^{\infty}$ which give rise to the sets $d_\infty(C)$ have the property that $d_n(C), d_n(C')$ belong to the same random system for all equivalence classes $C, C'$ and fixed $n$, i.e. $D_n(C) = D_n(C')$. Write simply $D_n = D_n(C)$

**Step 4** $D^* = \{d_\infty(C) \,/\, C \in m(\mathcal{D})/R\}$ is a random system.

By Lemma 6.6, it is clear that $d_\infty(C)$ is a cluster for $X^I$ for each $C \in m(\mathcal{D})/R$.

Let $d_\infty(C_1), ..., d_\infty(C_K) \in D^*$, and let $i_k \in d_\infty(C_k), k = 1, ..., K$. We have to prove that $x^I_{i_1}, ..., x^I_{i_K}$ are independent.

Let $n_1, ..., n_K$ be such that $i_k \in \cap_{m=n_k}^{\infty} d_m(C_k)$, $k = 1, ..., K$. Denote $n = \max\{n_1, ..., n_K\}$. Then, $i_k \in d_n(C_k), k = 1, ..., K$ and $d_n(C_k) \in D_n$, $k = 1, ..., K$. Thus, $x^I_{i_1}, ..., x^I_{i_K}$ are independent.

Finally, it is clear that $D^*$ is an upper bound for the chain $\mathcal{D}$. Thus, every chain in the set of all random systems for $X^I$ has an upper bound. Existence of randomness basis is then implied by Zorn's Lemma. ■

*Remark* 6.11. It is not true in general that $\mu(\cup_{d \in D} d) = 1$. This question will be addressed later.

# 7 Classification of population extensions

We are now ready to use randomness basis to gain an insight on the possible variety of population extensions, in terms of independence and correlation. We need some additional concepts to establish the link between the correlation structure in a population extension and the individual random variables it represents.

**Definition 7.1.** Let $D$ be a randomness basis for a population extension $X^I$. An element $d \in D$ is called *idiosyncratic if* $d = \{i\}$*, almost idiosyncratic if* $\mu(d) = 0$*, and significant if* $\mu(d) > 0$.

**Definition 7.2.** Let $D$ be a randomness basis for a population extension $X^I$. A *representation of* $D$ *is a subset* $M \subset I$ *such that, for every* $i \in M$*, there exists a unique* $d \in D$ *such that* $i \in d$*, and for every* $d \in D$*, there exists a unique* $i \in M$ *such that* $i \in d$*. Denote* $D(i) = d$ *if* $i \in d$.

*Remark* 7.3. A representation of a randomness basis is automatically a family of i.i.d. random variables, of the cardinality of the randomness basis. Thus, one natural question is which is the maximum cardinality of both.

The next theorem gives a classification of the possible types of population extensions, or, more accurately, of the associated randomness bases. Essentially, what we see is that the complexity of a population extension can be expressed in terms of the cardinality of the maximal sets of independent random variables that it represents. This complexity ranges, then, from easy examples like the Wheel Extension (cardinality 1) to fairly complex structures where this cardinality is uncountable.

To complete the existence part of the classification, examples are required. Some of them have already been presented. The remaining ones are detailed after the proof.

**Theorem 7.4.** *Let* $X^I$ *be a population extension with randomness basis* $D$*. Then,* $D$ *is necessarily of one of the following four types. Moreover, for each of these types, there exist examples of population extensions having such randomness basis.*

1. *Finite:* $D$ *is finite.*
2. *Essentially finite:* $D$ *is countably infinite, but only a finite number of its elements are significant.*
3. *Countable:* $D$ *is countably infinite, and an infinite number of its elements are significant.*

4. *Uncountable: $D$ is uncountable, but only a countable number of its elements are significant. Moreover, there is no representation of strictly positive measure.*

*Proof.* $D$ can be finite, countably infinite, or uncountable. Examples 3.1 and 3.2 belong to the first type. If $D$ is countably infinite, the number of significant elements can either be finite or countably infinite. Examples 7.5 and 7.6 show that both cases are possible.

If $D$ is uncountable, no representation can have positive measure to avoid FG impossibility result. Example 7.7 shows that it is possible to have a countably infinite number of significant elements. All what is left to do is to show that there cannot be an uncountable number of them. But this is immediate since $m(D)$ must be countable by Lemma 6.7. ■

*Example* 7.5. An essentially finite extension.

Consider any Z-extension, and afterwards endow each of the agents in $\{\frac{1}{n}\}_{n=1}^{\infty}$ with an i.i.d. random variable, independent also from the Z-extension. All these agents belong to singleton idiosyncratic elements of the randomness basis, while the remaining set is significant.

*Example* 7.6. A countable extension.

Choose the set of agents $M = \{\frac{1}{m}\}_{m=1}^{\infty}$ and apply to every interval $(\frac{1}{m+1}, \frac{1}{m}]$ an independent Wheel Extension. These intervals form a countably infinite randomness basis (with representation $M$), and all of them are significant.

*Example* 7.7. The Cantor Population Extension

Let $E_1^1 = (\frac{1}{3}, \frac{2}{3}], i_1^1 = \frac{2}{3}$. Apply to all the agents in $E_1$ a Wheel Extension, and remove it from $I$. Let $E_2^1 = (\frac{1}{9}, \frac{2}{9}], i_2^1 = \frac{2}{9}$ and $E_2^2 = (\frac{7}{9}, \frac{8}{9}], i_2^2 = \frac{8}{9}$. Apply to all the agents in $E_2^1$ a Wheel Extension and to all the agents in $E_2^2$ another Wheel extension. Proceed iteratively: at each step $k$, define $\{E_k^l\}_{l=1,..,k}$ as the (half-open) middle thirds of the intervals which remain after $\{E_t^l\}_{l=1,..,t}^{t=1,...,k-1}$ are removed. The set $C = \cup_{k=1}^{\infty}\{i_k^1, ..., i_k^k\}$ is obviously countable. The set $D = (0, 1]\backslash\cup_{k=1}^{\infty}\{\cup_{l=1}^{k} E_k^l\}$ is, analogously to the Cantor ternary set, uncountable, and has measure zero. Endow all agents in $D$ with independent random variables identical to $X$, and independent from those of agents in $C$.

This extension has an uncountable randomness basis with representation $C \cup D$. $C$ is a countable set of representatives of significant clusters, where $D(i_k^l) = E_k^l$, and $D$ is an uncountable set of idiosyncratic representatives. The idea which allows us to obtain an uncountable set of agents with i.i.d. random variables is the recourse to a nowhere dense set, namely the Cantor Set. As Theorem 5.1 shows, this is the only kind of examples which can be constructed with an uncountable randomness basis.

# 8 Separated Randomness Basis

In this section, we study a particular type of population extensions which turn out to be particularly well-behaved. In particular, we will see that randomness basis are essentially unique, and that the clusters are itself population extensions, in the sense of fulfilling the strong law of large numbers.

Given a randomness basis for a population extension, it is not guaranteed that the joint measure of the clusters is equal to the measure of the population.

Quite naturally, it is possible to have overlapping "sources of randomness" giving rise to much more complicated behavior out of the clusters. This area is defined now.

**Definition 8.1.** Let $D$ be a randomness basis for a population extension $X^I$. The residual of $X^I$ according to $D$ is defined as $V(D) = I - \cup_{d \in D} d$. If $\mu(V(D)) = 0$, then $D$ is called *separated.*

When the residual is negligible, the population extension has a number of appealing properties which add to the tractability of the process. The first one is the essential uniqueness of the randomness basis.

**Theorem 8.2.** *If a population extension $X^I$ has a separated randomness basis, then it is unique (up to null sets), i.e. $m(D) \succ m(D')$ and $m(D') \succ m(D)$ for any two separated randomness basis $D, D'$.*

*Proof.* Let $D, D'$ be separated randomness basis.

Let $d' \in m(D')$. Obviously, there exists a unique $d \in D$ such that $d \cap d' \neq \emptyset$. Since $\mu(V(D)) = 0$, it must be $d' \subset d$ a.e. Since $D'$ is a maximal random system, this also implies that $d \subset d'$ a.e., hence $d = d'$ up to almost inclusion. This also holds (reciprocally) for $D$, and thus $D = D'$ up to almost inclusion. ■

This property does not hold for non-separated randomness basis, as we see in the following counterexample.

*Example* 8.3. A population extension without separated randomness basis.

Consider a disjoint partition of $I$ into five intervals, called $A_1, A_2, B_1, B_2, B_3$. Consider six i.i.d. random variables uniformly distributed on $(0, 1]$, and denote their realizations by $\omega_j$, $j = 1, ..., 6$. Apply to each of the intervals a $Z$-extension based on a random variable obtained from these six, according to the scheme:

| $A_1$ | $A_2$ | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|
| $\frac{1}{3}(\omega_1 + \omega_3 + \omega_5)$ | $\frac{1}{3}(\omega_2 + \omega_4 + \omega_6)$ | $\frac{1}{2}(\omega_1 + \omega_2)$ | $\frac{1}{2}(\omega_3 + \omega_4)$ | $\frac{1}{2}(\omega_5 + \omega_6)$ |

There are two different randomness basis. The first is $D = \{A_1, A_2\}$, with $V(D) = B_1 \cup B_2 \cup B_3$. The second is $D' = \{B_1, B_2, B_3\}$, with $V(D') = A_1 \cup A_2$. Note that $|D| \neq |D'|$.

Another property of population extensions with separated randomness basis is that the clusters inherit the properties of the extension, which yields a further (partial) homogeneity result in addition to Proposition 4.3. This result is quite intuitive. If we have two clusters, all the variables of agents in one of them are independent of those of individuals in the second. Then the "law of large numbers" must hold in the clusters, because if the fraction of agents obtaining a given realization were lower than the corresponding probability in one cluster, then this fraction should be larger in the second cluster, so that both add up and the "law" is verified for the whole population. Hence we would (intuitively) have a correlation, which is against the definition of the clusters. This apparently trivial fact has a non-trivial proof, due to the difficulties of linking the independence among clusters with the fact that we have a continuum of variables in each of them.

**Theorem 8.4.** *Let $X^I$ be a population extension for a random variable $X$ and let $D$ be a randomness basis for $X^I$. Suppose $D$ is separated. Then,*

$$\mu(\{i \in d \;/\; X_i^I(\omega) = s\}) = \lambda_s \cdot \mu(d) \;\forall\; d \in D.$$

*Proof.* This property is trivial if $\mu(d) = 0$. Hence, assume $\mu(d) > 0$, and call $d' = \cup\{d'' \in D\backslash\{d\}\}$. By construction, $d$ and $d'$ verify that, for all $i \in d$ and $j \in d'$, $x_i^I$ and $x_j^I$ are independent. Moreover, $\mu(d \cup d') = 1$ and $d \cap d' = \emptyset$.

Consider a given realization, $s$, such that $\lambda = \lambda_s > 0$. By joint measurability, the set $E = x^{-1}(s) \cap (d \times \Omega)$ is measurable in the product $\sigma$-algebra $\mathcal{B} \times \mathcal{F}$. We want to prove that, for almost all $\omega$, $\mu(E^\omega) = \lambda \cdot \mu(d)$.

Suppose not. There exist $\Omega_-, \Omega_+ \in \mathcal{F}$ such that $P(\Omega_- \cup \Omega_+) = 1$ and $\mu(E^\omega) \leq \lambda \cdot \mu(d) \;\forall\; \omega \in \Omega_-$, $\mu(E^\omega) > \lambda \cdot \mu(d) \;\forall\; \omega \in \Omega_+$. Moreover, $\int_{\Omega_-} \mu(E^\omega) dP(\omega) < \lambda \cdot \mu(d) \cdot P(\Omega_-)$.

By Fubini's Theorem, $\int_\Omega \mu(E^\omega) dP(\omega) = \int_d P(E_i) d\mu(i) = \int_d \lambda d\mu(i) = \lambda \cdot \mu(d)$. Hence, it is immediate to see that $\int_{\Omega_-} (\lambda \cdot \mu(d) - \mu(E^\omega)) dP(\omega) = \int_{\Omega_+} (\mu(E^\omega) - \lambda \cdot \mu(d)) dP(\omega) = \eta > 0$.

Take $\varepsilon < \eta$ and consider the approximation described in Lemma A.1 (see Appendix A). Call $E_0 = E \backslash \cup_{l=1}^L (I \times F_l)$. We have that

$$(\mu \times P)(E \Delta \cup_{l=1}^L (B(F_l) \times F_l)) = (\mu \times P)(E_0) + \sum_{l=1}^L \int_{F_l} \mu(E^\omega \Delta B(F_l)) dP(\omega) < \varepsilon.$$

We prove now two claims.

**Claim 1** Let $\rho > 0$. A set $A \in \mathcal{F}$ is called $\rho$-homogeneous if $\int_A \mu(E^\omega) dP(\omega) = \rho \cdot P(A)$ and $\int_{\Omega \backslash A} \mu(E^\omega) dP(\omega) = \rho \cdot P(\Omega \backslash A)$ Then, if $A, B$ are $\rho$-homogeneous, $A \backslash B$ is also $\rho$-homogeneous.

Consider the partition of $\Omega$ given by $\{A \cap B, A \backslash B, B \backslash A, \Omega \backslash (A \cup B)\}$. For each $C$ in this partition, call $Q(C) = \int_C \mu(E^\omega) dP(\omega)$. Then, from the fact that both $A$ and $B$ are $\rho$-homogeneous we have the following system of equations:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} Q(A \cap B) \\ Q(A \backslash B) \\ Q(B \backslash A) \\ Q(\Omega \backslash (A \cup B)) \end{pmatrix} = \rho \cdot \begin{pmatrix} P(A) \\ P(\Omega \backslash A) \\ P(B) \\ P(\Omega \backslash B) \end{pmatrix} =$$
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} \rho \cdot P(A \cap B) \\ \rho \cdot P(A \backslash B) \\ \rho \cdot P(B \backslash A) \\ \rho \cdot P(\Omega \backslash (A \cup B)) \end{pmatrix}$$

Since the coefficient matrix is invertible, the conclusion of the first claim follows.

**Claim 2** For any $i \in d$, $E_i$ is $\lambda \cdot \mu(d)$-homogeneous.

Since for all $j \in d'$, $x_i^I$ and $x_j^I$ are independent, $P(E_i \cap (x_j^I)^{-1}(s)) = \lambda^2 \;\forall\; j \in d'$. Then, by Fubini's theorem,

$$\int_{E_i} \mu(\{i \in d' \;/\; x(i,\omega) = s\}) dP(\omega) = \int_{d'} P(E_i \cap (x_j^I)^{-1}(s)) d\mu(i) = \lambda^2 \cdot \mu(d').$$

By aggregate stability, $\lambda \cdot P(E_i) = \int_{E_i} \mu(\{i \in I \ / \ x(i,\omega) = s\})dP(\omega) = \int_{E_i} \mu(\{i \in d \ / \ x(i,\omega) = s\})dP(\omega) + \lambda^2 \cdot \mu(d')$.

It follows that $\int_{E_i} \mu(E^\omega)dP(\omega) = \int_{E_i} \mu(\{i \in d \ / \ x(i,\omega) = s\})dP(\omega) = \lambda^2 \cdot (1 - \mu(d')) = \lambda^2 \cdot \mu(d) = \lambda \cdot \mu(d) \cdot P(E_i)$, where the last equality holds because, by individual uncertainty, $P(E_i) = \lambda$. The conclusion of the second claim follows.

Consider now a given $F_l$. By construction (see Lemma A.1), there exist $\{j_1, ..., j_n\} \subset \{i_1, ..., i_N\}$ such that $F_l = ((E_{j_1} \backslash E_{j_2}) \backslash ...) \backslash E_{j_n}$. Combining the two claims above, and reasoning iteratively, it is easy to see that $F_l$ is $\lambda \cdot \mu(d)$-homogeneous.

This means that $\int_{F_l} \mu(E^\omega)dP(\omega) = \lambda \cdot \mu(d) \cdot P(F_l)$ and automatically yields that $\int_{F_l \cap \Omega_-} (\lambda \cdot \mu(d) - \mu(E^\omega))dP(\omega) = \int_{F_l \cap \Omega_+} (\mu(E^\omega) - \lambda \cdot \mu(d))dP(\omega) = \eta_l$.

We evaluate now the approximation restricted to $F_l \times d$. Let $\mu(B(F_l)) = a_l$, and let $F_l(a-) = \{\omega \in F_l \ / \ \mu(E^\omega) \leq a\}, F_l(a+) = \{\omega \in F_l \ / \ \mu(E^\omega) > a\}$ for any number $a$. Note that

$\int_{F_l} \mu(E^\omega \Delta B(F_l))dP(\omega) \geq \int_{F_l} |\mu(E^\omega) - a_l|dP(\omega) =$

$= \int_{F_l(a-)} (a_l - \mu(E^\omega))dP(\omega) + \int_{F_l(a+)} (\mu(E^\omega) - a_l)dP(\omega)$.

Suppose that $a_l \geq \lambda \cdot \mu(d)$. Then, $F_l(\lambda \cdot \mu(d)) \subset F_l(a-)$ and it follows that

$\int_{F_l(a-)} (a_l - \mu(E^\omega))dP(\omega) \geq \int_{F_l(\lambda \cdot \mu(d))} (a_l - \mu(E^\omega))dP(\omega) \geq$

$\int_{F_l(\lambda \cdot \mu(d))} (\lambda \cdot \mu(d) - \mu(E^\omega))dP(\omega) = \eta_l$.

Analogously, if $a_l \leq \lambda \cdot \mu(d)$ then $\int_{F_l(a+)} (\mu(E^\omega) - a_l)dP(\omega) \geq \eta_l$.

In any case, we have that $\int_{F_l} \mu(E^\omega \Delta B(F_l))dP(\omega) \geq \eta_l$.

Now consider the whole approximation:

$(\mu \times P)(E_0) + \sum_{l=1}^L \int_{F_l} \mu(E^\omega \Delta B(F_l))dP(\omega) \geq \sum_{l=1}^L \eta_l =$

$\sum_{l=1}^L \int_{F_l \cap \Omega_+} (\mu(E^\omega) - \lambda \cdot \mu(d))dP(\omega) = \int_{\Omega_+} (\mu(E^\omega) - \lambda \cdot \mu(d))dP(\omega) = \eta$, a contradiction. ■

# 9 Applications

## 9.1 Approximately idiosyncratic risk

A model for risk is always based on a fixed random variable, $x$, e.g. the probability of having an accident. A population extension of $X$ provides an immediate large population model for such risk framework, exhibiting the essential features one expects to capture. For example, Proposition 4.3 implies that, for a large number of traders, equal sharing of resources is a Core Allocation (defined as an allocation such that no measurable coalition would be prefer another one in expected terms).

The only thing we have renounced to, if a population extension is used to model risk in a large population, is full independence. While admittedly this could be unacceptable for some extreme situations, for many others, if independence is important at all, it would be enough to assume that correlation is *small enough*. The following theorem shows in which sense can this be formalized.

**Theorem 9.1.** *Given a simple random variable $X$ and given $\varepsilon > 0$, there exist population extensions $X^I$ with regular randomness basis such that, for every agent $i$,*

$$\mu(\{j \in I \mid x_i^I, x_j^I \text{ are not independent}\}) < \varepsilon.$$

*Proof.* Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and apply an independent Z-extension to every interval $(\frac{k}{n}, \frac{k+1}{n}]$, $k = 0, ..., n-1$. ■

Al-Najjar [2] constructs "finite-characteristic economies" where all the agents in pre-specified intervals obtain exactly the same realization. Thus, the intervals are actually atoms and the economies are easily reinterpreted as economies with a finite number of agents. Aggregate stability ("a law of large numbers") is obtained as a limit result as the size of the intervals approaches zero, although for any fixed finite-characteristic economy, the aggregate stability ("law of large numbers") condition is false.

Theorem 9.1 is unrelated to Al-Najjar's. It provides economies where agents obtain correlated but not identical realizations inside each of a number of intervals. Thus, these economies cannot be reinterpreted as economies with a finite number of agents. Moreover, for each $\varepsilon > 0$, the aggregate stability condition holds exactly. There is no need to reinterpret the result in terms of limits. The role of $\varepsilon$ here is radically different. If the measure of the elements of a randomness basis is interpreted as the size of "correlation areas", i.e. the clusters, then they would be a measure of the correlation in the economy. Thus, the theorem states that this measure can be made arbitrarily small.

## 9.2 Random Matching and other dynamic models

Random matching processes are a quite complex application of continuum families of random variables. It might seem that a random matching process could simply be viewed as a family of independent random variables, whose realizations are interpreted as the random name of the partner. This is not true because such variables could not be independent. If agent $i$ is matched with agent $j$, then agent $j$ has to be matched with agent $i$.

If agents can be of a finite number of types, then three properties are usually needed in economic applications. Firstly, for any given agent, the probability of being matched to agents of a certain type equals the proportion of such agents in the population. Secondly, the fraction of matches between agents of two given types equals the product of the population proportions of these two types (twice for different types). Thirdly, the probability of any two fixed agents to be matched is zero.

In Alós-Ferrer [5], it is shown that there exist random matching processes for a continuum of individuals, satisfying these properties. The key for the construction is to consider a random variable $X$ taking values on the set of types such that the probability of each type equals the proportion of agents of such type, and then construct a family of random variables which actually can be reinterpreted now as a population extension of $X$. The realizations of this population extension are interpreted as the type of the future partners, and then it is shown that this information is enough to construct a probability space on true matchings.

As a final example, consider Harrington [11]. A continuum of agents of different types are randomly matched each period. Additionally, agents face the realization of an environmental stochastic element (favorable or adverse environment, for instance) *each period.* Then, a law of large numbers is used to write down deterministic equations which characterize the evolution of the population.

In terms of population extensions, this model is now easy to understand. Random matching proceeds as explained above. Additionally, there exists a population extension of the random variable which determines the environment. Both elements give exact results, and hence are easily repeated in time. The whole dynamical system is easily constructed, and deterministic equations are indeed the result.

# 10 Concluding Remarks

We have shown that population extensions display enough regularities to provide a tractable framework for the study of stochastic mass phenomena in economics. The joint measurability condition provides analytical tractability, including the convenient consequence (Theorem 4.3) that the unavoidable pathologies cancel out in the aggregate. The individual uncertainty condition allows us to incorporate the usual models into this framework in a straightforward way. The lack of aggregate uncertainty reflects our desire to study population situations where aggregative effects are indeed observed.

In the framework of a population extension, full independence is necessarily lost (Theorem 8.4). Instead, potentially complex patterns of independence and correlation appear. Randomness bases are a first tool for the classification and understanding of what these patterns are. The fact that they *always* exist, as proven in Theorem 6.10, guarantees their usefulness. Its uniqueness in the class of population extensions with separated randomness basis allows the identification of the randomness basis with the population extension.

The two properties illustrated for population extensions with separated randomness basis (uniqueness of the randomness basis and cluster-homogeneity) single out this class of population extensions as a powerful modeling tool for economics.

The study of the residual is left for future research. While it is clear that we could ignore it and concentrate on separated randomness basis for most economic models, the possibilities opened by its appearance are, at the very least, intriguing.

Which is the relevance of population extensions for theoretical economists? The aim has been to provide and study a tractable framework for modeling purposes. Suppose a researcher wants to study an individual uncertainty situation in a large population framework. Then, the first step is to model this uncertainty by a random variable $X$. The second step is now simply to postulate a population extension of $X$, and the researcher has automatically a tractable model (where Fubini-type results hold) with individual uncertainty cancelling out exactly on aggregation. Not only that, but the sets of agents facing a given realization are measurable, allowing for the construction of dynamical systems where the measures of those sets are the relevant variables. The researcher can then regard the population extension as a convenient "black box," which, like reality, may exhibit varying degrees of internal complexity while still capturing the features that the researcher was interested in.

# A The approximation of measurable sets

The following technical property is used twice in the main text.

**Lemma A.1.** *Let $(I, \mathcal{B}, \mu)$ and $(\Omega, \mathcal{F}, P)$ be measurable spaces, and let $E \in \mathcal{B}\times\mathcal{F}$. Then, for all $\varepsilon > 0$,*

1. *there exist $N \in \mathbb{N}, B_1, ..., B_N \in \mathcal{B}$ pairwise disjoint, and $G_1, ..., G_N \in \mathcal{F}$ such that $(\mu \times P)(E\Delta \cup_{n=1}^{N} (B_n \times G_n)) < \varepsilon$*
2. *furthermore, it can be assumed that, for each $n = 1, ..., N$, there exists $i_n \in B_n$ such that $G_n = E_{i_n} = \{\omega \;/\; (i_n, \omega) = s\} = (x_i^I)^{-1}(s)$*
3. *there exist $F_1, ..., F_L \in \mathcal{F}$ pairwise disjoint such that $\cup_{n=1}^{N}(B_n \times G_n) = \cup_{l=1}^{L}(B(F_l) \times F_l)$, where $B(F_l) = \cup\{B_n \;/\; F_l \subset E_{i_n}\}$.*

*Proof.* The first part is a consequence of Halmos [10, S.13, Theorem D], taking into account that $\mathcal{B} \times \mathcal{F}$ is generated by the class of all finite, disjoint unions of rectangles. The sets $B_1, ..., B_N$ are ensured to be disjoint by appropriate refining. The third part is then easy, and the $F_l$ are obtained as $F_l = ((G_{j_1} \backslash G_{j_2})\backslash ...)\backslash G_{j_m}$ for appropriate $\{j_1, ..., j_m\} \subset \{1, ..., N\}$.

To see the second part, take the approximation prescribed by the first one for $\frac{\varepsilon}{3}$, and define

$E_n = (E \cap (B_n \times \Omega)), E_0 = E\backslash((\cup_{n=1}^{N} B_n) \times \Omega)$. Note that $E = \cup_{n=0}^{N} E_n$.

$\varepsilon_n = (\mu \times P)(E_n\Delta(B_n \times G_n)) = \int_{B_n} P(E_i\Delta G_n)d\mu(i)$

$\varepsilon_0 = (\mu \times P)(E_0)$

Then, $\sum_{n=0}^{N} \varepsilon_n = \frac{\varepsilon}{3}$

Given $n \in \{1, ..., n\}$, then there exists $i_n \in B_n$ such that $P(E_{i_n}\Delta G_n) \leq \frac{2\varepsilon_n}{\mu(B_n)}$. If not, $\int_{B_n} P(E_i\Delta G_n)d\mu(i) \geq 2\varepsilon_n$, a contradiction. Consider $B_n \times E_{i_n}$.

The symmetric difference operator verifies that, for any three sets $A, B, C$, $A\Delta C \subset (A\Delta B) \cup (B\Delta C)$. Using this property, $E_n\Delta(B_n \times E_{i_n}) \subset (E_n\Delta(B_n \times G_n)) \cup ((B_n \times G_n)\Delta(B_n \times E_{i_n}))$. Hence,

$(\mu \times P)(E_n\Delta(B_n \times E_{i_n})) \leq (\mu \times P)(E_n\Delta(B_n \times G_n)) +$
$(\mu \times P)(B_n \times (G_n\Delta E_{i_n})) = \varepsilon_n + \mu(B_n) \cdot P(G_n\Delta E_{i_n}) \leq 3\varepsilon_n$

Then, $(\mu \times P)(E\Delta \cup_{n=1}^{N} (B_n \times E_{i_n})) =$
$(\mu \times P)(E_0) + \sum_{n=1}^{N}(\mu \times P)(E_n\Delta(B_n \times E_{i_n})) \leq \varepsilon_0 + 3\sum_{n=1}^{N} \varepsilon_n \leq 3\frac{\varepsilon}{3} = \varepsilon$.

All what is left is to rename $G_n = E_{i_n}$. ■

# B A non-measurable extension

Consider a random variable $X$ taking the values $0, 1$ with probability $\frac{1}{2}$. Following Judd [12], we can show that,[8] given an interval, there exists a family of i.i.d. random variables identical to $X$, indexed on that interval, and such that, with probability one in the appropriate probability space, the measure of the agents rolling 0 in that interval is $r$, for any $r$ in $[0, 1]$. We call such a family an $r$-family.

Take a partition of $(0, 1]$ into six intervals of measure $\frac{1}{6}$, $A_1, ..., A_6$. Individuals in $C_1 = A_1 \cup A_2 \cup A_3$ will obtain realizations according to the following procedure. First, roll $k = 1, 2$, or 3, each with probability $\frac{1}{3}$. Then, assign to

[8] This result is more general than the one proved by Judd, but the proof is totally analogous.

the agents in $A_k$ an $r$-family of random variables, with $r = 1$. Apply to the union of the other two intervals a wheel extension. Do exactly the same for $C_2 = A_4 \cup A_5 \cup A_6$, but taking $r = 0$. Then, $C_1$ is a cluster of agents, each of them having probability $\frac{1}{2}$ of rolling a 0, but such that the measure of agents rolling a 0 is always $\frac{1}{3}$. Analogously, $C_2$ is a cluster where this measure is $\frac{1}{6}$. Thus, the total measure is always $\frac{1}{2}$, and we have a population extension with a separated randomness basis, but the associate constants for the clusters are not $\frac{1}{2}$, but $\frac{2}{3}$ and $\frac{1}{3}$.

Of course, this contradicts Theorem 8.4, a fact which shows that this "population extension" fails to be jointly measurable.

This example is created through a trick. Embedding a continuum of i.i.d. random variables in the construction, the relevant sets of realizations (in this case, those such that $\frac{1}{3}$ or $\frac{2}{3}$ of the agents obtain 0) turn out to be non-measurable in such a way that a new probability space can be created by assigning to them any measure we please. Obviously, the existence of such "pathological" extensions is more a technical problem than a conceptual one. Joint measurability excludes these examples.

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