

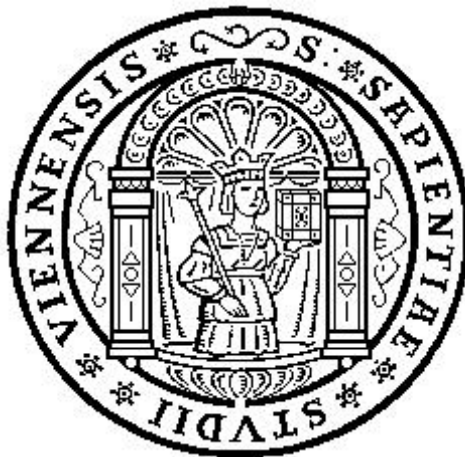
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# Capital price bubbles and dynamic inefficiency

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## **Abstract**

We demonstrate that the price of physical capital in standard neoclassical one-sector growth models can exceed its fundamental value, that is, a capital price bubble can exist. It is furthermore shown that the existence of a capital price bubble is in general unrelated to the dynamic inefficiency of equilibrium. We illustrate these results in the contexts of the Ramsey-Cass-Koopmans model with finitely many infinitely-lived dynasties of households, the Blanchard-Yaari model with infinitely many overlapping generations of finitely-lived households, and the Solow-Swan model without microfoundation. Our findings are in contrast to those derived by Tirole [17] and they are complementary to those from Kocherlakota [8] and Tirole [18].

**Keywords:** Capital price bubbles, dynamic inefficiency, one-sector growth model, Ramsey-Cass-Koopmans model, Blanchard-Yaari model, Solow-Swan model.

**JEL Classification:** O41, G10

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# 1 Introduction

Conventional wisdom about the occurrence of rational bubbles in infinite-horizon economies has been largely shaped by two seminal papers by Tirole [17, 18]. The first one argues that rational bubbles cannot occur if the economy is populated by finitely many infinitely-lived traders, whereas the second one demonstrates that rational bubbles can exist in an economy with overlapping generations of finitely-lived agents provided that the bubbleless equilibrium is dynamically inefficient. The purpose of the present paper is to reconsider these two propositions in the standard neoclassical one-sector growth model with physical capital as the only asset. We show that the claim from [17] does not hold in this setting and that the relation between dynamic inefficiency and the existence of rational bubbles is more subtle in our framework than in the model used in [18].

Kocherlakota [8] has already shown that the result in [17] is incorrect and that rational bubbles can exist in economies with a finite number of infinitely-lived agents. He points out that the flaw in Tirole's argument consists of his failure to impose wealth constraints or short sales constraints, without which the existence of equilibrium is not ensured.<sup>1</sup> The framework of the analysis in [8] is an endowment economy with one commodity and one asset in exogenously fixed supply. If the agents are subject to a wealth constraint (also referred to as the natural borrowing limit or as a no-Ponzi-game condition), Kocherlakota [8] shows that asset price bubbles can exist only if the asset is in zero net supply.

The argument against the validity of [17] put forward in the present paper is different from that in [8] in several respects. First of all, we consider a production economy, namely the standard Ramsey-Cass-Koopmans model. The population consists of a finite number of infinitely-lived dynasties, whose sizes grow at rate  $n$  and who are subject to no-Ponzi-game conditions. The only asset in the economy is physical capital and the price of this asset in terms of contemporaneous consumption is constant and equal to 1 (as it is the case in any one-sector economy). Now suppose that the population growth rate  $n$  is negative and that the pure rate of time-preference of the households,  $\rho$ , satisfies  $\rho \in (n, 0]$ . The effective rate of time-preference of each dynasty,  $\rho - n$ , is then strictly positive, which ensures existence, uniqueness, and dynamic efficiency of an equilibrium. However, the assumption  $\rho \leq 0$  implies that the real interest rate converges to a non-positive number. Combining this observation with the fact that the price of capital is positive, it follows that the fundamental value of physical capital falls short of its price, i.e., a capital price bubble exists. How does this finding fit together with the claim from Kocherlakota [8] that asset price bubbles can only occur if the asset is in zero net supply? To see the connection, just note that the asset supply in the Ramsey-Cass-Koopmans model

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<sup>1</sup> "It is true that no equilibrium with bubbles exists in this environment. However, no other equilibrium exists either [...]" [8, footnote 2]

is endogenously determined and that it converges *in per-capita terms* to the modified Golden Rule capital stock  $k^* > 0$ . In the situation described above, where  $n < 0$  holds, the *aggregate* capital stock therefore converges to 0, which is consistent with Kocherlakota's claim at least asymptotically.

Let us now turn to the second main result of the present paper, which deals with the relation between dynamic inefficiency of equilibria and the existence of asset price bubbles. The overlapping generations economy studied by Tirole [18] has two assets: physical capital and a useless asset. Under certain parameter restrictions, the model has a unique equilibrium and the price of the useless asset in terms of contemporaneous consumption is 0. Under alternative parameter constellations, however, the model admits infinitely many equilibria: in one of these equilibria the useless asset has the prize 0, whereas in all other equilibria it has a strictly positive price. What Tirole [18] shows is that there exist bubbly equilibria (equilibria in which the useless asset has a strictly positive price) if and only if the unique bubbleless equilibrium is dynamically inefficient.

Although we do not question the validity of the result in Tirole [18], we show that the relation between dynamic inefficiency and the existence of asset price bubbles is not as clear cut as in [18] if one adopts a different point of view. More specifically, we consider a general neoclassical one-sector economy (it could be one with infinitely-lived dynasties such as the Ramsey-Cass-Koopmans model, or one with overlapping generations of finitely-lived households such as the Blanchard-Yaari model, or a model without any specific microstructure or demographic structure such as the Solow-Swan model) and show that the conditions under which a feasible allocation is dynamically efficient are *different* from those under which the price of physical capital exceeds its fundamental value (i.e., a capital price bubble exists). In particular, we prove that in the case of a positive population growth rate  $n$ , the existence of a capital price bubble implies dynamic inefficiency, whereas in the case where  $n$  is negative, dynamic inefficiency of an allocation implies that it involves a capital price bubble. In general, however, all four combinations are possible: there are dynamically efficient equilibrium allocations with or without capital price bubbles, and there are dynamically inefficient equilibrium allocations with or without capital price bubbles. We illustrate these possibilities by means of the Blanchard-Yaari model with overlapping generations of finitely-lived households and by means of the Solow-Swan model. The difference between our results and the findings of Tirole [18] occurs because we assume physical capital to be the only asset of the economy, whereas Tirole considers a setting with a second, unproductive asset. The existence of a bubble in our framework does not imply that excess savings are diverted from their productive use and, as a consequence, the capital price bubbles considered in the present paper do not help to avoid the inefficient overaccumulation of capital.

The rest of the paper is organized as follows. Section 2 defines asset price bubbles in a very general setting and presents a simple characterization of bubbles, which goes back to Montrucchio [10]. Section 3 sets the stage for the core of our analysis. We start by reviewing the neoclassical one-sector economy and by defining dynamic (in)efficiency of allocations. We also derive simple conditions that can be used to verify whether certain allocations are dynamically efficient or bubbly, respectively. Finally, we present our first theorem which highlights that the relation between dynamic efficiency and the existence of capital price bubbles depends crucially on whether the population growth rate is negative or positive. It has to be emphasized that the material in section 3 does neither depend on the demographic structure of the population nor on the microfoundation of the aggregate saving rate. Section 4 presents the main results of the paper by treating, in turn, the Ramsey-Cass-Koopmans model, the Blanchard-Yaari model, and the Solow-Swan model. First we show that the Ramsey-Cass-Koopmans model admits bubbly equilibria despite the fact that the population consists of finitely many infinitely-lived decision makers. Then we demonstrate that the existence of capital price bubbles in the Blanchard-Yaari model or the Solow-Swan model, respectively, is not equivalent to the overaccumulation of capital (i.e., to dynamic inefficiency of the equilibrium). Section 5 summarizes our findings and discusses the relevance and accuracy of some of the underlying assumptions.

## 2 Asset prices, fundamental values, and bubbles

In this section we collect a number of definitions and results about asset price bubbles that are independent of the particular structure of the economy in which they arise. Consider any deterministic economy that evolves continuously over the time domain  $\mathbb{R}_+$  and in which the real interest rate at time  $t \in \mathbb{R}_+$  is given by  $r(t)$ . Suppose that there exists an asset which, at time  $t \in \mathbb{R}_+$ , pays dividend  $d(t)$  and is traded at price  $p(t)$ . Dividends and prices are expressed in terms of contemporaneous real consumption. We assume that  $d : \mathbb{R}_+ \mapsto \mathbb{R}$  and  $r : \mathbb{R}_+ \mapsto \mathbb{R}$  are measurable and that  $p : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is absolutely continuous and satisfies  $p(t) > 0$  for all  $t \in \mathbb{R}_+$ . Furthermore, we assume that the asset price satisfies the condition

$$r(t) = \frac{d(t) + \dot{p}(t)}{p(t)} \quad (1)$$

for almost all  $t \in \mathbb{R}_+$ . If the asset under consideration is the only one available in the economy, then (1) is simply the definition of the real interest rate. If there are other assets, then (1) is interpreted as a no-arbitrage condition.

The fundamental value of the asset under consideration at time  $t \in \mathbb{R}_+$ , which we denote by

$f(t)$ , is the present value (as of time  $t$ ) of all future dividend payments. Formally, it is given by

$$f(t) = \int_t^{+\infty} D(t, \tau) d(\tau) d\tau, \quad (2)$$

where the discount factor is defined as

$$D(t, \tau) = e^{-\int_t^\tau r(s) ds}. \quad (3)$$

We define the bubble component of the asset price at time  $t \in \mathbb{R}_+$  by  $p(t) - f(t)$  and say that the asset price contains a bubble at time  $t \in \mathbb{R}_+$  if  $p(t) \neq f(t)$  holds. We have the following result.

**Proposition 1** *Let equations (1)-(3) be satisfied.*

(a) *It holds for all  $t \in \mathbb{R}_+$  that*

$$f(t) = p(t) \left[ 1 - \lim_{\tau \rightarrow +\infty} e^{-\int_t^\tau d(s)/p(s) ds} \right]. \quad (4)$$

(b) *An asset price bubble exists at time  $t \in \mathbb{R}_+$  if and only if*

$$\lim_{\tau \rightarrow +\infty} \int_t^\tau d(s)/p(s) ds < +\infty.$$

(c) *There are no negative asset price bubbles, that is,  $p(t) \geq f(t)$  holds for all  $t \in \mathbb{R}_+$ .*

PROOF: We combine (1) and (2) to obtain

$$f(t) = \int_t^{+\infty} e^{-\int_t^\tau [d(s)+\dot{p}(s)]/p(s) ds} d(\tau) d\tau.$$

This equation can be rewritten as

$$\begin{aligned} f(t) &= \int_t^{+\infty} e^{-\int_t^\tau d(s)/p(s) ds - \ln p(\tau) + \ln p(t)} d(\tau) d\tau \\ &= p(t) \int_t^{+\infty} e^{-\int_t^\tau d(s)/p(s) ds} [d(\tau)/p(\tau)] d\tau \\ &= p(t) \left[ -e^{-\int_t^\tau d(s)/p(s) ds} \right]_{\tau=t}^{+\infty} \\ &= p(t) \left[ 1 - \lim_{\tau \rightarrow +\infty} e^{-\int_t^\tau d(s)/p(s) ds} \right], \end{aligned}$$

which proves (4). Statements (b) and (c) are immediate consequences of (4).  $\square$

Whereas we are not aware of any published version of statement (a), statement (b) is a continuous-time version of a condition derived by Montrucchio in [10]. Statement (c) says that the existence of a bubble involves always overpricing of the asset but never underpricing and can be found, for example, in [18, equation (13) and footnote 8] or in [13, proposition 2.1].

### 3 The one-sector economy, dynamic (in)efficiency, and capital price bubbles

We start the present section by reviewing the neoclassical one-sector economy, which forms the framework of the following analysis. Subsequently, we discuss under which conditions equilibria of this economy are dynamically efficient or not, and under which conditions there exist capital price bubbles.

Consider a one-sector economy that evolves in continuous time over the infinite time horizon  $\mathbb{R}_+$ . The population is assumed to grow at the constant rate  $n$  and its initial size is normalized to be equal to  $N(0) = 1$ . The population size at time  $t \in \mathbb{R}_+$  is therefore given by  $N(t) = e^{nt}$ .

There exists a single production sector consisting of a continuum of measure 1 of identical firms. These firms transform the input factors capital and labor into an output good which can be consumed or invested. We denote the intensive production function by  $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$  and assume that  $F$  is a continuous and strictly increasing function satisfying  $F(0) = 0$ . We assume furthermore that  $F$  is twice continuously differentiable on the interior of its domain, that the Inada conditions  $\lim_{k \rightarrow 0} F'(k) = +\infty$  and  $\lim_{k \rightarrow +\infty} F'(k) = 0$  are satisfied, and that  $F''(k) < 0$  holds for all  $k > 0$ . Finally, we assume that capital depreciates at the constant rate  $\delta$ .

We assume that both  $\delta$  and  $\delta + n$  are strictly positive numbers, but we do not require the population growth rate to be positive. It follows from the above assumptions that there exists a unique value  $K > 0$  satisfying  $F(K) = (\delta + n)K$ . This value  $K$  is the maximal sustainable per capita capital stock. The initial (per capita) capital endowment of the economy is denoted by  $\kappa$  and it is assumed that  $\kappa \in (0, K)$ .

We denote by  $k(t)$ ,  $c(t)$ , and  $i(t)$  the per capita capital stock, the per capita consumption rate, and the per capita investment rate at time  $t \in \mathbb{R}_+$ . Note that the functions  $k$ ,  $c$ , and  $i$  are defined on the time domain  $\mathbb{R}_+$  and take values in  $\mathbb{R}$ . The triple  $(k, c, i)$  is called a feasible allocation for the economy, if  $c$  and  $i$  are measurable, if  $k$  is absolutely continuous, and if the conditions

$$\dot{k}(t) = i(t) - (\delta + n)k(t), \quad k(0) = \kappa, \tag{5}$$

$$c(t) + i(t) = F(k(t)), \tag{6}$$

$$c(t) \geq 0, \quad k(t) \geq 0 \tag{7}$$

hold for almost all  $t \in \mathbb{R}_+$ . The first equation is the capital accumulation equation which says that net investment equals gross investment minus depreciation. The second one can be interpreted as an output market clearing condition and reflects the fact that final output can be used for consumption or for investment. The third line imposes non-negativity of consumption

and capital at all times. Note that investment is not required to be non-negative, which implies that output that has been installed as capital can be turned into the consumption good.

We assume that the factor markets are perfectly competitive. This implies that the rental rates for capital and labor at time  $t \in \mathbb{R}_+$  are given by  $q(t) = F'(k(t))$  and  $w(t) = F(k(t)) - F'(k(t))k(t)$ , respectively. The real interest rate is defined as the real return on physical capital and equals  $r(t) = q(t) - \delta = F'(k(t)) - \delta$ .

A feasible allocation  $(k, c, i)$  is called dynamically inefficient if there exists another feasible allocation  $(\tilde{k}, \tilde{c}, \tilde{i})$  such that  $\tilde{c}(t) \geq c(t)$  holds for almost all  $t \in \mathbb{R}_+$  and such that the set  $\{t \in \mathbb{R}_+ \mid \tilde{c}(t) > c(t)\}$  has positive measure. A feasible allocation  $(k, c, i)$  is called dynamically efficient if it is not dynamically inefficient. Translating the conditions derived by Cass [4] into the continuous-time setting of the present note it follows that a feasible allocation  $(k, c, i)$  is dynamically inefficient if and only if

$$\lim_{\tau \rightarrow +\infty} \int_0^\tau e^{\int_0^t [r(s) - n] ds} dt < +\infty \quad (8)$$

holds. The Golden Rule per capita capital stock  $\bar{k}$  is defined as the unique positive number satisfying  $F'(\bar{k}) = \delta + n$ . Note that  $\bar{k} \in (0, K)$  holds under the maintained assumptions, which include the requirement that  $\delta + n > 0$ . The inefficiency criterion of [4] stated in (8) gives rise to the following proposition, which can be used to verify dynamic efficiency or dynamic inefficiency, respectively, of allocations along which the per capita capital stock converges.

**Proposition 2** *Let  $(k, c, i)$  be a feasible allocation and assume that  $k_\infty = \lim_{t \rightarrow +\infty} k(t)$  exists.*

- (a) *If  $k_\infty < \bar{k}$ , then it follows that  $(k, c, i)$  is dynamically efficient.*
- (b) *If  $k_\infty > \bar{k}$ , then it follows that  $(k, c, i)$  is dynamically inefficient.*
- (c) *If  $k_\infty = \bar{k}$  and if there exist positive numbers  $M$  and  $\mu$  such that  $|k(t) - \bar{k}| \leq Me^{-\mu t}$  holds, then it follows that  $(k, c, i)$  is dynamically efficient.*

PROOF: (a) If  $k_\infty < \bar{k}$ , then it follows that  $F'(k_\infty) > F'(\bar{k}) = \delta + n$ . Because of  $\lim_{s \rightarrow +\infty} r(s) = F'(k_\infty) - \delta$ , there exist  $\varepsilon > 0$  and  $s_\varepsilon > 0$  such that  $r(s) - n > \varepsilon > 0$  holds for all  $s > s_\varepsilon$ . Obviously, this rules out that the limit in (8) is finite and it follows therefore from Cass [4] that the allocation  $(k, c, i)$  is dynamically efficient.

(b) Analogously to case (a) one can see that there exist  $\varepsilon > 0$  and  $s_\varepsilon > 0$  such that  $r(s) - n < -\varepsilon < 0$  holds for all  $s > s_\varepsilon$ . This implies that there exists a constant  $S > 0$  such that

$$e^{\int_0^t [r(s) - n] ds} \leq S e^{-(\varepsilon/2)t}.$$



Obviously, this makes the limit in (8) finite and dynamic inefficiency of  $(k, c, i)$  follows from Cass [4].

(c) Exponentially fast convergence of  $k(t)$  to  $\bar{k}$  implies that  $r(s) - n$  converges exponentially fast to 0. This property, in turn, shows that  $\int_0^t [r(s) - n] ds$  does not diverge to  $-\infty$  as  $t$  approaches  $\infty$ . This implies that  $e^{\int_0^t [r(s) - n] ds}$  does not converge to 0 and it follows that (8) is violated. Referring again to Cass [4] it follows that  $(k, c, i)$  is dynamically efficient.  $\square$

For the hairline case  $k_\infty = \bar{k}$  we have stated a sufficient efficiency condition only, which covers all cases considered in this paper. This condition, which requires exponentially fast convergence of  $k(t)$  to the Golden Rule stock  $\bar{k}$ , is not necessary for dynamic efficiency. On the other hand, convergence of  $k(t)$  to  $\bar{k}$  alone is not sufficient for dynamic efficiency. For example, if  $k(t)$  approaches  $\bar{k}$  slowly enough from above, the allocation  $(k, c, i)$  can be dynamically inefficient.

Let us now turn to capital price bubbles. If such a bubble exists, we say that capital is overpriced; see proposition 1(c). To get started, first note that the price of capital in the one-sector economy described above is equal to  $p(t) = 1$  no matter what microfoundation we would impose in order to determine market prices. The fact that  $p(t)$  is equal to 1 is simply a consequence of the assumption that consumption and investment are just two different uses of the single output good that is produced in the economy. The dividend of capital is equal to the real interest rate, that is,  $d(t) = r(t) = F'(k(t)) - \delta$ . Noting that condition (1) is satisfied, it follows therefore from proposition 1(b) that capital is overpriced if and only if

$$\lim_{t \rightarrow +\infty} \int_0^t r(\tau) d\tau < +\infty \quad (9)$$

holds. Now let us define  $k_\delta$  as the unique solution of the equation  $F'(k_\delta) = \delta$  and note that  $k_\delta$  may be larger than the maximum sustainable capital stock  $K$ . Using (9) it is now straightforward to derive the following result, which characterizes the existence of capital price bubbles in those allocations along which the per capita capital stock is convergent.

**Proposition 3** *Let  $(k, c, i)$  be a feasible allocation and assume that  $k_\infty = \lim_{t \rightarrow +\infty} k(t)$  exists.*

- (a) *If  $k_\infty < k_\delta$ , then it follows that there is no capital price bubble.*
- (b) *If  $k_\infty > k_\delta$ , then it follows that there exists a capital price bubble.*
- (c) *If  $k_\infty = k_\delta$  and if there exist positive numbers  $M$  and  $\mu$  such that  $|k(t) - k_\delta| \leq M e^{-\mu t}$  holds, then it follows that there exists a capital price bubble.*

PROOF: (a) If  $k_\infty < k_\delta$ , then it follows that  $F'(k_\infty) > F'(k_\delta) = \delta$ . Because of  $\lim_{\tau \rightarrow +\infty} r(\tau) = F'(k_\infty) - \delta$ , there exist  $\varepsilon > 0$  and  $t > 0$  such that  $r(\tau) > \varepsilon$  holds for all  $\tau > t$ . Obviously, this contradicts (9) and we conclude that there is no capital price bubble.

(b) Analogously to the case (a), we see that there exist  $\varepsilon > 0$  and  $t > 0$  such that  $r(\tau) < -\varepsilon$  holds for all  $\tau > t$ . This proves that the limit in (9) is  $-\infty$  and it follows that capital is overpriced.

(c) Exponentially fast convergence of  $k(t)$  to  $k_\delta$  implies that the interest rate  $r(t)$  converges exponentially fast to  $F'(k_\delta) - \delta = 0$ . This property, in turn, shows that the limit in (9) is finite and it follows that capital is overpriced in this case as well.  $\square$

Comparing propositions 2 and 3 it becomes clear that dynamic inefficiency (capital overaccumulation) and the existence of capital price bubbles (overpricing of capital) are determined by different conditions. Whereas the dynamic inefficiency of a convergent allocation depends on whether the long-run per capita capital stock  $k_\infty$  exceeds the Golden Rule per capita capital stock  $\bar{k}$  or not, the existence of a capital price bubble depends on whether  $k_\infty$  exceeds  $k_\delta$  or not. This difference suggests that the existence of capital price bubbles is not tightly linked to the dynamic inefficiency of the allocation. Nevertheless, we have the following general result which, again, is only stated for allocations along which the limit  $\lim_{t \rightarrow +\infty} k(t)$  exists.

**Theorem 1** *Let  $(k, c, i)$  be a feasible allocation and assume that  $k_\infty = \lim_{t \rightarrow +\infty} k(t)$  exists.*

(a) *If  $n > 0$  holds, then it follows that the existence of a capital price bubble implies that the allocation is dynamically inefficient.*

(b) *If  $-\delta < n < 0$  holds, then it follows that dynamic inefficiency of the allocation implies the existence of a capital price bubble.*

PROOF: (a)  $n > 0$  implies  $\bar{k} < k_\delta$ . If an equilibrium involves a capital price bubble, then it follows from proposition 3 that  $k_\infty \geq k_\delta$  holds. Thus, we obtain  $k_\infty > \bar{k}$  which, according to proposition 2, implies that the allocation is dynamically inefficient.

(b)  $-\delta < n < 0$  implies  $\bar{k} > k_\delta$ . If an allocation is dynamically inefficient, then it follows from proposition 2 that  $k_\infty \geq \bar{k}$  holds. Consequently, it follows that  $k_\infty > k_\delta$  and proposition 3 shows that capital is overpriced.  $\square$

It cannot be emphasized enough that all the results stated in this section, including theorem 1, hold for any model that is consistent with the feasibility conditions (5)-(7) of the neoclassical one-sector economy. This includes models with infinitely-lived households such as the Ramsey-Cass-Koopmans model and models with finitely-lived households such as the Blanchard-Yaari model. It also includes settings that lack a microfoundation such as the Solow-Swan model. In order to analyze the relation between dynamic inefficiency of allocations and the existence of capital price bubbles further, we need to know the relative positions of  $k_\infty$ ,  $\bar{k}$ , and  $k_\delta$ . This

requires to be more specific about how  $k_\infty$  is determined, i.e., it requires a description of the mechanisms that determine how output is allocated between consumption and investment. We illustrate various possibilities in the following section.

## 4 Applications

The allocation of output between its two uses consumption and investment can be determined in different ways. In the present section we discuss three popular examples. Since these examples are well-know models from the literature, we skip many details and focus on those aspects that are relevant for the purpose of our study.

### 4.1 The Ramsey-Cass-Koopmans model

The first example is the neoclassical one-sector growth model with infinitely-lived households going back to the contributions by Ramsey [11], Cass [3], and Koopmans [9].<sup>2</sup> In this model the population of the economy is formed by a large but finite number  $H$  of identical and infinitely-lived dynasties. Initially at time  $t = 0$ , every dynasty consists of a continuum of measure  $1/H$  of individuals and is endowed with  $\kappa/H$  units of physical capital. At time  $t \in \mathbb{R}_+$ , the representative dynasty consists of  $e^{nt}/H$  individuals, each of which is endowed with one unit of labor per time-unit. If an individual consumes  $c(t)$  units of output at time  $t \in \mathbb{R}_+$  it derives instantaneous utility  $U(c(t))$ , where  $U : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$  is a strictly increasing and strictly concave utility function, which is twice continuously differentiable on the interior of its domain and which satisfies  $\lim_{c \rightarrow 0} U'(c) = +\infty$ . The decision makers are the dynasties (not the individuals) and they seek to maximize the aggregate lifetime utility of their members subject to a lifetime budget constraint. Denoting by  $\rho$  the common pure time-preference rate of all dynasties, the objective function of the representative dynasty is

$$\frac{1}{H} \int_0^{+\infty} e^{-(\rho-n)t} U(c(t)) dt,$$

where the appearance of the multiplicative factor  $e^{nt}$  in the integrand reflects the assumption that the dynasty aggregates individual instantaneous utilities in a utilitarian way. The lifetime budget constraint of the dynasty is

$$\frac{1}{H} \int_0^{+\infty} D(0, t) e^{nt} c(t) dt \leq \frac{\kappa}{H} + \frac{1}{H} \int_0^{+\infty} D(0, t) e^{nt} w(t) dt,$$

where the discount factor  $D$  has been defined in equation (3). The budget constraint says that the present value of lifetime consumption expenditures of the dynasty (left-hand side)

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<sup>2</sup>The presentation here follows Romer [12, chapter 2A]; see also Acemoglu [1, chapter 8].

must not exceed its initial capital endowment plus the present value of its lifetime wage income (right-hand side).

Throughout this subsection we assume (in addition to the parametric assumptions introduced in the previous section) that the time-preference rate  $\rho$  exceeds the population growth rate  $n$ . Note that this does not rule out a negative time-preference rate; indeed it could be that  $-\delta < n < \rho < 0$  holds. The assumption  $\rho > n$  is sufficient to ensure finiteness of all three integrals in the above two displayed formulas. Moreover, it is known that under the assumption  $\rho > n$  there exists a unique equilibrium of the economy, which has the property that the per capita capital stock converges exponentially fast to  $k^*$ , where  $k^*$  is the unique value satisfying  $F'(k^*) = \delta + \rho$ .<sup>3</sup> The assumption  $\rho > n$  implies furthermore that  $k_\infty = k^* < \bar{k}$  is satisfied and we conclude from proposition 2 that the equilibrium is dynamically efficient. If  $\rho > 0$  holds, then it also follows that  $k_\infty < k_\delta$  is true and proposition 3 implies that there is no capital price bubble. However, if  $n < \rho \leq 0$  is satisfied, then we obtain  $k_\delta \leq k_\infty < \bar{k}$ . In this situation it follows from proposition 3 that there exists a capital price bubble. We summarize these findings in the following theorem.

**Theorem 2** *Consider the Ramsey-Cass-Koopmans model under the parameter assumptions  $\delta > 0$ ,  $\delta + n > 0$ , and  $\rho > n$ .*

- (a) *There exists a unique equilibrium of the economy. This equilibrium is dynamically efficient and it holds in equilibrium that  $\lim_{t \rightarrow +\infty} k(t) = k^*$ .*
- (b) *The unique equilibrium of the economy involves a capital price bubble if and only if  $-\delta < n < \rho \leq 0$ .*

Part (a) of the theorem is of course well known, but part (b) may come as a surprise. In order to understand the result, let us therefore consider the case where  $n < \rho < 0$  holds.<sup>4</sup> If  $\rho$  is negative, the real interest rate  $r(t)$  must be negative for all sufficiently large  $t$ . Since the price of capital is equal to  $p(t) = 1$  for all  $t$ , we see from proposition 1(a) that the fundamental value of capital is  $f(t) = -\infty$ . Hence, there exists a huge capital price bubble. Nevertheless, the dynasties are willing to hold on to their capital stock because, due to the negative time-preference rate  $\rho$ , they have a strong desire to transfer wealth from the present into the future, and holding physical capital is the only way to achieve that. Contrary to the situation discussed by Kocherlakota [8], where the agents would benefit from selling their assets but the existence of wealth or short sales constraints prevents them from doing that, here the agents do not even

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<sup>3</sup>The value  $k^*$  is often referred to as the modified Golden Rule stock.

<sup>4</sup>The borderline case  $\rho = 0$ , which is covered by the theorem, will not be separately discussed here.

want to sell their assets. The existence of a capital price bubble does not represent an arbitrage opportunity.

It is often claimed that the existence of asset price bubbles in models with infinitely-lived households is ruled out by the transversality condition

$$\lim_{t \rightarrow +\infty} D(0, t)e^{nt}k(t) = 0.$$

It is therefore worth mentioning that this condition holds even in the situation described in part (b) of theorem 2. Indeed, convergence of  $k(t)$  towards  $k^*$  implies that the interest rate  $r(t)$  converges to  $F'(k^*) - \delta = \rho$  which, together with  $\rho > n$ , implies that the transversality condition stated above holds true.

## 4.2 The Blanchard-Yaari model

This model was developed by Blanchard [2] building on Yaari [20].<sup>5</sup> The key difference to the Ramsey-Cass-Koopmans model is that households have finite but random lifetimes. The initial population of the economy consist of a unit interval of identical households and has the size  $N(0) = 1$ . Denoting the population size at time  $t \in \mathbb{R}_+$  by  $N(t)$ , it is assumed that the measure of newborn households in any time interval  $[t, t + \Delta)$  is given by  $bN(t)\Delta + o(\Delta)$ , where  $b > 0$  is referred to as the crude birth rate. The lifetime of every household is an exponentially distributed random variable with expected value  $1/m$ , where  $m > 0$  is the mortality rate. Applying an exact law of large numbers, it follows that the measure of dying households in the time interval  $[t, t + \Delta)$  is  $mN(t)\Delta + o(\Delta)$ . The population growth rate is therefore given by  $n = b - m$  and we have  $N(t) = e^{nt}$  as before.

Since households have random lifetimes, there is a demand for life annuities. These insurance contracts are offered by a unit interval of identical insurance companies which can produce them at no cost. The contracts are actuarially fair, which implies that the insurance companies pay a real rate of return equal to  $r(t) + m$  at time  $t \in \mathbb{R}_+$  during the lifetime of the household and seize the entire financial wealth of the household upon its death. Households hold their wealth solely in the form of annuities. The insurance companies, on the other hand, own the physical capital stock and rent it out to the firms at the real interest rate  $r(t)$  at time  $t \in \mathbb{R}_+$ .

The initial population of households is endowed with financial wealth  $\kappa$ , which consists of annuities. New households are born without any wealth but offer one unit of labor per time period to the firms. The flow budget constraint of a household born at time  $\tau \in \mathbb{R}_+$  is given by

$$\dot{a}_\tau(t) = [r(t) + m]a_\tau(t) + w(t) - c_\tau(t)$$

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<sup>5</sup>A very readable and detailed presentation is Groth [7, chapter 12]; see also Acemoglu [1, chapter 9.8].

for all  $t \geq \tau$  with initial condition  $a_\tau(\tau) = 0$ . Here,  $a_\tau(t)$  and  $c_\tau(t)$  denote wealth and consumption at time  $t \in \mathbb{R}_+$  of a household which was born at time  $\tau \in [0, t]$ . The household seeks to maximize its expected lifetime utility, which is given by<sup>6</sup>

$$\int_{\tau}^{+\infty} e^{-(\rho+m)(t-\tau)} \ln(c_\tau(t)) dt,$$

subject to the flow budget constraint stated above and the no-Ponzi game condition

$$\lim_{t \rightarrow +\infty} a_\tau(t) e^{-\int_{\tau}^t [r(s)+m] ds} \geq 0.$$

As in the Ramsey-Cass-Koopmans model, one needs to impose a parameter restriction in order to ensure that the expected lifetime utility and the present value of the labor endowment remain finite. This restriction is  $\rho + m > 0$ . Furthermore, in order for the modified Golden Rule capital stock  $k^*$  to be well defined, we need to assume that  $\rho + \delta > 0$  holds.

The model is closed by requiring that aggregate household wealth at time  $t \in \mathbb{R}_+$ , which is given by  $e^{-mt} a_0(t) + \int_0^t b e^{n\tau} e^{-m(t-\tau)} a_\tau(t) d\tau$ , coincides with the aggregate capital stock  $k(t) e^{nt}$ . Due to the assumed demographic structure, aggregation over all cohorts is tractable, and the equilibrium conditions can be formulated in the form of the following system of two differential equations in the per capita variables  $k(t)$  and  $c(t)$ :

$$\begin{aligned} \dot{k}(t) &= F(k(t)) - (\delta + b - m)k(t) - c(t), \\ \dot{c}(t) &= [F'(k(t)) - (\delta + \rho)]c(t) - b(\rho + m)k(t). \end{aligned}$$

Phase diagram analysis shows that there exists a unique equilibrium and that the per capita capital stock converges exponentially fast to  $k_\infty$ , where the pair  $(k_\infty, c_\infty)$  is the unique positive fixed point of the above system of differential equations. It is moreover easily seen that  $k_\infty < k^*$  must hold. This allows us to derive the following result.

**Theorem 3** *Consider the Blanchard-Yaari model under the parameter assumptions  $b > 0$ ,  $m > 0$ ,  $n = b - m$ ,  $\delta > 0$ ,  $\delta + n > 0$ ,  $\rho + m > 0$ , and  $\rho + \delta > 0$ .*

- (a) *There exists a unique equilibrium of the economy. In this equilibrium it holds that  $\lim_{t \rightarrow +\infty} k(t) = k_\infty < k^*$ .*
- (b) *If  $\rho \geq b - m$  holds, then it follows that the equilibrium is dynamically efficient.*
- (c) *If  $\rho \geq 0$  holds, then there does not exist a capital price bubble.*

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<sup>6</sup>The logarithmic instantaneous utility function is assumed for simplicity. The multiplicative factor  $e^{-m(t-\tau)}$  in the integral is the probability that a household born at time  $\tau$  is still alive at time  $t$ .

(d) Depending on the parameter constellation, the unique equilibrium can have each of the following properties: (i) it is dynamically efficient and there is no capital price bubble, (ii) it is dynamically efficient and there is a capital price bubble, (iii) it is dynamically inefficient and there is no capital price bubble, and (iv) it is dynamically inefficient and there is a capital price bubble.

PROOF: (a) This claim is well known; see the references stated in footnote 5.

(b) The assumption  $\rho \geq b - m$  is equivalent to  $k^* \leq \bar{k}$ . Together with  $k_\infty < k^*$  we obtain  $k_\infty < \bar{k}$  and the claim follows from proposition 2.

(c) The assumption  $\rho \geq 0$  is equivalent to  $k^* \leq k_\delta$ . Together with  $k_\infty < k^*$  we obtain  $k_\infty < k_\delta$  and the claim follows from proposition 3.

(d) We prove this claim by considering the special case where the production function is of the Cobb-Douglas form  $F(k) = k^\alpha$  with  $\alpha \in (0, 1)$ . In this situation it is straightforward to verify that

$$\bar{k} = \left( \frac{\alpha}{\delta + b - m} \right)^{1/(1-\alpha)}, \quad k_\delta = \left( \frac{\alpha}{\delta} \right)^{1/(1-\alpha)}, \quad k^* = \left( \frac{\alpha}{\delta + \rho} \right)^{1/(1-\alpha)}.$$

Moreover,  $k_\infty$  and  $c_\infty$  must satisfy the equations

$$\begin{aligned} c_\infty &= k_\infty^\alpha - (\delta + b - m)k_\infty, \\ [\alpha k_\infty^{\alpha-1} - (\delta + \rho)]c_\infty &= b(\rho + m)k_\infty. \end{aligned}$$

Substituting the former into the latter, dividing by  $k_\infty$ , and defining  $z = k_\infty^{\alpha-1}$  one obtains the quadratic equation

$$[\alpha z - (\delta + \rho)][z - (\delta + b - m)] = b(\rho + m).$$

Note that the requirement  $k_\infty < k^*$  translates into  $z > (\delta + \rho)/\alpha$ . The above quadratic equation has two real roots, but only the larger one satisfies the condition  $z > (\delta + \rho)/\alpha$ . Hence, it holds that  $k_\infty = z^{-1/(1-\alpha)}$ , where

$$z = \frac{\delta + \rho + \alpha(\delta + b - m) + \sqrt{[\delta + \rho + \alpha(\delta + b - m)]^2 + 4\alpha(m - \delta)(\delta + b + \rho)}}{2\alpha}.$$

According to proposition 2, the equilibrium is dynamically inefficient if and only if  $k_\infty > \bar{k}$  holds, which is equivalent to  $\alpha z < \delta + b - m$ . Using the above expression for  $z$  this inequality can be rewritten as

$$\sqrt{[\delta + \rho + \alpha(\delta + b - m)]^2 + 4\alpha(m - \delta)(\delta + b + \rho)} < (1 - \alpha)(\delta + b - m) + b - m - \rho.$$

We know from part (b) of the theorem that this inequality cannot be true if  $\rho \geq b - m$  holds. If  $\rho < b - m$ , on the other hand, then the right-hand side of the inequality is non-negative and

we can take squares on both sides. This yields after some rearrangement

$$\rho < \frac{(1 - \alpha)[b^2 + (\delta - m)(b - m)] - bm}{\alpha b + (1 - \alpha)(\delta + b - m)}. \quad (10)$$

The right-hand side of (10) is always smaller than  $b - m$  so that we conclude that the unique equilibrium is dynamically inefficient if and only if condition (10) is satisfied.

According to proposition 3, the equilibrium involves a capital price bubble if and only if the condition  $k_\infty \geq k_\delta$  holds, which is equivalent to  $\alpha z \leq \delta$ . Substituting the value of  $z$  from above, this inequality can be rewritten as

$$\sqrt{[\delta + \rho + \alpha(\delta + b - m)]^2 + 4\alpha(m - \delta)(\delta + b + \rho)} \leq (1 - \alpha)\delta + \alpha m - \alpha b - \rho.$$

If  $\rho > (1 - \alpha)\delta + \alpha m - \alpha b$  holds, this inequality is obviously false. Otherwise, its right-hand side is non-negative and we can take squares on both sides. This yields after rearrangement  $\rho \leq -\alpha b m / [(1 - \alpha)\delta + \alpha m]$ . To summarize, we have proved that the equilibrium involves a capital price bubble if and only if

$$\rho \leq \min \left\{ (1 - \alpha)\delta + \alpha m - \alpha b, -\frac{\alpha b m}{(1 - \alpha)\delta + \alpha m} \right\}. \quad (11)$$

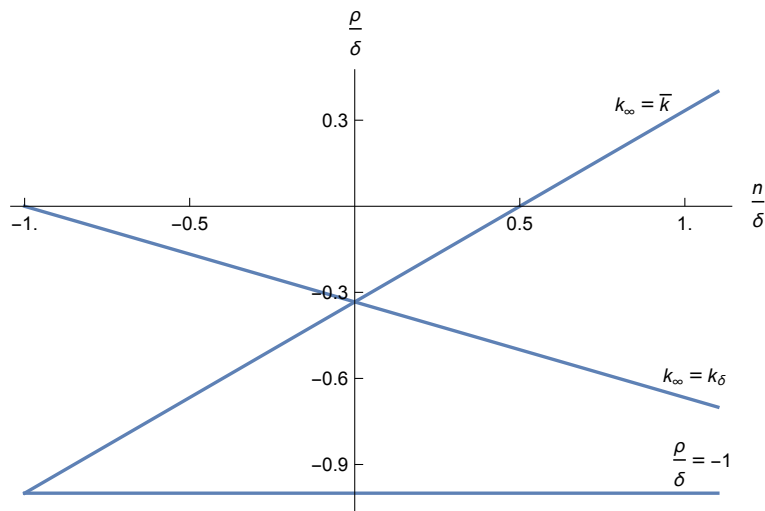


Figure 1: The  $(n/\delta, \rho/\delta)$ -parameter space for the example from the proof of theorem 3(d).

To verify that all four scenarios mentioned in part (d) of the theorem are possible, we set  $\alpha = 1/3$  and  $m/\delta = 1$ . The general assumptions of the theorem then require that  $n/\delta = (b - m)/\delta > -1$  and  $\rho/\delta > -1$ . The set of all pairs  $(n/\delta, \rho/\delta)$  satisfying these two restrictions is shown in figure 1. The figure also shows the curves along which the inequalities (10) and (11) hold with equality. The curve labeled  $k_\infty = \bar{k}$  is the locus of all points where (10) holds with equality. Parameter constellations above this curve correspond to dynamically efficient equilibria and those below



the curve to dynamically inefficient ones. The curve labeled  $k_\infty = k_\delta$  is the locus of points where (11) holds as an equality. Above the curve there do not exist capital price bubbles, below the curve capital is overpriced. We see that the two curves partition the feasible parameter space into four non-empty regions that correspond to the four cases mentioned in part (d) of the theorem. This concludes the proof of the theorem.  $\square$

The most interesting statement in theorem 3 is part (d). First of all, it says that capital price bubbles can occur in this model. From part (c) we can see that - as in the Ramsey-Cass-Koopmans model - this requires that the pure rate of time-preference  $\rho$  is negative. Contrary to the Ramsey-Cass-Koopmans model, however, capital price bubbles do not require a negative population growth rate. Indeed, in the example used in the proof of the theorem, there exist capital price bubbles for all parameter combinations below the line  $k_\infty = k_\delta$  and above the line  $\rho/\delta = -1$  in figure 1. Apparently, this includes economies with a positive population growth rate. Part (d) of the theorem illustrates also another important point, namely that there is no tight relation between the existence of capital price bubbles and the dynamic inefficiency of the equilibrium. Indeed, figure 1 shows that nothing more can be said about this relation than what was already stated in theorem 1: if  $n > 0$ , then the existence of a capital price bubble implies the dynamic inefficiency of the equilibrium, and if  $n < 0$ , then dynamic inefficiency of the equilibrium implies the existence of a capital price bubble.

The above observations do not contradict the results derived by Tirole [18]. Note that in the model used in [18], there exist two assets: physical capital and an unproductive asset. Furthermore, that model admits multiple equilibria. The main result of [18] states that the existence of an equilibrium featuring a bubble in the price of the unproductive asset is conditioned on the dynamic inefficiency of the unique bubbleless equilibrium. In the present model there exists only a single asset (physical capital) and, under the general assumption stated in theorem 3, there exists a unique equilibrium. What theorem 3 then demonstrates is that this unique equilibrium can be dynamically efficient or not, that it can contain a capital price bubble or not, and that all four combinations of these two properties are possible.

Finally, let us mention that there is (as in the Ramsey-Cass-Koopmans model) no contradiction between the existence of capital price bubbles and the transversality condition

$$\lim_{t \rightarrow +\infty} a_\tau(t) e^{-\int_\tau^t [r(s)+m] ds} = 0. \quad (12)$$

Indeed, as  $k(t)$  approaches  $k_\infty$ , it follows that  $r(t)$  approaches  $F'(k_\infty) - \delta$ . Consequently, the exponential term in the transversality condition grows asymptotically at the rate  $-[F'(k_\infty) - \delta + m] < -[F'(k_\infty) - (\delta + \rho)]$ , where the last inequality follows from the assumption  $\rho + m > 0$ . It is also possible to show that  $a_\tau(t)$  grows asymptotically at the rate  $F'(k_\infty) - (\delta + \rho)$ .<sup>7</sup> Combining

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<sup>7</sup>See Groth [7, chapter 12] for a formal proof.

these observations, it is clear that the transversality (12) holds.

### 4.3 The Solow-Swan model

Finally, let us consider the Solow-Swan model, which was derived by Solow [15] and Swan [16].<sup>8</sup> This model is not microfounded. Instead, the allocation of output between consumption and investment is determined by the behavioral rule  $i(t) = sF(k(t))$ , where  $s \in (0, 1)$  is an exogenously given saving rate. It is well known that there exists a unique equilibrium and that the per capita capital stock  $k(t)$  converges exponentially fast to  $k_\infty$ , where  $k_\infty$  is the unique positive solution of the equation  $sF(k_\infty) = (\delta + n)k_\infty$ . Defining  $\bar{s} = (\delta + n)\bar{k}/F(\bar{k})$  and  $s_\delta = (\delta + n)k_\delta/F(k_\delta)$ , it follows that  $k_\infty < \bar{k}$  holds if and only if  $s < \bar{s}$  and that  $k_\infty < k_\delta$  is equivalent to  $s < s_\delta$  (and analogously for the weak inequalities). These observations allow for an easy characterization of the various possibilities that can occur. Instead of presenting this characterization in the form of a theorem, we simply provide an illustrative example.

**Example 1** Suppose that the technology is described by a Cobb-Douglas production function  $F(k) = k^\alpha$  with  $\alpha \in (0, 1)$ . In this case it is straightforward to verify that  $\bar{s} = \alpha$  and  $s_\delta = \alpha(\delta + n)/\delta$ . The parameter region for which the equilibrium of the Solow-Swan model involves a capital price bubble is therefore given by  $s \geq s_\delta = \alpha(\delta + n)/\delta$ , and the parameter region in which the equilibrium is dynamically efficient is given by  $s \leq \bar{s} = \alpha$ . Fixing the value of  $\alpha$ , these regions are illustrated in the  $(n/\delta, s)$ -parameter space in figure 2.

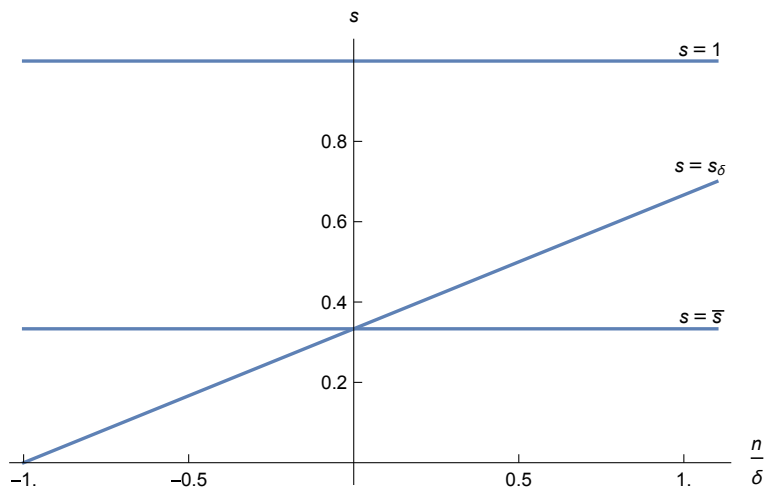


Figure 2: The  $(n/\delta, s)$ -parameter space in example 1 for fixed  $\alpha$ .

The upward sloping line defined by  $s = \alpha[1 + (n/\delta)]$  is labelled as  $s = s_\delta$  and separates the parameter region with capital price bubbles from the one where there is no overpricing.

<sup>8</sup>For modern presentations we refer to Romer [12, chapter 1] or Acemoglu [1, chapter 2].

The horizontal line  $s = \alpha$  is labelled  $s = \bar{s}$  and separates the parameter region in which the equilibrium is dynamically efficient from the area corresponding to a dynamically inefficient equilibrium. It is clearly seen that, whenever  $n > 0$ , the absence of a capital price bubble does neither imply dynamic efficiency of the equilibrium nor its dynamic inefficiency. In the case  $n < 0$ , on the other hand, dynamic efficiency of the equilibrium allocation does neither imply the absence of a capital price bubble nor its presence. Similarly to figure 1, the result shown in figure 2 serves also as a very good illustration of theorem 1.  $\square$

## 5 Concluding remarks

We have shown that capital can be overpriced (i.e., it can contain a positive bubble component) in standard neoclassical growth models such as the Ramsey-Cass-Koopmans model, the Blanchard-Yaari model, or the Solow-Swan model. The existence of capital price bubbles in the Ramsey-Cass-Koopmans model contradicts the results in Tirole [17] but it corroborates and complements those in Kocherlakota [8]. Moreover, we have reconsidered the relation between dynamic inefficiency of an equilibrium and the existence of rational bubbles. If one considers neoclassical growth models in which physical capital is the only asset, then it turns out that this relation is not as simple as one would expect on the basis of Tirole [18]. More specifically, it depends crucially on the population growth rate whether dynamic inefficiency is necessary for the existence of bubbles or vice versa.

None of the results presented in this paper depends on the continuous-time formulation that we have chosen. Modeling time as a continuous variable was simply a matter of convenience. A result like proposition 1(a), for example, is very easy to prove in continuous time, whereas it takes a bit more work to derive it in a discrete-time model. Also the treatments of the Ramsey-Cass-Koopmans model and the Solow-Swan model seem to be somewhat simpler in a continuous time setting than in discrete time. As for the overlapping generations model, it is of course the case that the popular discrete-time version of Diamond [5] is simpler to analyze than the continuous-time version of Blanchard [2]. On the other hand, the Blanchard-Yaari model used in this paper has the advantage that it does not require a choice of the time unit that it is unsuitable for quantitative exercises. Furthermore, the consistency of asset price bubbles with the transversality condition is more apparent in the continuous-time Blanchard-Yaari model than in the discrete-time Diamond model; see equation (12).

The paper may be criticized because it shows the existence of capital price bubbles in micro-founded one-sector growth models only if the time-preference rate is non-positive. Our primary line of defence against this criticism is that the main points of the paper are of a theoretical

nature. We wanted to prove that the existence of a bubble does not necessarily represent an arbitrage opportunity for an infinitely-lived agent and that those conditions that are sufficient for existence and uniqueness of equilibrium do not rule out the occurrence of capital price bubbles. Moreover, we wanted to highlight that the causes of dynamic inefficiency are not equivalent to those of asset price bubbles. From a purely empirical point of view, we agree of course that the negativity of the pure rate of time-preference is hard to justify, although it is not completely unconceivable that in a model with finite lifetimes more weight is given to the utility derived in old-age than to the utility derived during the early stages of life. As for the population growth rate  $n$ , which needs to be negative for the occurrence of bubbles in the Ramsey-Cass-Koopmans model, one should not discard the case of negative  $n$  as unrealistic. Indeed, according to the United Nations [19], the populations of more than 50 countries or regions of the world are expected to shrink from now until the year 2050. Correspondingly, the academic literature on economic growth is already responding to this development; see Ferrara [6] or Sasaki and Hoshida [14].

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