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On the Clock of the Combinatorial Clock Auction

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Abstract

The Combinatorial Clock Auction (CCA) has been frequently used in recent spectrum auctions. It combines a dynamic clock phase with subsequent VCG pricing in order to maximize price discovery and efficiency. We inquire into the role of the clock when bidders have lexicographic preferences for raising rivals' costs. All equilibria of the CCA are inefficient if there is substantial room for price discovery, that is, if there is large uncertainty concerning the competitor's type. Conversely, in all efficient equilibria price discovery is limited. Qualitative features of our equilibria are in line with evidence concerning bidding behavior in some recent CCAs.

JEL Classification: D440, D470, L960
Key Words: Combinatorial auction, Spectrum auction, Spiteful bidding, Raising rival's cost

1 Introduction

In recent years, the Combinatorial Clock Auction (CCA) has been used in many countries around the world to allocate telecommunication spectrum. The CCA is basically a dynamic version of the Vickrey-Clarke-Groves (VCG) mechanism and consists of two integrated phases.[1] Throughout the auction bidders bid on packages, thereby solving the exposure problem associated with the simultaneous ascending auction (SAA). The first phase of the CCA is a clock phase where bidders express their demand on packages at given prices in every round. In the second, supplementary, phase bidders can bid on as many additional packages as they like and may raise bids on packages they have bid on in the clock phase, subject to some constraints that are derived from the behavior in the clock phase. At the end of the supplementary phase, goods are allocated and prices are determined according to the (second-price) VCG rules (see, e.g. Milgrom, 2004). Thus, the CCA is a kind of VCG mechanism preceded by a clock phase. The main rationale for this complex auction format that can be found in the literature is that the clock phase facilitates price and package discovery (see, e.g. Ausubel, Cramton and Milgrom, 2006),[2] while the VCG pricing rule is chosen to foster an efficient allocation and to provide bidders with incentives to bid truthfully (Cramton, 2013).

In this paper we want to better understand how the clock phase changes the incentives of bidders to bid strategically. Thus, we provide a full equilibrium analysis of the bidding behavior in the two stages of the CCA and compare the equilibria to the equilibria of the VCG mechanism. We do so in a simple, elegant set-up that was introduced by Levin and Skrzypacz (forthcoming) where two bidders with private information bid how to divide a perfectly divisible object.

To reduce the number of equilibria, we assume that *ceteris paribus* bidders value outcomes where the competitor pay more. That is, bidders have

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[1]In practice there is a third phase - the assignment phase. In this phase generic packages are allocated. We abstract away from this phase since it does not effect our analysis.

[2]In large combinatorial auctions, computing the values of packages can be costly. A dynamic auction can lead bidders in focusing on relevant packages and might help forecasting final prices.

(weak) spiteful preferences to raise rival's costs, where the spite motive is modeled in a lexicographic manner implying that if two strategies yield the same expected surplus to a bidder, the bidder chooses the strategy that increases the price of the other bidder. Thus, the analysis with lexicographic preferences can be considered a robustness check on the equilibria under standard preferences: equilibria under our preferences are also equilibria under standard preferences, but the reverse does not necessarily hold true. The raising rivals' cost motive is an important concern of bidders in real-world auctions. Janssen and Karamychev (2014) argue that the raising rivals' cost motive may stem from (i) principal-agent issues within a firm (bidder)[3] or from (ii) the fact that (in spectrum auctions) bidders face weaker competitors in the market after an auction if competitors have paid more for their licenses. If firm A makes B pay more for spectrum, B's credit rating will fall and its cost of capital will go up, weakening its strategic position. Milgrom (2004) and Cramton and Ockenfels (2014) mention fairness as a reason for why bidders may want to raise rivals' cost.[4]

The raising rivals' cost motive is also of concern to regulators designing auctions.[5] After the 2013 auction the Austrian regulator RTR attributed the high revenue to overly aggressive behavior by bidders: during the clock phase, bidders were bidding very offensively and the majority of the supplementary bids were on very large packages that had a low probability of winning but played a crucial role in determining other bidders' prices. The fact that payments in the Austrian auction were essentially the same as the final clock prices, is a clear signal of aggressive bidding: with Vickrey pricing and "downward sloping demand", one would not expect marginal and average prices to be identical. Moreover, regulators usually publicly discuss

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[3]In spectrum auctions, given the large amount of uncertainty concerning future technological developments and uptake of data services, it is difficult for bidders to evaluate what the spectrum is really worth. Valuations are highly subjective. Accordingly, if a bidder wants to have a more objective evaluation measure of its bidding team's performance, it is better to evaluate performance relative to other bidders, than it is to evaluate relative to the uncertain and subjective own valuation.

[4]In a part of their paper, Levin and Skrzypacz (forthcoming) also consider bidders that have a lexicographic preference to raise rivals' cost.

[5]See, e.g., (i) BAKOM (2012) on the outcome of the Swiss auction and subsequent discussion on why Sunrise paid much more for comparable spectrum than other bidders, (ii) Ofcom (2012, page 122, point 7.9.) in response to an earlier consultation on the UK LTE auction in 2013.

with stakeholders different features of a particular design before the start of the auction. In a consultation document on the award of the 2.3 GHz and 3.4 GHz bands, the British regulator Ofcom (2014, p. 38, 6.73-6.77) explicitly mentions the possibility of price driving by placing "risk-free bids" in the supplementary phase as a problematic aspect of the CCA. Some of the potential bidders' responses share this concern.[6] Although, none of these arguments for raising rivals' cost implies that bidders should have a lexicographic preference for doing so, lexicographic preferences are a useful modeling approach to inquire into the effects of this motive.

Our main result is that under raising rivals' cost preferences if the uncertainty concerning the competitor's type is sufficiently large, all equilibria of the CCA are inefficient, while the VCG mechanism always has efficient equilibria. In this sense, the clock phase may cause the allocation mechanism to loose its efficiency properties. The source of inefficiency can be understood along the following lines. With a potentially large difference between the valuation of bidders, there does not exist a final clock price that is such that, first, weak bidders would like to stay active in the clock phase, and, second, the truthful demand of strong bidders is smaller than half of the available perfectly divisible object. In this case, if the clock phase continues, a weak bidder is able to infer that their competitor is a relatively strong bidder. This learning, or price discovery, creates the opportunity for weak bidders to make their supplementary round behavior conditional on the price at which the clock phase stops. Knowing the competitor is strong, they can raise the rival's cost more (without running the risk of winning more spectrum than they would like to win) than when they do not know. In order to prevent weak bidders to raise their cost, strong bidders find it optimal to reduce their demand towards the end of the clock phase in such a way that the CCA rules prevent strong bidders to express their true valuation for all possible allocations. This creates the possibility of an inefficient final allocation.

We also provide some other interesting characterization results. First, all symmetric equilibria have bidders bidding for the full spectrum in the beginning of the clock phase, with sudden drops in demand at later moments. At lower prices, where no bidder wants to drop demand and leave

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[6]See, e.g. the response of BT (2015).

the auction, bidders find it optimal to bid for the full amount to be allocated in order to be able to maximally raise rivals' cost in the supplementary round.[7] This is in line with, for example, the Austrian 2013 auction where (as mentioned above) bidders were bidding very aggressively in the clock phase. An extreme form of such equilibrium behavior is when all bidders bid for the full spectrum until the last clock round price and then drop demand to at most half of the available spectrum. In this case, there is no price discovery whatsoever during the clock phase.

Second, in all (efficient) equilibria the clock phase stops with excess supply with positive probability.[8] This result is also in line with some real world auctions.[9] Many observers of the CCA have argued that excess supply at the end of the clock phase severely limits the possibility of bidders to raise rivals' cost.[10] We show that despite the existence of excess supply, bidders are still able to raise rival's cost considerably. By bidding for the full spectrum until the last clock round, bidders are able to express their true marginal values in the supplementary phase on all allocation shares they potentially could win. As winning the full spectrum is not feasible if both bidders are active in the final clock round, bidders can maximally raise their bid on the full spectrum to raise rivals' cost.

Third, there are equilibria where two almost identical bidders pay sig-

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[7]Thus, truthful bidding cannot be sustained as equilibrium behavior.

[8]This is in stark contrast with the fact that the clock would always end with market clearing under truthful bidding in our framework.

[9]The Austrian mobile network operator Telekom Austria (2013) indicates in a press release after the auction that the clock phase ended with excess supply in key spectrum bands.

[10]See, e.g., Levin and Skrzypacz (forthcoming, remark 2 on page 19) where they observe that "If we allowed bidder 2 to create excess supply at the end of the clock phase, she could increase bidder 1 payment even more. ... Such extreme predatory behavior is even more difficult to execute and even more risky for player 2 than what we describe. Moreover, analyzing equilibria in this case is difficult, so we maintain the assumption that player 2 is not allowed to create excess supply in the clock phase." Similarly, Kroemer et al. (2015, p.6) observe that "In recent spectrum auction implementations, the regulator decided not to reveal excess supply in the last round, in order to make spiteful bidding risky. It depends on the market specifics, if this risk is high enough to eliminate spiteful bidding". In a recent consultation document on annual license fees, the UK regulator Ofcom (2015, A8.48 on page 16) also writes in a similar vein when they consider the Austrian 2013 CCA outcome: "We also noted that at the end of the clock rounds there was an excess supply of 2x10 MHz in each of the 900 MHz and 1800 MHz bands. ... This further suggested a possible reason why bidders may have considered price driving in the supplementary bids to be a risky strategy, ..."

nificantly different amounts for an almost identical share (half) of the spectrum. The reason for this is as follows. In this type of equilibrium, there is a critical type of bidder that is such that all bidders with values lower than this type drop demand at a certain clock price in such a way that for many type combinations of bidders the clock will stop. High valuation bidders above this critical type continue to bid for the full spectrum until a much larger clock price and drop demand discontinuously at this higher price. If the two bidders in the auction happen to have valuations close to either side of this critical type, then the efficient allocation is implemented because of the bidding behavior in the supplementary round, but the (marginally) higher valuation bidder has a much higher ability to raise rivals' cost.

Finally, despite the introduction of a lexicographic preference for raising rival's cost, to resolve potential indeterminacies in bidder's optimal strategies, there is a continuum of equilibria in the CCA, like in the VCG. Equilibrium revenue in the CCA is never larger than in a "corresponding" equilibrium of the VCG, but sometimes strictly smaller as the equilibrium bidding behavior in the clock phase may introduce restrictions that severely limit the ability of bidders to raise rivals' cost.

As discussed above, the model we use is essentially that of Levin and Skrzypacz (forthcoming). Levin and Skrzypacz (forthcoming) present a sequence of three closely related and stylized models of the CCA. There are two bidders with a quadratic utility function over a divisible object. In all of these models the clock is required to end with market clearing and at least one out of two bidders restricts himself to linear proxy strategies. Levin and Skrzypacz (forthcoming) are mostly interested in how different (expected) bidding behaviors in the supplementary round may affect bidding behavior in the clock. They do not consider, however, whether any bidder wants to use a continuous clock demand function. That is, they do not provide a full equilibrium analysis of their model. In contrast to Levin and Skrzypacz (forthcoming), we do provide an equilibrium analysis and the constraints on bidders' bidding behavior in the supplementary round are endogenously determined by their bidding behavior in the clock phase (as in real CCAs). Bidders are not restricted to linear proxy strategies and we do not insist in the clock phase ending with market clearing.

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In practice, CCAs have different regimes concerning the information that is released to the bidders in the clock phase. In one regime, bidders are only informed about the fact that there is still excess demand and that the clock phase continues. In another regime, bidders are informed about total demand at every clock price. The first regime was used in the first part of the Austrian auction and seems to be favored in case there is some suspicion that collusion between bidders may be something to worry about.[11] This is also the information regime we focus on in this paper. As indicated above, even though no direct information is revealed to bidders, bidders may infer information about their competitors' type as the length of the clock phase may differ depending on whether or not the competitor is a strong bidder.

Auctions where bidders do not know competitors' demand are easier to analyze as bidders cannot condition their demand on what rivals have demanded in previous clock rounds and can only condition their behavior on the prices they observe. In the consultation document on the award of the 2.3 GHz and 3.4 GHz bands, Ofcom (2014) proposes to use either the CCA or the SAA without demand disclosure. In a reaction for Hutchinson 3G, Power Auctions LLC (2015) claims that a dynamic auction with no demand disclosure is basically a sealed-bid auction. We show, however, that the equilibria that can be sustained in a CCA without demand disclosure during the clock phase can fundamentally differ from the equilibria of the VCG.

This paper contributes to the growing literature that explores real-world auction mechanisms. Ausubel et al. (2014) analyze the discriminatory and the uniform price auction in a similar framework. Goeree and Lien (2014a) derive equilibria for the SAA and find that the exposure problem is indeed problematic for efficiency and revenue. However, as the number of items grows large, outcomes converge to VCG outcomes. Bichler et al. (2013) report experimental evidence on the CCA and present a simple example in which one bidder submits a spiteful bid. Gretschko et al. (forthcoming) discuss why bidding can be complicated in a CCA. Ausubel and Baranov (2014) discuss the evolution of the CCA. A variant of the CCA has first

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[11]Not revealing the final clock demand was done for example in the Canadian 700 MHz spectrum auction (Power Auctions LLC 2015, p. 3).

been suggested by Ausubel et al. (2006) and further developed in Cramton (2013).

The rest of this paper is organized as follows. Section 2 describes the different auction models and the environment we analyze. Section 3 analyzes equilibrium behavior of the VCG mechanism. Section 4 presents an example of an efficient equilibrium of the CCA where the clock phase does not provide reveal any information to bidders. In Section 5 we prove our main result and discuss other general properties of the CCA. Section 6 provides other examples of efficient and inefficient equilibria and demonstrates that bidders may pay very different amounts for an almost identical share of the total supply. Section 7 concludes with a discussion. Proofs are in the Appendix.

2   The auction models

We use the same set-up as in Levin and Skrzypacz (forthcoming). There is one divisible good in unit mass supply to be allocated over two bidders. Bidders have a strictly increasing quadratic utility function of the form

$$U(a_i, x) = a_i x - \frac{b}{2} x^2, \tag{1}$$

where a_i is randomly drawn from an atom-less distribution with support $[\underline{a}, \overline{a}]$, with $\underline{a} \geq b > 0$ and $x \in [0, 1]$. The intercept of the marginal utility function a_i is private information, while (to have a one dimensional type space) the slope is the same for both players.[12] The condition $\underline{a} \geq b$ guarantees that the utility function is increasing in x for all types. The support of the distribution of types is common knowledge among the bidders. When it is convenient we write $U_i(x)$ instead of $U(a_i, x)$. Throughout the paper we denote level functions with capital letters and the respective derivative with small letters. For example, we write U_i for the utility function and u_i for marginal utility. Also, $\underline{U} = U(\underline{a}, \cdot)$ ($\overline{U} = U(\overline{a}, \cdot)$) denotes the utility function of the weakest (strongest) possible bidder.

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[12]Following Levin and Skrzypacz (forthcoming), we could allow for the two bidders having different, but commonly known, values of b. This would only complicate the analysis without adding new insights.

Given the type profile (a_i, a_j), the efficient share of bidder i is

$$x_i^*(a_i, a_j) = \frac{a_i - a_j + b}{2b} \in \arg\max_x U(a_i, x) + U(a_j, 1 - x)$$

We adopt the assumption of Levin and Skrzypacz (forthcoming) that $\bar{a} - \underline{a} < b$, which guarantees that the efficient allocation is always in the interior of $(0, 1)$ as $u_i(0) > u_j(1), j \neq i$.

The largest and the smallest efficient shares of bidder i are denoted by $\bar{x}_i = x_i^*(a_i, \underline{a})$ and $\underline{x}_i = x_i^*(a_i, \bar{a})$ respectively. The smallest efficient share of any bidder is $\underline{x} = x_i^*(\underline{a}, \bar{a})$. When there is no danger of causing confusion, we sometimes drop the i in x_i^*.

Besides these standard preferences, the two bidders have a spite motive. Like Janssen and Karamychev (2014) and in some of the models in Levin and Skrzypacz (forthcoming), we model this spite motive in a lexicographic way. In the first dimension, bidders maximize their surplus from the auction and in the second dimension they want to maximize the other bidder's price. This spite motive is relatively weak since bidders do not want to damage the other bidder if this implies getting a lower surplus themselves. Introducing a spite motive in a lexicographic manner resolves indifferences concerning auction outcomes in favor of those outcomes that harm the other bidder most. Consequently, studying auction outcomes under lexicographic preferences can be seen as a robustness check for equilibria under standard preferences. Some equilibria under standard preferences stop being equilibria under lexicographic preferences, but all equilibria under lexicographic preferences are equilibria with standard preferences.[13]

VCG Rules

In the VCG mechanism, every bidder submits a bidding function $S_i : [0, 1] \rightarrow \mathbb{R}_+$. The auctioneer chooses the allocation (x_1, x_2) such that[14]

$$(x_1, x_2) \in \arg\max_{x_1 + x_2 \leq 1} S_1(x_1) + S_2(x_2).$$

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[13]When we talk about efficiency we talk about efficiency in the first dimension of the preferences.

[14]If two allocations are optimal, the auctioneer implements the allocation in which the distance to the allocation $\left(\frac{1}{2}, \frac{1}{2}\right)$ is minimized.

Bidder i gets x_i and pays the VCG price $\max_y S_j(y) - S_j(x_j)$, the opportunity cost he imposes on the other bidder. If bidders submit nondecreasing bidding functions, the payment of bidder i is given by $S_j(1) - S_j(x_j)$. If the final allocation is $(x, 1-x)$, then bidder i's surplus from the auction is given by

$$U_i(x) - \max_y S_j(y) + S_j(1-x).$$

In Section 3 our solution concept is Bayesian Nash equilibrium. In a Bayesian Nash equilibrium players choose a pure strategy based on their type and the information they have about the other player. The strategy is optimal given the believe about the other bidder's chosen strategy.

CCA Rules

Next consider the CCA rules. The CCA is a two stage auction with a clock and a supplementary phase. In the clock phase bidders express a (weakly) decreasing demand for every price, while in the supplementary phase a VCG auction is held where bidders can submit many bids subject to certain activity rules that are described below. Different real-world CCAs have had different rules concerning the information that is disclosed in the clock phase: the auctioneer can disclose individual demands to all bidders, only aggregate demand or no demand information at all. In the case of two bidders the first and the second case are equivalent. As recent auctions (e.g. in Austria) did not provide any direct information concerning demand to bidders and as this case is also easier to analyze, we focus on that case.

The details of the auction rules are as follows. The clock phase begins at an initial price $p_0 = 0$. At each price, both bidders demand a certain share $x_i \in [0, 1]$. The price is increased continuously if there is excess demand, i.e. if $x_1 + x_2 > 1$. Bidders cannot increase their demand at higher prices. Since bidders neither learn individual nor aggregate demand, they can only condition their actions on the current price and their own past demand. Bidder i's action in the clock phase is a weakly decreasing demand function $x_i : \mathbb{R}_+ \to [0, 1]$ that maps prices to demand.[15]

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[15]An alternative way is to disclose information in the clock, but restrict bidders to proxy strategies. This approach was used by Levin and Skrzypacz (forthcoming).

In the subsequent supplementary phase, bidders submit bidding functions $S_i : [0,1] \to \mathbb{R}_+$. The auctioneer chooses the allocation $x = (x_1, x_2)$ that maximizes the total value, $x \in \arg\max S_1(x_1) + S_2(x_2)$ and bidder i pays the VCG payment $\sup_{y_j} S_j(y_j) - S_j(x_j)$ for his share x_i.[16] The choice of the supplementary bidding function S_i is constrained by three activity rules. First, the clock bids are still valid so that if bidder i demanded x at clock price p, then $S_i(x) \geq px$. Second, supplementary bids must satisfy the *final cap*, that is, revealed preference with respect to the final clock round: $S_i(x) \leq S_i(\tilde{x}_i) + \tilde{p}(x - \tilde{x}_i), x \neq \tilde{x}_i$, where $\tilde{p}$ is the final clock round price at which i demanded $\tilde{x}_i$. Lastly, if in the clock phase bidder i was demanding x at a price p, then in the supplementary round for any $x' > x$, bidder i cannot express an incremental value for x above p , i.e., $S_i(x) - px \geq S_i(x') - px'$, or $s_i(x') \leq p$, if S_i differentiable. Levin and Skrzypacz (forthcoming) call this the 'local revealed preference rule,' but it is also known as the *relative cap*.

In our analysis we derive Bayesian Nash equilibria for the CCA. A strategy consists of a non-increasing clock demand function $x : \mathbb{R}_+ \to [0,1]$ and a supplementary bidding function $S : [0,1] \to \mathbb{R}_+$ for every possible final clock price. Any supplementary bidding function S needs to satisfy the activity rules with respect to the clock demand x and at the materialized final clock price. Moreover, in equilibrium bidders use the fact that the clock has not ended and the final clock price to update their prior about the other bidder.

3 Equilibria in the VCG mechanism

We first examine the opportunities that arise in equilibrium to raise rival's cost in the VCG auction. The supplementary phase of the CCA is a VCG auction in which the bidders are constrained by the activity rule. It is therefore a natural starting point to investigate the equilibrium outcomes if

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[16]The real payment rule is slightly more complicated ("core-selecting"), but for the environment we consider here this would not make a difference. The core restricting elements in the pricing rule of real CCA auctions (see, e.g. Day and Milgrom (2008), Day and Cramton (2012), and Erdil and Klemperer (2010), as well as Goeree and Lien (2014b) and Ausubel and Baranov (2013)) are not binding in our case with only two bidders as in our case the CCA rule is exactly equal to the pricing rule of the CCA.

no constraints apply. This sets the natural benchmark to study the impact of the clock phase on the equilibrium outcomes of the auction. Given the assumption that $\overline{a} - \underline{a} < b$, if bidders bid their true marginal values, they know that the final allocation is such that all bidders are winners. This implies that certain bids on large packages cannot be winning and they can choose bidding strategies with high bids on large packages to raise rival's cost.

With standard preferences it is well-known that bidding value is a weakly dominant strategy (e.g. Milgrom 2004). With lexicographic preferences, however, it is straightforward to see that this is not true anymore. Suppose that bidder j bids truthfully and consider bidder i. By bidding $S_i(x) = U_i(x)$ bidder i wins an amount in the interval $[\underline{x}_i, \overline{x}_i]$ as defined in the previous section, with $\overline{x}_i < 1$. If bidder i slightly lowers his bid on all shares in $[\underline{x}_i, \overline{x}_i]$ by the same amount, then he does not change the winning allocation, but he increases the difference between his winning bid and the highest bid in the bid strategy, thereby increasing the payment of the rival bidder. Thus, under a lexicographic preference to raise rivals' cost, this strategy gives a higher pay-off if bidder j bids truthfully. On the other hand, like in the standard analysis without a lexicographic preference to raise rivals' cost, it may also be that by bidding in this way a bidder will loose the auction. Thus, none of these two strategies dominates the other. In what follows we therefore perform an equilibrium analysis, rather than an analysis in terms of weak dominance.

We first show that in any equilibrium where bidders use a continuously differentiable and increasing bidding function, each bidder i's marginal bids have to be equal to marginal value over the relevant domain $[\underline{x}_i, \overline{x}_i]$. To see this, note that given the bidding functions $S_i(x), S_j(x)$, the auctioneer implements the allocation $(x, 1 - x)$ such that $s_i(x) = s_j(1 - x)$. In equilibrium, a bidder wants to maximize his surplus given the strategy of the other bidder, i.e., bidder i wants to maximize

$$U_i(x) - \max S_j(y) + S_j(1 - x).$$

A necessary condition for this to be the case is that $u_i(x) = s_j(1 - x)$. Since $s_j(1 - x) = s_i(x)$, it follows that over the relevant domain $s_i(x) =$

$u_i(x)$ has to hold in any equilibrium. Note that this implies that an equilibrium where bidders use continuously differentiable and increasing bidding functions is efficient. Outside this domain, the bidder's strategy is undetermined. Without lexicographic preferences, bidder i is indifferent between many bids on $(\bar{x}_i, 1]$. The lexicographic bidder j knows, however, that he can increase the price bidder i has to pay by raising the bid on shares that cannot be winning. The easiest way to do so is to increase the bid $S_i(1)$ as much as possible under the constraint that it is not winning.[17] He never wins the full supply if for all $a_j \in [\underline{a}, \overline{a}]$

$$S_i\left(x_i^*\left(a_i, a_j\right)\right) + S_j\left(1 - x^*\left(a_i, a_j\right)\right) \geq S_i(1). \tag{2}$$

Lemma 1. *The value of the efficient allocation*

$$U(a_i, x_i^*(a_i, a_j)) + U(a_j, 1 - x_i^*(a_i, a_j))$$

is increasing in a_i and in a_j.

Using Lemma 1 and the fact that in equilibrium the marginal bids are equal to marginal utilities, the left-hand side of (2), is increasing in a_j. Therefore bidder i can use his private information and the information on the lowest possible type to raise the bid on the full supply. To protect themselves against the possibility for others to raise their price, bidders can also reduce their bids over the relevant domain of possibly efficient allocations without affecting the marginal bid. Thus, as the following result formalizes, we have a continuum of equilibria.

Proposition 1. *For any $(c_1, c_2) \in \left[0, \frac{(\underline{a} - \overline{a} + b)^2}{4b}\right]^2$ there exists a Bayesian Nash equilibrium, where, for $i = 1, 2$,*

$$S_i(x) = \begin{cases} U_i(x) - c_i & \text{if } x \in [\underline{x}_i, \overline{x}_i] \\ U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) - c_i - c_j & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Each of these equilibria implements the efficient allocation.

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[17]He could also increase his bid on other $x \in (\overline{x}_i, 1)$, but this does not create any benefit.

It is interesting to note the nature of the multiplicity of efficient equilibria and the boundaries on how far bidders can go in their bid shading. The raising rivals' cost motive makes that bidders want to increase their bid on the full supply as much as possible without winning it. If a bidder bids above value in the relevant interior domain, others can raise this bidder's payment above value so that a bidder would make a loss in the auction. Thus, bidders do not bid above value. However, if bidder i shades his bids a little, then bidder j cannot raise rival's cost as much as before without winning everything at a price that is too high to be worthwhile. Reversely, if bidder j does not raise rival's cost to the full extent, then bidders are indifferent between further shading their bids and not doing so. Together these arguments provide scope for multiple equilibria, resulting in different revenues. If bidder i would shade his bid too much, however, then bidder j prefers to win everything. This imposes the upper bound on c_i as mentioned in the Proposition. For $c = 0$, the revenue is the largest that is consistent with an ex-post efficient allocation.[18] In the Online Appendix we provide some indication how much bidders can raise rivals' cost. To this end, we compare the highest possible revenue in the VCG auction to the revenue under truthful bidding.

Next, we show by means of an example that inefficient equilibria also exist. Consider the following symmetric functions

$$S_i(x) = \begin{cases} p & \text{if } x = 1 \\ \frac{p}{2} & \text{if } x = \frac{1}{2} \\ 0 & \text{else.} \end{cases} \tag{3}$$

and suppose that ties are broken in favor of equal allocations. If bidders follow these strategies, the tie-breaking rule implies that the allocation $(1/2, 1/2)$ is implemented and bidders pay the VCG price $p/2$. we will argue that these strategies constitute an equilibrium if $\bar{a} - \underline{a} < \frac{b}{2}$ and $\bar{a} - \frac{3}{4}b < p < \underline{a} - \frac{b}{4}$. To see this, note that if a bidder wants to get more than $1/2$, the VCG price becomes p. Since the utility function is strictly

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[18]Note that in general lower values of c_i increase the chance of winning without affecting the allocation provided it is in the interior. However, depending on the other bidders' bids, a too low value of c_i may make the bid on the full supply winning.

increasing and the price for shares larger than $1/2$ is always p, this bidder wants to win either $1/2$ or 1. Winning $1/2$ is strictly preferred to winning 1 for all types since $p > \overline{a} - \frac{3}{4}b$.[19] Moreover, all types prefer to win $1/2$ above not winning anything.[20] Bidders cannot raise each other's prices because every bidder already pays his bid.

4 Clock-pooling Equilibrium in the CCA

We now come to the main body of the paper and describe when, and if so how, because of the clock phase equilibrium behavior and equilibrium outcomes in the CCA are different from those in the VCG mechanism. We proceed as follows. To gain some understanding what equilibrium bidding behavior in the CCA may look like, we start in this Section by considering a simple example of an efficient equilibrium where in the clock phase, all types of bidders choose the same (clock-pooling) strategies. We use the term "clock-pooling" as the clock phase does not reveal any information concerning opponents' types. We show this type of equilibrium exists when the uncertainty concerning competitor's type is small, i.e., $\overline{a} - \underline{a} < \left(\frac{1}{2} - \frac{1}{4}\sqrt{2}\right)b$. The next Section subsequently shows that efficient equilibria only come in two different varieties of which the clock-pooling equilibrium is one. We then show that if the uncertainty concerning bidders' types is relatively large, i.e., $\overline{a} - \underline{a} > b/2$ all equilibria must be inefficient. Section 6 then provides examples of the second equilibrium variety and characterizes properties of equilibrium behavior.

The main idea behind the examples of efficient equilibria in this and later Sections is that if given the proposed bidding strategy of the opponent, bidders can figure out that the clock will not stop below a certain threshold price, then bidders can demand 1 without the risk of obtaining it at any price up to the threshold price. This provides them with the flexibility to fully raise their bid on the full supply in the supplementary round in order to raise rivals' cost maximally. At the threshold price, they may drop demand to being truthful at that price. It is clear that such bidding

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[19]To see this, note that $U_i\left(\frac{1}{2}\right) - \frac{p}{2} > U_i(1) - p \Leftrightarrow p > a_i - \frac{3}{4}$.

[20]This follows from consider the low type $\underline{a}$ and the upper bound on p, i.e., $\underline{U}\left(\frac{1}{2}\right) - \frac{p}{2} > 0 \Leftrightarrow \underline{a} - \frac{b}{4} > p$.

strategies involve jumps and are discontinuous and non-linear.[21]

In the clock-pooling equilibrium, all bidders bid on the full supply until a certain threshold price is reached and bid truthfully at this threshold price. The clock phase ends at this price, because all type's truthful demand is less than or equal to half of the full supply. In the subsequent supplementary phase, bidders bid true marginal values on possibly efficient shares and maximally raise the bid on the full supply such that it does not become winning and it satisfies the activity rule. Note that in this equilibrium, the clock phase ends with excess supply with probability 1.

Pooling in the clock phase can nevertheless lead to efficient outcomes if all types are active in the last clock round and the price is such that all types are able to bid their marginal values in the supplementary round. This is the case if the final clock price is sufficiently high, i.e. if $\tilde{p} \geq \overline{a} - b/2$ (the marginal utility of the highest type at his smallest efficient share, which equals $1/2$). This will allow all bidders to bid their true marginal values in the relevant demand range $[\underline{x}_i, \overline{x}_i]$ in the supplementary round. The flexibility in the supplementary phase and the incentive structure makes that bidders bid in such a way that the final allocation is efficient.

The threshold price must not be too high either. All types are required to demand more than $1/2$ of the supply until the threshold price $\tilde{p}$ is reached, because otherwise the clock may end earlier for some combination of types. If the threshold price is too high, then bidders are forced to bid above utility such that the efficient allocation becomes winning and the activity rule is satisfied. As in the VCG mechanism it is necessary that bidders do not bid above their true utility level since this allows the other bidder to raise their payments even more, yielding negative surplus. Therefore it must be that $\underline{U}(1/2) \geq \tilde{p}/2$. This condition is satisfied whenever $\underline{a} - b/4 \geq \tilde{p}$.

Formally, in an efficient clock-pooling equilibrium, bidders demand the full supply throughout the clock until the price reaches $\tilde{p}$ and then drop demand to their true demand $\tilde{x}_i = \frac{a_i - \tilde{p}}{b}$, i.e. they use the clock demand

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[21]Levin and Skrzypacz (forthcoming) restrict bidders to linear strategies in many of their equilibria. However, any of these bidders can raise rival's costs by expanding demand in early phases of the clock.
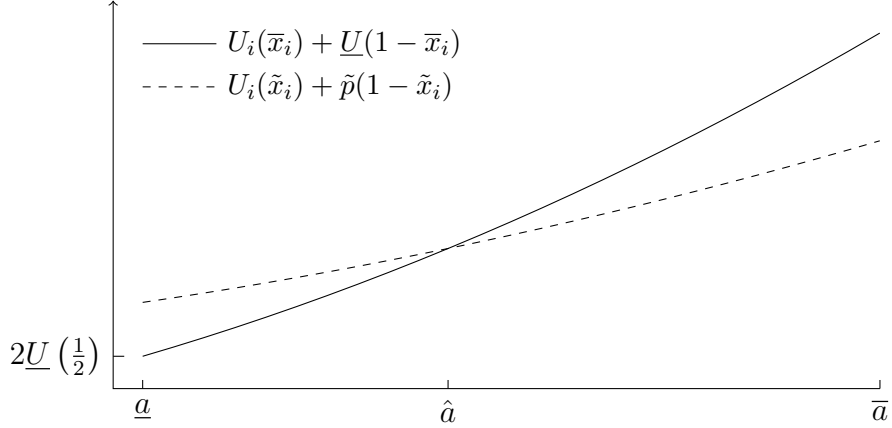
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Figure 1: Constraints in the supplementary bidding function

function

$$x_i(p) = \begin{cases} 1 & \text{if } p < \tilde{p} \\ \max\left\{\frac{a_i - p}{b}, 0\right\} & \text{if } p \geq \tilde{p} \end{cases} \tag{4}$$

After $\tilde{p}$ bidders drop demand to zero.

Thus, to have an efficient clock-pooling equilibrium for all distributions of types over $[\underline{a}, \overline{a}]$ the final clock price $\tilde{p}$ at which bidders drop demand should satisfy two conditions: $\underline{a} - \frac{b}{4} \geq \tilde{p} \geq \overline{a} - \frac{b}{2}$. If the difference $\overline{a} - \underline{a}$ is sufficiently small in relation to b, then the clock-pooling price is so high that all types demand less than their lowest efficient share if they demand truthfully at this price.

In the supplementary phase, bidders' marginal bids on units in the interval $[\tilde{x}_i, \overline{x}_i]$ are equal to their marginal values. Their bids on $[0, \tilde{x}_i)$ and $(\overline{x}_i, 1)$ are irrelevant as at the end of the clock all bidders know they will not win units in this interval. The only thing that is left to be determined is how they can raise rivals' cost as much as possible without running the risk of winning the full supply and satisfying the activity rule. Depending on the final clock price $\tilde{p}$ and on the type of bidder, bidders either use the opportunity to maximally raise their bid on the full supply (or decreasing their bids on interior shares) or not. Whatever they bid in the supplementary round on their last clock round package $\tilde{x}_i$, by the final and the relative cap, they can bid maximally $\tilde{p}(1 - \tilde{x}_i)$ more on the full supply. However, if their bid on the full supply is more than $S_i(\overline{x}_i) - S_i(\tilde{x}_i) + \underline{S}(1 - \overline{x}_i)$ larger than their bid on $\tilde{x}_i$ they run the risk of winning the full supply if the rival

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bidder is of low type. In the equilibrium that we consider, bidders raise the supplementary bids to the level of true utility.

We will now show that bidders do not fully raise their bids on 1. They bid less than they are allowed by the activity rule, because the expressed value of the efficient allocation might be less. Therefore, we equate

$$U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) = U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i),$$

which gives the following two roots in a_i,

$$\hat{a}_1(\tilde{p}) = 2\tilde{p} + b - \underline{a} + \sqrt{2}(\tilde{p} - \underline{a}),$$
$$\hat{a}_2(\tilde{p}) = 2\tilde{p} + b - \underline{a} - \sqrt{2}(\tilde{p} - \underline{a}),$$

when $0 < \tilde{p} < \underline{a}$. In all equilibria we consider in this Section, we have that $\tilde{p} \geq \min(\overline{a} - \frac{b}{2}, \underline{a} - \frac{b}{4})$. It is easy to see that $\hat{a}_2(\tilde{p}) > \overline{a}$ for these values of $\tilde{p}$ and that we effectively only have to consider whether or not $a_i < \hat{a}_1(\tilde{p})$. We drop the subscript 1 and simply write $\hat{a}(\tilde{p})$. If $\hat{a}(\tilde{p}) < a_i$ then

$$U_i(\tilde{x}_i) + \tilde{p}(1 - \tilde{x}_i) < U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i),$$

that is, the activity rule constrains type a_i from (i) bidding value in the relevant interior $[\tilde{x}_i, \overline{x}_i]$ *and* (ii) maximally raising rivals' cost. On the other hand, if $a_i < \hat{a}(\tilde{p})$ then bidder i can bid value in the relevant interior and maximally raise rivals' cost. In particular, if $\overline{a} \leq \hat{a}(\tilde{p})$ then this condition is satisfied for all types. This inequality can be transformed into a lower bound on $\tilde{p}$:

$$\frac{\overline{a} + \underline{a}(1 + \sqrt{2}) - b}{2 + \sqrt{2}} \leq \tilde{p}. \tag{5}$$

Thus, after the proposed equilibrium path of the clock phase bidders will submit a supplementary round bidding function of the form:

$$S_i(x) = \begin{cases} 0 & \text{for } x < \tilde{x}_i \\ U_i(x) & \text{for } x \in [\tilde{x}_i, \overline{x}_i] \\ 0 & \text{for } x \in (\overline{x}_i, 1) \\ U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) & \text{for } x = 1. \end{cases} \tag{6}$$

A necessary condition for a clock-pooling equilibrium to exist is that $\overline{a} - \underline{a} < \frac{b}{4}$, because then $\overline{a} - \frac{b}{2} < \underline{a} - \frac{b}{4}$. Under this constraint of the support of the type distribution, the lower bound from equation (5) is always higher than $\overline{a} - \frac{b}{2}$. Therefore, we restrict the threshold prices to be in the interval

$$\frac{\overline{a} + \underline{a}(1 + \sqrt{2}) - b}{2 + \sqrt{2}} \leq \tilde{p} \leq \underline{a} - \frac{b}{4}.$$

In the following Proposition we give a sufficient condition on the size of the support such that an efficient clock-pooling equilibrium exists. If this condition is satisfied, then $\overline{a} - \underline{a} < b/4$ is also satisfied. If this condition is not satisfied, then there may still exist clock-pooling equilibria, but then the high types will shade their supplementary bids in the interior to be able to fully raise rivals' cost. As the necessary calculations are then somewhat more cumbersome and the idea here is to provide examples of possible equilibria, we do not provide the full range of parameter values where a clock-pooling equilibrium exists.

Proposition 2. *If $\overline{a} - \underline{a} < \left(\frac{1}{2} - \frac{1}{4}\sqrt{2}\right) b$, there exists a continuum of symmetric efficient clock-pooling equilibria, where in the clock bidders demand according to function (4), that is, they demand the full supply for prices lower than $\tilde{p}$, with $\underline{a} - \frac{b}{4} \geq \tilde{p} \geq \frac{\overline{a} + \underline{a}(1 + \sqrt{2}) - b}{2 + \sqrt{2}}$, and then drop demand to $\tilde{x}_i = \frac{a_i - \tilde{p}}{b}$. In the supplementary phase bidders submit the bidding function (6).*

To see that this behavior can be supported as an equilibrium, the proof in the Appendix specifies some remaining parts of the strategy profiles.

The final equilibrium allocation of these clock-pooling equilibria is always ex-post efficient and bidders are able to fully raise their rivals' cost without knowing the type of their rival. Thus in the parameter range specified in the Proposition, the CCA yields the same revenue as the maximum revenue in the unconstrained VCG mechanism. There is, however, no price discovery in the clock, since bidders do not reveal any private information.

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5 Equilibrium behavior in the CCA: general properties

In the previous Section we have provided one example of a symmetric efficient equilibrium of the CCA. In this Section we show our main result, namely that the CCA does not have efficient equilibria when the uncertainty concerning competitor's type is too large. To show this, we denote by $p(a_i)$ the lowest clock price at which along the equilibrium path the clock phase *may* end for type a_i. Formally, let $p : [\underline{a}, \overline{a}] \rightarrow \mathbb{R}_+$. In the clock-pooling equilibrium, we have that $p(a_i) = \tilde{p}$ for all types $a_i \in [\underline{a}, \overline{a}]$.[22] In the second example of the next Section we have that the image of the function p is a pair of two prices. To show our main result, we first show that there do not exist symmetric efficient equilibria with three or more lowest possible final clock prices or where the function p is continuously increasing on some interval. In this sense, the examples in the previous and the next Section are not arbitrary: other symmetric efficient equilibria do not exist. We then use this result to show that efficiency requires that the uncertainty concerning rivals' types is relatively small, i.e., $\overline{a} - \underline{a} \leq \frac{b}{2}$.

Lemma 2. *In any symmetric efficient equilibrium of the CCA, the image of the function $p(a_i)$ is either a singleton or a pair.*

The main idea that is exploited in the proof is as follows. It is clear that the function $p(a_i)$ is non-decreasing such that in equilibrium if the clock phase is over for type combinations (a'_i, a'_j), then it is also finished for all type combinations (a_i, a_j) with $a_i < a'_i$ and $a_j < a'_j$ as otherwise type a'_i would like to imitate the behavior of the low type bidder a_i. If there are three or more prices $\tilde{p}^k$ then there must be three or more intervals A^k of bidders' types such that $p(a_k) = \tilde{p}^k$ for all $a_k \in A^k$, with higher intervals representing higher types and $\tilde{p}^1 < \tilde{p}^2 < ...$. If the clock phase is over for a certain type of bidder a_k bidding against a type a_j, then because bidders' demands are weakly increasing in their types, it must also be the case that the clock phase is over for the same bidder a_k bidding against

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any type $a_i < a_j$. Thus, we can define a set $\overline{a}^k$ of critical types such that these are the highest types for which the clock possibly ends at $\tilde{p}^k$ and a set of $\overline{x}_i^k = x_i((\overline{a}^k + a_i - b)/2)$ which is the maximally efficient share knowing that your competitor is of a type at least as large as $\overline{a}^k$. A set of lowest type bidders will then learn from the price at which the clock phase is over that their rival bidder is of a type in class A^k and not in class A^{k-1}. These lowest types of bidders can then adjust their clock phase bidding and their supplementary phase bidding such that they raise their rivals' cost by a discrete amount more if they learn they bid against a type in interval A^3 compared to if they learn they bid against a type in interval A^2. In a two-step equilibrium, there is no possibility for weak bidders to use the information that their competitor is strong as the lowest type wants to bind true marginal values on all possibly efficient shares smaller than $\tilde{x}_i^1$. This is no longer true in a three step-equilibrium, where $\tilde{x}_i^2 > \underline{x}_i$ and upon learning that the clock did not stop at prices at or close to $\tilde{p}^2$ the lowest types may actually increase their supplementary bids in the interval between $\overline{x}_i^2$ and $\tilde{x}_i^2$. The lowest types in interval A^3 would then prefer to deviate and pretend they are in class A^2 and can do so by reducing their bids in the clock phase such that the clock phase is also over for them at prices $\tilde{p}^2$ if their rival is of the lowest possible types. By doing so, they give up getting the efficient share against all rival types in order to pay less. An example of an inefficient equilibrium that shares this feature of high types reducing demand in the clock phase to pay less is in the next Section.

Our main result is that efficient equilibria do not exist if $\overline{a} - \underline{a} > \frac{b}{2}$.

Proposition 3. *A necessary condition for an efficient equilibrium of the CCA to exist is $\overline{a} - \underline{a} \leq \frac{b}{2}$.*

Given Proposition 3, the only thing to check is that efficient equilibria where the image of $p(a)$ is a singleton or a pair require that $\overline{a} - \underline{a} \leq \frac{b}{2}$. At the highest possible clock price in both cases it should be true that (i) all types, including type $\overline{a}$, demand not less than their truthful demand (to be able to implement the efficient allocation) and (ii) even if both bidders are of the high type $\overline{a}$, the clock phase is over so that together they demand not more than the full supply. This implies that $2\frac{\overline{a}-\tilde{p}^2}{b} \leq 1$ for the two-step equilibrium, or $\tilde{p}^2 \geq \overline{a} - \frac{b}{2}$ (and similarly for the clock-pooling equilibrium).

On the other hand, as in both cases, the lowest possible type $\underline{a}$ is required to be still active at the highest possible clock price, we need that $\underline{a} \geq \tilde{p}^2$ as otherwise the lowest type bidder $\underline{a}$ will not want to be active in the clock phase and the high types win (undesirably and inefficiently) the full supply. Together these constraints imply that one can only find relevant clock prices if $\bar{a} - \underline{a} \leq \frac{b}{2}$.[23]

Finally, we show that demand expansion in an early stage of the clock phase is an essential ingredient of any equilibrium, in the sense that bidders start by maximally expanding demand and demand the full supply at low prices.

Proposition 4. *In any symmetric efficient equilibrium of the CCA, bidders bid on the full supply when the clock price is smaller than $\underline{a} - b/2$.*

The idea behind this Proposition is clear. When the clock price is still smaller than $\underline{a} - \frac{b}{2}$, no bidder will want to drop out of the clock phase as this would seriously restrict the bidder's behavior in the supplementary round. When the other bidder is active in the clock phase, however, the clock phase will never finish if you demand the full supply, and you want to do so to maximally be able to raise rival's cost.

6 Two-step equilibria

To argue that efficiency in the CCA requires that the uncertainty concerning competitors' types is not too large, we have proved in the previous Section an intermediate result, namely that in any efficient equilibrium the lowest price at which the clock phase may stop for type a_i is either the same for all types, or there are at most two such prices $\tilde{p}^1$ and $\tilde{p}^2$ so that we have that $p(a_i) = \tilde{p}^1$ for all types $a_i \in [\underline{a}, \bar{a}^1)$ and $p(a_i) = \tilde{p}^2$ for all types $a_i \in [\bar{a}^1, \bar{a}]$. In this Section, we provide more detail into how such an equilibrium with two clock prices may look like. We first consider an efficient equilibrium in this class and provide more intuition why this equilibrium only exist for $\bar{a} - \underline{a} < \frac{b}{2}$. We also show here that two types that are very close to each other may en up paying very different prices for almost

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[23]For the clock-pooling equilibrium the constraint is even more severe than what we have indicated here.

identical shares of the total supply. Finally, we show that in the parameter region where efficient equilibria do not exist, inefficient equilibria exist that share many of the features of the efficient two-step equilibria.

6.1 Efficient two-step equilibria: $b(\sqrt{2}-1) < \overline{a} - \underline{a} < \frac{b}{2}$

The main feature of the two-step equilibrium is that it classifies bidders' types into two categories: *weak* and *strong* types. Weak types demand the full supply for all prices $p < \tilde{p}^1$ and demand truthfully at $\tilde{p}^1$. Weak types are types in $[\underline{a}, \overline{a}^1)$ such that $\overline{a}^1$ is the highest type for which the clock can possibly end if both bidders demand truthfully at $\tilde{p}^1$: $\overline{a}^1 = 2\tilde{p}^1 + b - \underline{a}$. Strong types demand the full supply for all prices $p < \tilde{p}^2$, where $\tilde{p}^2 > \tilde{p}^1$ and demand truthfully at $\tilde{p}^2$. The price $\tilde{p}^2$ can be chosen such that under truthful bidding the clock ends for all types at this price and all types are active in the clock phase for all prices $p < \tilde{p}^2$. If $\overline{a} - \underline{a} < \frac{b}{2}$, which is what we assume in this subsection, $\tilde{p}^2$ can be any price in the interval $[\overline{a} - \frac{b}{2}, \underline{a}]$. For simplicity, and to be easily characterize the structure of this equilibrium, we choose $\tilde{p}^2 = \underline{a}$. The price $\tilde{p}^1$ is chosen such that no weak type can further raise rivals' cost if he learns that the rival is a strong bidder. This is the case if $\hat{a}(\tilde{p}) \leq \underline{a}$ and we choose $\tilde{p}^1$ to be the largest price that satisfies the inequality:

$$\tilde{p}^1 = \underline{a} - \frac{b}{2 + \sqrt{2}}. \tag{7}$$

It follows that $\overline{a}^1 = \underline{a} + (b\sqrt{2})/(2 + \sqrt{2}) = \underline{a} + b\left(\sqrt{2} - 1\right)$, which is smaller than $\overline{a}$ given the parameter space in this subsection.

To finish the description of the clock phase, we still have to describe how weak types bid for prices p such that $\tilde{p}^1 < p < \tilde{p}^2$. We specify that they bid according to true marginal values until they learn that the other bidder is strong. Given this strategy, a weak type a_i learns that the rival bidder is a strong bidder if the clock phase is not over at a price $(\overline{a}^1 + a_i - b)/2$.[24] Once they learn, their rival is strong, they keep their demand at that level $\overline{x}_i^1 = x_i((\overline{a}^1 + a_i - b)/2)$ to be maximally able to raise their rival's cost as long as $p \leq \frac{U_i(x)}{x}$. At this price they demand their truthful demand.

This description of clock phase behavior is summarized in Figure 2. If

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[24]As $(\overline{a}^1 + a_i - b)/2 < \underline{a}$ all weak types learn this before the clock reaches price $\tilde{p}^2$.
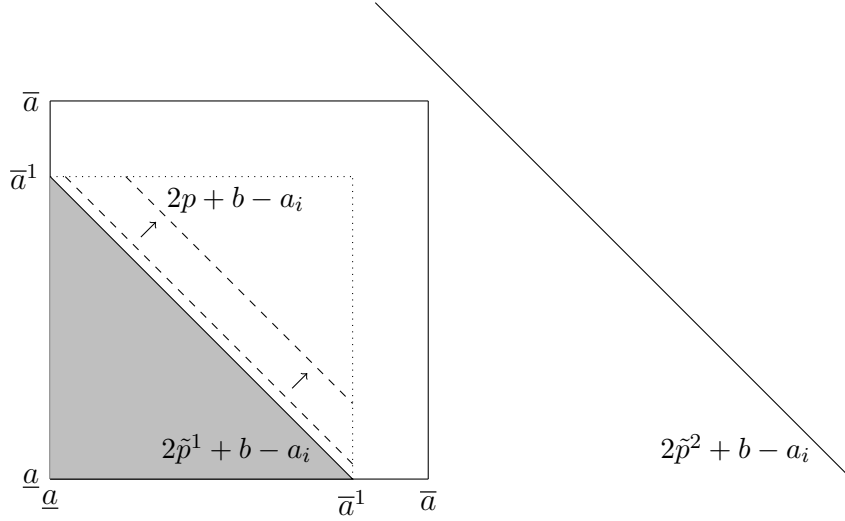
Figure 2: Illustration of the two-step equilibrium

the types are such that they are in the gray area, the clock phase stops at price $\tilde{p}^1$. If the types are such that each $a_i, a_j < \overline{a}^1$, but (a_i, a_j) is not in the gray area, then the clock phase stops at a price p such that $\tilde{p}^1 < p < \tilde{p}^2$. If at least one type is strong, then the clock phase stops at price $\tilde{p}^2$.

We will now specify the behavior in the supplementary phase. If the clock ends at $\tilde{p}^1$ by both bidders demanding truthfully, then these weak bidders submit the supplementary bidding function

$$
S_i^{\tilde{p}^1}(x) = \begin{cases} 0 & \text{if } x < \tilde{x}_i^1 \\ U_i(x) & \text{if } \tilde{x}_i^1 \le x \le \overline{x}_i \\ 0 & \text{if } \overline{x}_i < x < 1 \\ U_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) & \text{if } x = 1, \end{cases} \tag{8}
$$

where $\tilde{x}_i^i = x_i(\tilde{p}^i), i = 1, 2$. If the clock ends at $\tilde{p}^1$ along the equilibrium path, weak bidders know that the efficient share lies in the interval $[\tilde{x}_i^1, \overline{x}_i]$ and want to bid true marginal values in this area and maximally raise rivals' bids by increasing their bid on 1 maximally. This is what is achieved by the bidding strategy in (8). For future reference, the location of different relevant cut-off values are presented in Figure 3.

If the clock phase ends at price p^* such that $\tilde{p}^1 < p^* < \tilde{p}^2$, it ends with market clearing and bidder i believes that the efficient share x_i^* has

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been implemented. Since they have submitted positive, truthful, bids for $x \in [x_i^*, \tilde{x}_i^1]$, they submit the bidding function

$$
S_i^{p^*}(x) = \begin{cases} 0 & \text{if } x < x^* \\ U_i(x) & \text{if } x \in [x_i^*, \tilde{x}_i^1] \\ 0 & \text{if } x \in (\tilde{x}_i^1, 1) \\ U_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) & \text{if } x = 1. \end{cases} \tag{9}
$$

Bidders cannot further raise the competitor's cost compared to the situation when the clock ends at $\tilde{p}^1$, since the relative cap was already binding at $\tilde{p}^1$.

Finally, if the clock ends at $\tilde{p}^2$ strong bidders submit the following supplementary bidding function

$$
S_i^{\tilde{p}^2}(x) = \begin{cases} 0 & \text{if } x < \tilde{x}_i^2 \\ U_i(x) & \text{if } \tilde{x}_i^2 \leq x \leq \overline{x}_i \\ 0 & \text{if } \overline{x}_i < x < 1 \\ U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) & \text{if } x = 1. \end{cases} \tag{10}
$$

If the clock ends at $\tilde{p}^2$ along the equilibrium path, strong bidders have received no information where their efficient share may lie and any share in the interval $[\underline{x}_i, \overline{x}_i]$ is possible. If, $\tilde{x}_i^2 < \underline{x}_i$, which is certainly the case for all i if $\overline{a} - \underline{a} < \frac{b}{2}$, then the clock phase behavior allows them to bid their true value on the whole domain by bidding true value on $[\tilde{x}_i^2, \overline{x}_i]$. Hence, they want to bid true marginal values in this interval. Moreover, as it is well possible that their competitor is the weakest possible, they will not use the ability to fully raise rivals' cost as they risk winning the full supply at too high a price. If the clock ends at $\tilde{p}^2$, weak bidders bid

$$
S_i^{\tilde{p}^2}(x) = \begin{cases} 0 & \text{if } x < \tilde{x}_i^2 \\ U_i(x) & \text{if } \tilde{x}_i^2 \leq x \leq \tilde{x}_i^1 \\ 0 & \text{if } \tilde{x}_i^1 < x < 1 \\ U_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) & \text{if } x = 1. \end{cases} \tag{11}
$$

Figure 3: Prominent shares in the two-step equilibrium

Note that knowing that the clock phase finished at $\tilde{p}^2$ and that despite the fact that they kept their demand at $\overline{x}_i^1$ for all prices p with $(\overline{a}^1 + a_i - b)/2 < p < \tilde{p}^2$ to be maximally able to raise their rival's cost, weak bidders bid truthfully for all $\tilde{x}_i^2 \leq x \leq \overline{x}_i^1$ as this maximizes their primary surplus. Together with the fact that they were bidding truthfully in the clock phase on the interval $\overline{x}_i^1 \leq x \leq \tilde{x}_i^1$ implies they will their true value on the whole interval $\tilde{x}_i^2 \leq x \leq \tilde{x}_i^1$.

Given the nature of the supplementary bidding functions it is not difficult to see that these candidate equilibrium strategies produces an ex post efficient outcome. Thus, we can claim the following:

Proposition 5. *If $b(\sqrt{2} - 1) < \overline{a} - \underline{a} < \frac{b}{2}$, then there exists an efficient two-step equilibrium where the clock demand functions are specified above and the corresponding supplementary bidding functions are (8), (9), (10) and (11).*

Two issues of the current equilibrium require further discussion: (i) to understand why strong bidders do not want to reduce demand to stop the clock phase earlier, and (ii) to understand why weak bidders do not want to raise the cost of the strong bidders further. The first issue follows from the fact that we have constructed $\tilde{p}^1$ in such a way that $\hat{a}(\tilde{p}) = \underline{a}$, i.e., in the equilibrium all weak types already raise rivals' cost to the maximal extent possible and they cannot increase this cost further even if they learn the rival is a strong bidder. Strong bidders therefore have nothing to gain by reducing demand.

The argument that weak bidders do not want to raise the cost of the strong bidders further by extending their demand on the full supply at prices $p > \tilde{p}^1$ is more involved. Consider a type $a_i = \underline{a} + b(\sqrt{2} - 1) - 2b\epsilon < \overline{a}^1$ for some $\epsilon > 0$. We will argue that there are some types of the rival bidder such that the only time type a_i can get the efficient share is by bidding truthfully at $\tilde{p}^1$. It is clear that the bidder does not want to reduce demand as this will prevent him from always getting the efficient share at a price he

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wants to pay for it. Compare then the situation where he bids truthfully at $\tilde{p}^1$ and one where he expands demand. If he bids truthfully and the rival is of type $a_j = \underline{a} + b\epsilon$, then their truthful demands at $\tilde{p}^1$ are

$$\tilde{x}_i^1 = \frac{\underline{a} + b(\sqrt{2} - 1) - 2b\epsilon - \underline{a} - b\left(\frac{1}{\sqrt{2}} - 1\right)}{b} = \frac{1}{\sqrt{2}} - 2\epsilon$$

$$\tilde{x}_j^1 = \frac{\underline{a} + b\epsilon - \underline{a} - b\left(\frac{1}{\sqrt{2}} - 1\right)}{b} = 1 - \frac{1}{\sqrt{2}} + \epsilon$$

and under truthful demand the clock ends at $\tilde{p}^1$ with excess supply, i.e.

$$\tilde{x}_i^1 + \tilde{x}_j^1 = \frac{1}{\sqrt{2}} - 2\epsilon + 1 - \frac{1}{\sqrt{2}} + \epsilon = 1 - \epsilon < 1.$$

Importantly, note that the efficient share $x_j^* = \frac{\underline{a} + b\epsilon - \underline{a} - b(\sqrt{2} - 1) + 2b\epsilon + b}{2b}$ for bidder j is larger than his demand $\tilde{x}_j^1$. Given that the supplementary bidding function (8) applies in this case, the efficient allocation will be implemented. If, however, bidder i expands demand so that the clock phase does not end at $\tilde{p}^1$, bidder j believes that $a_i > \underline{a} + b(\sqrt{2} - 1) - b\epsilon$ and that his efficient share is smaller than $\tilde{x}_j^1$. Given the specification of the supplementary bidding function (9), he will only bid on shares that are smaller than $\tilde{x}_j^1$, while the true efficient share is larger. Thus, the only time in the auction when a_j submits a positive bid for x_j^* is when the clock ends at $\tilde{p}^1$. Consequently, if bidder a_i does not drop demand to $\tilde{x}_i^1$, there is a positive probability he misses the chance of acquiring the efficient share and this reduces his expected surplus since $U_i(x_i^*) + U_j(x_j^*) > U_i(1 - \tilde{x}_j^1) + U_j(\tilde{x}_j^1)$. Thus, as the other weak bidder bids truthfully at $\tilde{p}^1$, all weak bidders want to bid truthfully too, at least until they learn that the other bidder is not weak. This argument does not hold true for types $a_i \geq \overline{a}^1$ as they have zero probability that the clock ends at $\tilde{p}^1$ under truthful bidding.

Note that again there is a multiplicity of two-step equilibria, as $\tilde{p}^1$ and $\tilde{p}^2$ can be chosen smaller than we have chosen here. In order not to complicate the argument too much, however, we have focused this example on precise values of these two clock prices.

Finally, we show that in a two-step equilibrium types that are close to each other may pay very different amounts of money for very similar

shares of the full supply. Let ε be positive but arbitrarily small. Suppose the two bidders have very similar types, but one is strong, while the other is weak, i.e. they have types $\bar{a}^1 + \varepsilon, \bar{a} - \varepsilon$, respectively. Along the equilibrium path, the clock will end at $\tilde{p}^2$ and both will express true utility for the efficient share, which is roughly $1/2$. However, the two bidders face different constraints on the bid on 1 and will consequently submit different bids. The CCA price for the strong bidder $\bar{a}^1 + \varepsilon$ is approximately equal to $U_i(\bar{x}_i) + \underline{U}(1 - \bar{x}_i) - U_i(\frac{1}{2})$, while the price for the weak bidder $\bar{a}^1 - \varepsilon$ is approximately equal to $U_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) - U_i(\frac{1}{2})$. Thus, the relative price of the strong bidder $\bar{a}^1 + \varepsilon$ is approximately

$$\frac{U_i\left(\bar{x}_i\right) + \underline{U}\left(1 - \bar{x}_i\right) - U_i\left(\frac{1}{2}\right)}{U_i\left(\tilde{x}_i^1\right) + \tilde{p}^1\left(1 - \tilde{x}_i^1\right) - U_i\left(\frac{1}{2}\right)} - 1 = 2 - \frac{4\sqrt{2}}{3} \approx 0.1143$$

lower than what type $\bar{a}^1 - \varepsilon$ pays. In a symmetric equilibrium of the VCG auction the price for both bidders would be the same.

Finally, we show that in a two-step equilibrium types that are close to each other may pay very different amounts of money for very similar shares of the full supply. Given the equilibrium strategies, the clock phase does not finish for a type $\bar{a}^1 + \varepsilon$ until the clock price $\tilde{p}^2$ is reached. On the other hand, the clock phase may stop at clock price $\tilde{p}^1$ (or just above it) for type $\bar{a}^1 - \varepsilon$ if the rival is weak. If the clock stops at (much) lower prices than $\tilde{p}^2$ these weak types are restricted in raising rivals' cost. If $b = \underline{a}$, then we can compute how large this effect can be. We consider the difference in the CCA price if a type $\bar{a}^1 - \varepsilon$ faces a type $\bar{a}^1 + \varepsilon$ when ε is arbitrarily small. To see this, note that the CCA price for the strong bidder $\bar{a}^1 + \varepsilon$ (facing a weak bidder $\bar{a}^1 - \varepsilon$) is approximately equal to $U_i(\bar{x}_i) + \underline{U}(1 - \bar{x}_i) - U_i(\frac{1}{2})$, where $a_i = \bar{a}^1$, while the same price for the weak bidder $\bar{a}^1 - \varepsilon$ (facing a strong bidder $\bar{a}^1 + \varepsilon$) is approximately equal to $U_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) - U_i(\frac{1}{2})$. Thus, the relative price of the strong bidder $\bar{a}^1 + \varepsilon$ is approximately

$$\frac{U_i\left(\bar{x}_i\right) + \underline{U}\left(1 - \bar{x}_i\right) - U_i\left(\frac{1}{2}\right)}{U_i\left(\tilde{x}_i^1\right) + \tilde{p}^1\left(1 - \tilde{x}_i^1\right) - U_i\left(\frac{1}{2}\right)} - 1 = 2 - \frac{4\sqrt{2}}{3} \approx 0.1143$$

lower than what type $\bar{a}^1 - \varepsilon$ pays.

6.2 An Inefficient Equilibrium: $\frac{b}{2} < \overline{a} - \underline{a} < b(2 - \sqrt{2})$

We finally show that inefficient equilibria exist for parameter values for which symmetric efficient equilibria do not exist. We do not want to give a full characterization of inefficient equilibria, but instead just show how inefficiencies may naturally arise.

The inefficient equilibrium we construct has the same features as the efficient two-step equilibrium, but now the uncertainty concerning rival types is so large that high types prefer to reduce demand to half the full supply even if this is smaller than their truthful demand at the last final clock price. That is, there is a positive mass of types whose truthful demand at $\tilde{p}^2 = \underline{a}$, $(a_i - \underline{a})/b$, is more than $1/2$ but in the equilibrium, these types reduce demand to $1/2$ in order to end the clock for sure at $\tilde{p}^2$. To distinguish these types $a_i > \underline{a} + \frac{b}{2}$ from the strong bidders that drop demand truthfully to less than $1/2$ at $p^2 = \underline{a}$, we call these types "super-strong". All other types behave in both phases of the auction in exactly the same way as in the efficient two-step equilibrium. This leads to the clock demand function

$$x_i(p) = \begin{cases} 1 & \text{if } p < \tilde{p}^2 \\ \min\left\{\tilde{x}_i^2, \frac{1}{2}\right\} & \text{if } p = \tilde{p}^2 \\ 0 & \text{if } p > \tilde{p}^2 \end{cases} \tag{12}$$

of strong and super-strong types.

In the supplementary phase strong types bid supplementary bidding function (10) and super-strong types bid according to the following strategy:

$$S_i^{\tilde{p}^2}(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} \\ U_i(\tilde{x}_i^2) + \tilde{p}^2(x - \tilde{x}_i^2) & \text{if } x \in [\frac{1}{2}, \tilde{x}_i^2) \\ U_i(x) & \text{if } x \in [\tilde{x}_i^2, \overline{x}_i] \\ 0 & \text{if } x \in (\overline{x}_i, 1) \\ U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) & \text{if } x = 1. \end{cases} \tag{13}$$

Super-strong bidders cannot bid true marginal values in the supplementary round that are higher than $\tilde{p}^2$ as their clock phase behavior in combination with the local revealed preference rule does not allow them to do so. This

is the source of inefficiency.

To complete the description of the equilibrium strategies, we should specify some aspects of bidder behavior if the clock continues out-of-equilibrium at prices $p > \tilde{p}^2$. The main thing that is important here is that we specify that bidders believe that it is the super-strong bidders that have deviated and demand more than $1/2$ and that this is the reason why the clock did not stop at $\tilde{p}^2$. Given this belief the strong and super-strong bidders that did not deviate will respond to this deviation by adapting their bid on the full supply from what is specified in (13) to at least $U_i(\tilde{x}_i^2) + \tilde{p}^2(1 - \tilde{x}_i^2)$.

Using these strategies, we can state the following Proposition.

Proposition 6. *If* $b/2 < \overline{a} - \underline{a} < b(2 - \sqrt{2})$, *then there exists an inefficient two-step equilibrium.*

There are two possible final allocations that lead to inefficiencies. First, if both bidders are super-strong, then the allocation is $(1/2, 1/2)$ as both bidders demanded $1/2$ in the final clock round. The final cap rule implies that this is the final allocation independent of the bidders' true types. This is clearly inefficient. Second, it can be the case that a super-strong bidder i meets another bidder j and that the sum of their truthful demands $(a_i - \tilde{p}^2)/b + (a_j - \tilde{p}^2)/b$ is larger than 1. In this case the final allocation is $(1 - \tilde{x}_j^2, \tilde{x}_j^2)$, where $1 - \tilde{x}_j^2 < \tilde{x}_i^2$. The efficient allocation would be to give bidder i more than $1 - \tilde{x}_j^2$ and bidder i less than $\tilde{x}_j^2$. However, bidder i cannot express a marginal bid that is larger than $\tilde{p}^2$ and at this expressed marginal bid of bider i, it is optimal for bidder j to acquire $\tilde{x}_i^2$.[25] By restricting ourselves to the parameter region $\overline{a} - \underline{a} < b(2 - \sqrt{2})$ we simplify the proof as for these parameter values if the super-strong bidder were to deviate to his true demand, the clock phase would still be finished if his competitor were a weak type. In that case, the deviation to demand truthfully at $\tilde{p}^2$ does not benefit the super-strong bidder. The proof shows that given the continuation strategies after a deviation (and the belief that the deviation is from a super-strong type), the additional efficiency gains to be achieved by deviating do not outweigh the cost of having to pay discontinuously higher prices.

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[25]The condition $\overline{a} - \underline{a} < b(2 - \sqrt{2})$ implies that if one bidder is weak, then the final allocation is efficient in equilibrium. In the proof of the Proposition we will see that this rules out a case where the existence of the equilibrium depends on the type distribution.

7 Discussion and Conclusion

This paper provides a full equilibrium analysis of the CCA where the strategic interaction between the clock phase and the supplementary round is studied in an environment where bidders not only care about their own pay-off, but also (lexicographically) about how much rivals pay. In an environment where bidders only care about own pay-off there are many (undominated) equilibria due to the second-price rule implying that bidders only affect the allocation, but not their payment (given the same allocation). The lexicographic preferences provide a robustness check in the sense that all of our environment are also equilibria when bidders only care about own pay-offs, but not vice versa. Previous studies using the lexicographic preference ordering mainly considered the supplementary round (Janssen and Karamychev (2014) or exogenously assumed how much bidders can raise their bids in the supplementary round without taking into account the rules that relate clock phase bids to supplementary bids (Levin and Skrzypacz (forthcoming)).

We arrive at the following surprising results. First, the CCA is inefficient when there is quite some uncertainty concerning the rivals' type. If this uncertainty is small, then there are efficient equilibria that take on unusual forms. For example, we have characterized an equilibrium where no information is revealed during the clock phase and where bidders bid on the full supply before dropping demand truthfully. In this equilibrium, the clock phase ends with excess supply with positive probability. We have also derived a "two-step" equilibrium where weak bidders drop much earlier than strong bidders and where two bidders of almost equal type pay quite different amounts for the same allocation. Finally, we see that the clock phase imposes constraints on how much bidders are able to raise rival's cost, implying that the CCA often results in lower payments than in the VCG mechanism, but in (much) higher payments than the ones if bidders would bid truthfully.

Ausubel and Baranov (2015) have worked on alternative activity rules with the purpose of providing bidders with stricter incentives to bid according to their intrinsic preferences. They propose to replace the relative cap we used in this paper by GARP (the generalized axiom of revealed

preference). We observe that in none of the three equilibria we constructed bidders violate GARP, and we conclude therefore that our conclusion continue to hold if we would adopt the GARP activity rule.

We think it is very natural for bidders to care about what others pay. In the introduction we have provided some of the arguments. It is even likely that they are able to give up some own pay-off in exchange for being able to substantially raise rivals' cost. We expect this would further reinforce our results. In our case, bidders would not like to raise their bid on the full spectrum if there is a chance this becomes winning. Information about how weak their competitor(s) can be is important in this respect. In real-world auctions, bidders may never know for sure how weak their competitor is. Raising rival's cost in these situations is still feasible if the probability that a rival is really very weak is small and bidders have more than just a lexicographic preference for raising rivals' cost. As it is difficult to know how bidders would make the trade-off between own surplus and being able to raise rivals' cost, we think the analysis with lexicographic preferences provide a good reference point of what we may expect in real world auction. Bidders paying different amounts for similar spectrum (as in our two-step equilibria and in the 2012 Swiss auction) or clock phases ending with excess supply (as in our clock-pooling equilibria and in the 2013 Austrian auction) are examples of phenomena where our equilibria clearly coincide with the limited information we have about real-world spectrum auctions.

This paper has studied a simplified model where additional units have a decreasing marginal value and complementarities between units do not play a role. We think this is a useful step in better understanding the equilibrium properties of the CCA in a more complicated (multi-band) setting. Obviously, to what extent the results of this paper extend to more complicated settings should be seen in follow-up research. Also, we want to stress that this paper should not be read as saying that the CCA should not be used in practice, as it may be the case that there is no alternative mechanism that performs better than the CCA. The main aim of the current paper is to contribute to an understanding of the relative advantages of the CCA versus other auction formats by studying the properties of the equilibria of the CCA under more realistic assumptions concerning bidders' preferences.

A   Proofs

Lemma 1. *The value of the efficient allocation*

$$U(a_i, x_i^*(a_i, a_j)) + U(a_j, 1 - x_i^*(a_i, a_j))$$

is increasing in a_i and in a_j.

Proof. Let $V(a_i, a_j) = U(a_i, x_i^*(a_i, a_j)) + U(a_j, 1 - x_i^*(a_i, aj))$. The gradient of V is

$$\nabla V(a_i, a_j) = (x_i^*, 1 - x_i^*) > 0$$

by virtue of the assumption that the efficient allocation is interior. □

Proposition 1. *For any $(c_1, c_2) \in \left[0, \frac{(a - \bar{a} + b)^2}{4b}\right]^2$ there exists a Bayesian Nash equilibrium, where, for $i = 1, 2$,*

$$S_i(x) = \begin{cases} U_i(x) - c_i & \text{if } x \in [\underline{x}_i, \overline{x}_i] \\ U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) - c_i - c_j & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Each of these equilibria implements the efficient allocation.

Proof. The proof is by construction. We show that the bidding function

$$S_i(x) = \begin{cases} 0 & \text{if } x < \underline{x}_i \\ U_i(x) - c_i & \text{if } x \in [\underline{x}_i, \overline{x}_i] \\ 0 & \text{if } x \in (\overline{x}_i, 1) \\ U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) - c_1 - c_2 & \text{if } x = 1. \end{cases}$$

is an equilibrium bidding function for every $(c_1, c_2) \in \left[0, \frac{(a - \bar{a} + b)^2}{4b}\right]^2$. Fix (c_1, c_2) in this interval. First, we will show that the auctioneer implements the efficient allocation. Second, we will point out that bidder's surplus is maximized globally by the efficient allocation. Third, we will demonstrate that no bidder can further raise the VCG price the other bidder has to pay.

First, we consider the implemented allocation by the auctioneer. The auctioneer maximizes the function $S_i(x_i) + S_j(1 - x_i)$. Whenever this func-

tion is differentiable, $s_i(x) = s_j(1-x)$ must hold in the optimal allocation. Since bidders bid true marginal values on relevant interior shares, the necessary condition for an interior allocation is satisfied by the efficient allocation. However, the bidding function is not concave, since bidders raise their bid on 1 relative to interior shares. As a result, the auctioneer has to check whether the value of the efficient allocation is higher than $S_i(1)$. The bid on $S_i(1)$ is constructed, however, such that it is the minimal value of the efficient allocation. Therefore it is always true that $S_i(x_i^*) + S_j(1 - x_i^*) \geq S_i(1)$.

Second, the efficient allocation maximizes bidder i's surplus, given bidder j bids according to the proposed equilibrium behavior. If bidder i wins x then either $S_j(1 - x) = 0$ or $S_j(1 - x) > 0$. In the last case, the surplus

$$U_i(x) - S_j(1) + S_j(1 - x) = U_i(x) - U_j(\overline{x}_j) - \underline{U}(1 - \overline{x}_j) + c_j + U_j(1 - x)$$

is maximized if $s_i(x) = u_i(x) = u_j(1-x) = s_j(1-x)$. The efficient allocation satisfies this condition. Now we consider the case where $S_j(1 - x) = 0$. If bidder i wins x such that $S_j(1 - x) = 0$, then bidder i pays the VCG price $S_j(1)$ for x. Thus, if bidder i does not want to win 1 at a VCG price of $S_j(1)$ then he also does not want to win any smaller share at this VCG price. It remains to compare the surplus from the efficient outcome to the surplus from winning 1. For the bidding function to constitute an equilibrium, it must be true that for all types i and j the surplus from winning 1 must be smaller than the surplus from the efficient share, that is,

$$U_i(x_i^*) - S_j(1) + S_j(1 - x_i^*) \geq U_i(1) - S_j(1)$$

must be true. This inequality can be transformed to

$$U_i(x_i^*) + U_j(1 - x_i^*) - U_i(1) \geq c_j.$$

As the LHS of this inequality is minimized if bidder i has the highest possible type and bidder j is as weak as possible, it follows that this inequality holds if

$$\overline{U}(\overline{x}) + \underline{U}(\underline{x}) - \overline{U}(1) = \overline{U}(1 - \underline{x}) + \underline{U}(\underline{x}) - \overline{U}(1) \geq c_j.$$

As the LHS equals $\frac{(\underline{a}-\overline{a}+b)^2}{4b}$ this condition holds if the values of c_j are in the interval. In conclusion, bidder j never shades the bids too much such that bidder i wants to win everything and therefore the efficient allocation is the surplus maximizing allocation for bidder i.

Third, no bidder can further raise the rival's cost since the value of the efficient allocation equals $U_i(x_i^*) + U_j(1 - x_i^*) - c_i - c_j$, which could be as low as $U_i(\overline{x}_i) + \underline{U}(1 - \overline{x}_i) - c_i - c_j$. If a bidder would increase the bid on the full supply, there would be a positive probability for this bid to become winning, resulting in a decrease in the bidder's expected surplus from the auction.

$\square$

Proposition 2. *If* $\overline{a} - \underline{a} < \left(\frac{1}{2} - \frac{1}{4}\sqrt{2}\right) b$, *there exists a continuum of symmetric efficient clock-pooling equilibria, where in the clock bidders demand according to function (4), that is, they demand the full supply for prices lower than* $\tilde{p}$, *with* $\underline{a} - \frac{b}{4} \geq \tilde{p} \geq \frac{\overline{a} + \underline{a}(1+\sqrt{2}) - b}{2+\sqrt{2}}$, *and then drop demand to* $\tilde{x}_i = \frac{a_i - \tilde{p}}{b}$. *In the supplementary phase bidders submit the bidding function (6).*

Proof. If the clock ends at $\tilde{p}$, no deviation is profitable in the supplementary phase. The efficient share is implemented and this share maximizes surplus in the interior. No bidder wants to win everything under truthful bidding, since

$$U_i(x_i^*) + U_j(1 - x_i^*) - S_j(1) \geq U_i(1) - S_j(1),$$

by construction of the efficient allocation.

No deviation in the clock phase results in a higher expected surplus. At $\tilde{p}$, no bidder can improve his surplus by not demanding truthfully, since the efficient allocation is implemented and the other bidder has to already pay the highest possible price at this level of information. Ending the clock by dropping demand to 0 at $p < \tilde{p}$ yields zero surplus. Dropping demand to some positive share limits the possibilities to raise $S_i(1)$ and does not yield a higher surplus. Bidders cannot improve on the final allocation by expanding demand until a price higher than $\tilde{p}$. However, they might try to learn something new about the other bidder's type and use this information to raise the other bidder's price. Since the other bidder is bidding truthfully, the only way to get new information is to keep demand at 1 and wait until

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the lowest type drops demand to 0. The deviating bidder cannot demand 1 at a price of at least $\underline{a}$, but he still needs to create excess demand in order to learn something about the other bidder's type. However, he cannot create excess demand without the risk of winning the demanded share by the clock ending with market clearing.

As a result, neither ending the clock earlier nor later pays off. No deviation in the clock and the supplementary phase is therefore profitable.

□

Lemma 2. *In any symmetric efficient equilibrium of the CCA, the image of the function $p(a_i)$ is either a singleton or a pair.*

Proof. We look at monotone equilibria, that is, $S_i^p(x) \geq S_j^p(x)$ for all x and $a_i \geq a_j$ if $p(a_i) \geq p(a_j)$.

Before we prove the Lemma, we proof an auxiliary Lemma. The Lemma says that there cannot be an efficient equilibrium if the price $p(\underline{a})$ is so high that, if the clock ends at this price, the efficient share for low types a_i is less than the true demand. We will show that if the true demand is indeed less than the lowest efficient share, then these types can make the CCA price dependent on the final clock round price and this induces other types to deviate from equilibrium play.

Lemma 3. *Let $p([\underline{a}, \overline{a}')) = \tilde{p}^1$ and $p([\overline{a}', \overline{a}'']) = \tilde{p}^2$, where $\underline{a} < \overline{a}' < \overline{a}'' \leq \overline{a}$. There is no efficient equilibrium if there exists a $\delta > 0$ such that for all $a_i \in [\underline{a}, \underline{a} + \delta)$ the truthful demand at $\tilde{p}^1$ is less than the lowest efficient share at $\tilde{p}^1$, i.e. if $\tilde{x}_i^1 = \frac{a_i - \tilde{p}^1}{b} < \frac{a_i - \overline{a}' + b}{2b} = x_i'$.*

Proof. Suppose there is an efficient equilibrium where this is the case. We consider two cases. First, it can be the case that $x_i'' = \frac{a_i - \overline{a}'' + b}{2b} < \tilde{x}_i^1 < x_i'$ for types close to $\underline{a}$. Otherwise, it has to be that case that $\tilde{x}_i^1 < x_i'' < x_i'$ for types close to $\underline{a}$. In these two cases bidders have different opportunities to raise rivals' cost. In any case, the boundary type $\overline{a}'$ must be indifferent between meeting a weak type at $\tilde{p}^1$ and $\tilde{p}^2$ in the first dimension of the preferences. If the boundary type were not indifferent, there would be an open set of types that wants to deviate by dropping demand earlier or later.

We first consider the case $x_i'' < \tilde{x}_i^1 < x_i'$. The structure of the argument is as follows. First, we show what the supplementary bidding function at $\tilde{p}^2$

for weak types must look like. As the boundary type $\bar{a}'$ must be indifferent between dropping demand at $\tilde{p}^1$ and $\tilde{p}^2$, low types must basically use the same supplementary bidding functions at $\tilde{p}^1$ and $\tilde{p}^2$. The supplementary bidding function after the clock finishes at $\tilde{p}^2$ then determines the supplementary bidding function after the clock finishes at $\tilde{p}^1$. Second, low types want to raise rival's cost at $\tilde{p}^1$ and we use this to pin down the supplementary bidding function of the lowest type $\underline{a}$. Third, we show that other low types can raise rival's cost by lowering the bids on possibly efficient shares. This is allowed by the activity rule and does not change the final allocation. The CCA price rivals have to pay when the clock finishes at $\tilde{p}^1$ can then be raised relative to the price rivals have to pay when the clock finishes at $\tilde{p}^2$ implying the boundary type $\bar{a}'$ is no longer indifferent.

First, we will argue how the supplementary bidding function of low types must look like. At $\tilde{p}^2$, in order to implement all possible efficient allocations low types have to bid true marginal values on $[x_i'', x_i']$. Note that in this first case $\tilde{x}_i^1 \in (x_i'', x_i')$ and that the relative cap prescribes $S_i(1) \leq S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1)$. It should hold that $S_i(1) = S_j(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1)$, as otherwise the bid on 1 would be different relative to the bid on the efficient share at $\tilde{p}^1$ and $\tilde{p}^2$ respectively, and $\bar{a}'$ will not be indifferent between dropping demand at $\tilde{p}^1$ and $\tilde{p}^2$. In order to implement the same CCA price for the rival bidder after the clock stops at $\tilde{p}^1$, bidder i has, in addition, to bid

$$S_i(x) = U_i(x) - U_i(\tilde{x}_i^1) + S_i(\tilde{x}_i^1) \tag{14}$$

for $x \in [\tilde{x}_i^1, \overline{x}_i]$. In this case, the relation of the bid $S_i(1)$ to the bid on the efficient share is the same.[26] Note that the level of the bidding function can be different, but this does not play a role in the determination of prices.

Second, we do now pin down the level of the bidding function for the lowest type $\underline{a}$. If two bidders with type $a_j = \underline{a}$ meet, then $2S_j(\frac{1}{2}) = S_j(1)$ must hold. If $2S_j(\frac{1}{2}) < S_j(1)$, the efficient allocation would not be implemented. If $2S_j(\frac{1}{2}) > S_j(1)$, low types would want to lower bids on efficient shares in order to raise the competitor's price. From equation (14) it follows

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[26]Types in $[\underline{a} + \delta, a')$ make supplementary bids in $S_i(1) = S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1)$ and $S_i(x) = U_i(x) - U_i(\tilde{x}_i^1) + S_i(\tilde{x}_i^1)$ for the relevant shares afte the clock finishes at $\tilde{p}^1$ and $\tilde{p}^2$, respectively. The CCA price the rival has to pay is consequently the same at both endings of the clock.

that

$$2S_j\left(\frac{1}{2}\right) = 2\left(U_j\left(\frac{1}{2}\right) - U_j\left(\tilde{x}_j^1\right) + S_j\left(\tilde{x}_j^1\right)\right) = S_j(1)$$

$$= S_j\left(\tilde{x}_j^1\right) + \tilde{p}^1\left(1 - \tilde{x}_j^1\right) \Leftrightarrow$$

$$S_j(\tilde{x}_j^1) = \tilde{p}^1(1 - \tilde{x}_j^1) - 2\left(U_j\left(\frac{1}{2}\right) - U_j(\tilde{x}_j^1)\right).$$

Note that this bid satisfies the activity rule, since it is higher than the expressed value for $\tilde{x}_j^1$ in the clock, i.e.,

$$S_j(\tilde{x}_j^1) = \tilde{p}^1(1 - \tilde{x}_j^1) - 2\left(U_j\left(\frac{1}{2}\right) - U_j\left(\tilde{x}_j^1\right)\right) > \tilde{p}^1\tilde{x}_j^1 \Leftrightarrow$$

$$\tilde{p}^1\left(\frac{1}{2} - \tilde{x}_j^1\right) > U_j\left(\frac{1}{2}\right) - U_j(\tilde{x}_j^1).$$

Third, we now consider bidder i with type $a_i > \underline{a}$, but sufficiently close to $\underline{a}$ so that $\tilde{x}_i^1 < x_i'$ continues to hold. As the value of the efficient allocation is increasing in types, if he bids in such a way that he still gets an interior solution when meeting the weakest possible type $\underline{a}$, then the same would hold when meeting other types. If he meets the lowest possible type $a_j = \underline{a}$ and both bidders use bidding function (14), then bidder i can lower the bids on $[x_i', \overline{x}_i]$, because the efficient allocation would still be implemented, but the CCA price would be higher. To see this, note that

$$S_i(1) = S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) < S_i(x_i^*) + S_j(1 - x_i^*) \Leftrightarrow$$

$$S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) < U_i(x_i^*) - U_i(\tilde{x}_i^1) + S_i(\tilde{x}_i^1) + U_j(1 - x_i^*) - U_j(\tilde{x}_j^1) + S_j(\tilde{x}_j^1) \Leftrightarrow$$

$$\tilde{p}^1(\tilde{x}_j^1 - \tilde{x}_i^1) + 2U_j\left(\frac{1}{2}\right) < U_i(x_i^*) + U_j(1 - x_i^*) + U_j(\tilde{x}_j^1) - U_i(\tilde{x}_i^1) \Leftrightarrow$$

$$\tilde{p}^1\frac{a_j - a_i}{b} + a_j - \frac{b}{4} < \frac{a_i^2 - 2a_ia_j + a_j^2 + 2a_ib + 2a_jb - b^2}{4b} + \frac{a_j^2 - a_i^2}{2b} \Leftrightarrow$$

$$4\tilde{p}^1(a_j - a_i) + 2ba_j < -2a_ia_j + 2a_ib + 3a_j^2 - a_i^2 \Leftrightarrow$$

$$(a_i - a_j)(a_i + 3a_j - 2b - 4\tilde{p}^1) < 0 \Leftrightarrow$$

$$a_i + 3\underline{a} < 2b + 4\tilde{p}^1.$$

As the right-hand side of the last inequality is increasing in $\tilde{p}^1$ the inequality will certainly hold if we plug in the lower bound $(a_i + a' - b)/2$. Doing so, the

inequality simplifies to $3\underline{a} < a_i + 2a'$, which is certainly true. As a result, types $a_i \in [\underline{a}, \underline{a} + \delta)$ can lower the bids on the efficient allocation without changing the final allocation. By doing so they raise the competitor's price. At $\tilde{p}^2$, however, they cannot do this, since they have to bid true marginal values around $\tilde{x}_i^1$. Thus, the CCA prices bidders have to pay against a rival with type around a' are different for when the clock finishes at $\tilde{p}^1$ and $\tilde{p}^2$, respectively, and type $\overline{a}'$ cannot be indifferent in the first dimension of the preferences between dropping demand at $\tilde{p}^1$ and at $\tilde{p}^2$.

Now consider the second case, where $\tilde{x}_i^1 < x_i'' < x_i'$ for certain low enough types $[\underline{a}, \underline{a} + \delta)$. Let $S^2(\overline{a}', x)$ be the equilibrium bidding function of type $\overline{a}'$ if the clock ends at $\tilde{p}^2 = p(\overline{a}')$. Call bidders in $[\underline{a}, \underline{a} + \delta)$ weak and bidders who drop demand at $\tilde{p}^2$ strong.

Weak bidders demand truthfully at $\tilde{p}^1 = p(\underline{a})$ because this allows them to raise the CCA price as much as possible. Demand at $\tilde{p}^1$ is increasing in type and all weak types demand less than 1/2 since $\frac{\overline{a}' - \tilde{p}^1}{b} < \frac{\overline{a}' - \overline{a}' + b}{2b} = \frac{1}{2}$. Therefore, if the clock does not end at $\tilde{p}^1$, then all weak types know that the other bidder's type is at least $\overline{a}'$.

After the clock ends at $\tilde{p}^2$ weak types bid true marginal utilities on $[x_i'', x_i']$, because these are the efficient shares that are possible. The difference with the previous case is that now $\tilde{x}_i^1$ is not in this interval. By bidding $S_i(x_i') = S_i(1) - S^2(\overline{a}', 1 - x_i')$ in the supplementary round weak types maximally increase the CCA price for the strong types. If the bids in the range $(\overline{x}_j, 1)$, i.e. $S_j(1 - \tilde{x}_i^1)$ for all a_j and a_i, are low enough, then the bid $S_i(\tilde{x}_i^1)$ can be chosen high enough such that the activity rule is satisfied. However, it can also be true that the activity rule is not satisfied by the desired bid. Thus, we have to consider two sub-cases. In both sub-cases, we set $S_i(1) = S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1)$ and vary the bid $S_i(\tilde{x}_i^1)$ such that the efficient allocation is always implemented and the activity rule is satisfied. The weak bidder's bids on $x \in [x_i'', x_i']$ are

$$S_i(x) = \min\left\{S_i(\tilde{x}_i^1) + \tilde{p}^1(x_i'' - \tilde{x}_i^1), U_i(x_i'') - U_i(x_i') + S_j(1 - x_i')\right\} + U_i(x) - U_i(x_i''),$$

if the clock ends at $\tilde{p}^2$, where $S_j(1 - x_i') = S^2(\overline{a}', 1 - x_i')$, that is, where $a_j = \overline{a}'$.

Consider first the sub-case where the minimum is attained for $U_i(x_i'') -$

$U_i(x_i') + S_j(1 - x_i')$. In this case, weak bidders use the supplementary bidding function

$$S_i(x) = U_i(x) - U_i(x_i') + S_j(1 - x_i')$$

for $x \in [x_i'', x_i']$, where $a_j = \overline{a}'$ and $S_i(1) = S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1)$. Using this supplementary bidding function raises the CCA price at $\tilde{p}^2$ of strong types as much as possible. If the clock ends at $\tilde{p}^2$ and a weak bidder a_i meets the boundary type $a_j = \overline{a}'$, then

$$S_i(x_i^*) + S_j(1 - x_i^*) = S_i(1).$$

If the other bidder has type $a_j > \overline{a}'$, then the left-hand side of the last equation is higher as it is increasing in a_j. Weak bidders cannot further lower the bids on $[x_i'', x_i']$ relative to the bid on 1 because otherwise they risk winning the full supply.

Recall that weak types must bid in such a way that the same CCA price is implemented after the clock stops at $\tilde{p}^1$. That is, $S_i(1)$ must have the same relation to $S_i(x_i^*)$ after the clock stops at $\tilde{p}^1$ and $\tilde{p}^2$. As after the clock stops at $\tilde{p}^1$ all types (know that all types) are smaller than $\overline{a}'$, the efficient allocation cannot be implemented at $\tilde{p}^1$ if weak types bid in the same way at both prices.

Consider now the sub-case where $S_i(\tilde{x}_i^1) + \tilde{p}^1(x_i'' - \tilde{x}_i^1)$ is the strict minimum for $\underline{a}$. By continuity, it is also the minimum for types a_i close enough to $\underline{a}$. We will look at the bidding behavior of these types. The sub-case occurs when the activity rule cannot simply be satisfied by raising $S_i(\tilde{x}_i^1)$, because the other bidder's bids on $(\overline{x}_j, 1)$ are too high, so that the allocation $(\tilde{x}_i^1, 1 - \tilde{x}_i^1)$ may become winning. In this case $S_i(\tilde{x}_i^1) + \tilde{p}^1(x_i'' - \tilde{x}_i^1)$ can be the minimum the bidder can bid. By the indifference argument, weak types have to use the bidding function $S_i(1) = S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1)$ and

$$S_i(x) = U_i(x) - U_i(x_i'') + S_i(\tilde{x}_i^1) + \tilde{p}^1(x_i'' - \tilde{x}_i^1)$$

for $x \in [x_i', \overline{x}_i]$ if the clock ends at $\tilde{p}^1$.

The proof follows the following two steps. First, we use the supplementary bidding behavior of the weakest type to pin down the level of the bidding function of the weakest type at $\tilde{p}^1$. Second, we show that at

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this level of the bidding function weak types want to change their bidding function after the clock ends at $\tilde{p}^1$ because they can further raise rival's cost. As a result, there is no bidding function of weak types such that the efficient allocation is always implemented and they fully raise rival's cost at both endings of the clock phase.

In order to pin down the level of the bidding function at $\tilde{p}^1$ consider the case when two identical weak types $a_j = \underline{a}$ meet. It must be true that

$$2 S_j \left(\frac{1}{2} \right) = S_j(1) \Leftrightarrow$$

$$2 \left(U_j \left(\frac{1}{2} \right) - U_j(x_j'') + S_j(\tilde{x}_j^1) + \tilde{p}^1(x_j'' - \tilde{x}_j^1) \right) = S_j(\tilde{x}_j^1) + \tilde{p}^1(1 - \tilde{x}_j^1)$$

because otherwise the types want to lower the interior of the bidding function. This equality can be transformed into

$$S_j(\tilde{x}_j^1) = \tilde{p} \left(1 + \tilde{x}_j - 2x_j'' \right) - 2 \left(U_j \left(\frac{1}{2} \right) - U_j(x_j'') \right), \tag{15}$$

which pins down the bid of the lowest type on the share $\tilde{x}_j^1$.

To show that weak types want to change their bidding function after the clock ends at $\tilde{p}^1$, note that as the value of the efficient allocation is increasing in types, if a weak bidder bids in such a way that he still gets an interior solution when meeting the weakest possible type $\underline{a}$, then the same would hold when meeting other types. Thus, consider the case where a weak type $a_i < \underline{a} + \delta$ meets $a_j = \underline{a}$. Again, from the raising rivals' cost motive it must be that $S_i(\overline{x}_i) + S_j(1 - \overline{x}_i) = S_i(1)$. If the joint bid on the efficient allocation is higher than the bid on 1, then the type a_i wants to lower the bids in the interior at $\tilde{p}^1$, i.e. the bids on $(x_i', \overline{x}_i]$. Lowering these

bids is always possible by the activity rule. Using (15) it turns out that

$$S_i(\overline{x}_i) + S_j(1 - \overline{x}_i) > S_i(1) \Leftrightarrow$$

$$S_i(\tilde{x}_i^1) + \tilde{p}^1(x_i'' - \tilde{x}_i^1) + U_i(\overline{x}_i) - U_i(x_i'') + S_j(\tilde{x}_j^1) + \tilde{p}^1(x_j'' - \tilde{x}_j^1) + U_j(1 - \overline{x}_i) - U_j(x_j'') >$$

$$S_i(\tilde{x}_i^1) + \tilde{p}^1(1 - \tilde{x}_i^1) \Leftrightarrow$$

$$U_i(\overline{x}_i) + U_j(1 - \overline{x}_i) + \tilde{p}(x_i'' - x_j'') > 2U_j\left(\frac{1}{2}\right) + U_i(x_i'') - U_j(x_j'') \Leftrightarrow$$

$$\frac{(a_i - a_j)^2 + 2a_ib + 2a_jb - b^2}{4b} - \frac{(a_i - a_j)(3a_i + 3a_j - 2a'' + 2b)}{8b} > a_j - \frac{b}{4} - \frac{\tilde{p}(a_i - a_j)}{2b} \Leftrightarrow$$

$$2(a_i - a_j)^2 + 4b(a_i - a_j) + 4\tilde{p}(a_i - a_j) > (a_i - a_j)(3a_i + 3a_j - 2a'' + 2b) \Leftrightarrow$$

$$4\tilde{p} + 2b + 2a'' > a_i + 5\underline{a}.$$

The left-hand side is increasing in $\tilde{p}^1$ implying that if the condition holds at the lower bound $(a_i + \overline{a}'' - b)/2$, then it always holds. Plugging in the lower bound gives $5\underline{a} < 4\overline{a}'' + a_i$, which is certainly true. Thus, weak bidders want to lower their bids on efficient shares relative to the bid on 1, thereby raising rivals' cost. Consequently, the boundary type is no longer indifferent between dropping demand at $\tilde{p}^1$ and $\tilde{p}^2$ and there is an open set of types who want to deviate.

$\square$

We can use Lemma 3 to prove the next Lemma.

Lemma 4. *In any efficient equilibrium, if $p(\underline{a}) > \underline{a} - b/2$, then $p(a_i) = p(\underline{a})$ must hold for all $a_i \in [\underline{a}, \overline{a}')$, where $\overline{a}' = 2p(\underline{a}) + b - \underline{a}$.*

Proof. The case of p being flat around $\underline{a}$ follows from the Lemma: suppose $a' = \sup\{a : p(a) = p(\underline{a})\} < \overline{a}'$. Then $\tilde{x}_i < x_i^*(a_i, a')$ for types sufficiently close to $\underline{a}$ and the Lemma applies. The Lemma deals with the case of p being flat around $\underline{a}$. However, the same rational also applies if p is strictly increasing. If p is strictly increasing and $p(\underline{a}) > \underline{a} - b/2$, then the lowest type drops demand truthfully at $p(\underline{a})$ to a share that is less than the lowest possible efficient share, which is $1/2$. By continuity, sufficiently weak types do the same in a neighborhood of $p(\underline{a})$, they all drop demand to less than the lowest efficient share. Since they know that the other bidder drops demand to less than $1/2$, too, they can keep demand constant at $\tilde{x}_i^{p(a_i)}$. Therefore, very low types can condition the CCA price on the ending of

the clock price like in the second case of the proof of the Lemma 3 ($\tilde{x}_i < x_i''$) and there are low types who want to end the clock earlier. □

We can now return to Lemma 2. We distinguish the following two cases. First, it can be that the case that the function p is continuous at $\overline{a}'$. It does not really matter for this case whether $p(\underline{a}) = \underline{a} - b/2$ (in which case $\overline{a}' = \underline{a}$) or $p(\underline{a}) > \underline{a} - b/2$. This is the case, for example, if bidders bid truthfully in the clock phase. Second, it can be the case that the function p is discontinuous at $\overline{a}'$. In this case it must be that $p(\underline{a}) > \underline{a} - b/2$ (as otherwise it is the case that the clock will only finish with probability 0 at prices close to $p(\underline{a})$).

We first consider the case of p being continuous at $\overline{a}'$. In this case the function must be strictly increasing at $\overline{a}'$, because otherwise the function would not be continuous or the equilibrium would not always be efficient. Let $\overline{a}'' > \overline{a}'$ be a type such that for all a_i, a_j such that $\overline{a}' \leq a_j < a_i \leq \overline{a}''$, $p(a_j) < p(a_i)$. Let $\tilde{p}^1 = p(\underline{a})$ and $\tilde{p}^2 = p(\overline{a}'')$. First, we argue that types in $[\underline{a}, \overline{a}')$ have to demand truthfully after $p(\underline{a})$. Second, we show that types in $[\overline{a}', \overline{a}'')$ want to expand demand further in order to raise rival's cost more. This can be done without a decrease of surplus in the first dimension of the preferences.

First, we show that types in $[\underline{a}, \overline{a}')$ demand truthfully after $\tilde{p}^1 = p(\underline{a})$. Suppose types close to $\underline{a}$ keep demand constant for prices slightly higher than $p(\underline{a})$. Then the clock does not end if types above $\overline{a}'$ drop demand (truthfully). If they reduce demand, then the clock might end too soon and only an inefficient allocation can be implemented. Therefore, they have to lower demand truthfully. But if types close to $\underline{a}$ lower demand truthfully, other types have to lower demand truthfully, too, because otherwise the final allocation might be inefficient. Second, consider the following deviation by $a_i \in [\overline{a}', \overline{a}'')$. Instead of demanding truthfully at $p(a_i)$, type a_i expands demand further and drops demand only at $\tilde{p}^2$. By doing so he further weakens the constraints of the activity rule and can increase his utility in the second dimension of the lexicographic preferences. The surplus in the first dimension of the preferences remains unchanged, however. If the clock ends at $\tilde{p}^2$, then the clock would have ended without demand expansion at a price in the interval $[p(a_i), \tilde{p}^2]$ with market clearing. The

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other bidder j demanded his efficient share at this price and expressed true marginal utilities for shares in $[x_j(\tilde{p}^2), x_j(\tilde{p}^1)]$ throughout the clock. When the clock ends, bidder j will fully raise the bids on these shares in order to raise rival's costs most. Thus, he bids true marginal utilities on these shares in the supplementary phase. Bidder i bids true marginal values on shares in $[\tilde{x}_i^2, \overline{x}_i]$, which, together with the right level of the bidding function of bidder i, implements the efficient share. The price bidder i has to pay must not change, because otherwise bidder j is able to make the CCA price dependent on the final clock price. In this case high types would have an incentive to reduce demand in order to avoid the raise in the CCA price. To summarize, there is an open set of types that wants to expand demand further in order to raise rival's cost more without changing the surplus in the first dimension of the preferences.

The second case is when the function p is discontinuous at $\overline{a}'$. We will show there cannot be more than two prices in the image of $p(a_i)$. Three prices in the image of p are enough to show that there are high types that want to reduce demand to an inefficient share. Let $\tilde{p}^1 < \tilde{p}^2 < \tilde{p}^3$, where $p([\underline{a}, \overline{a}^1)) = \{\tilde{p}^1\}$, $p([\overline{a}^1, \overline{a}^2)) = \{\tilde{p}^2\}$, and $p([\overline{a}^2, \overline{a})) = \{\tilde{p}^3\}$. By Lemma 3 we know that all types in $[\underline{a}, \overline{a}^1)$ must demand at least the lowest possible efficient share at $\tilde{p}^1$.

For δ small enough, there must exist a $p' < \tilde{p}^2$ such that all types in $[\underline{a}, \underline{a} + \delta)$ learn that if the clock does not finish at a price $\tilde{p} \leq p'$ their opponent is a type $a_i > \overline{a}'$. This is, for example, the case in our two-step equilibrium where bidders bid truthfully and the market clears at prices just above $\tilde{p}^1$. In that case types in the interval $[\underline{a}, \underline{a}+\delta)$ will continue to demand x' (which is what they were demanding at price p') as long as $p \in [p', \tilde{p}^2)$. In the supplementary round, this bidder will use different supplementary round bid functions, depending on whether the clock phase stopped at a price $\tilde{p} \leq \tilde{p}^2$ or whether the clock phase stopped at a price $\tilde{p}^3 > \tilde{p}^2$. In particular, if the clock phase stopped at a price $\tilde{p} \leq \tilde{p}^2$ these types will want to achieve an efficient allocation for all possibly efficient shares $\tilde{x}_i^2 \leq x \leq \tilde{x}_i^1$ at that price and raise their CCA price to be $\tilde{p}^1(1 - \tilde{x}_i^1) + U_i(x^*) - U_i(\tilde{x}_i^2)$. On the other hand, if the clock phase stopped at a price $\tilde{p}^3 > \tilde{p}^2$ these types will want to achieve an efficient allocation only on shares $x < \tilde{x}_i^2$ the CCA price will be raised to $U_i(x^*) - U_i(\tilde{x}_i^3) + \tilde{p}^2(\tilde{x}_i^1 - \tilde{x}_i^2) + \tilde{p}^1(1 - \tilde{x}_i^1)$. This will

give types just above $\bar{a}^2$ an incentive to reduce their bid in the clock phase to imitate types just below $\bar{a}^2$.

□

Proposition 4. *In any symmetric efficient equilibrium of the CCA, bidders bid on the full supply when the clock price is smaller than $\underline{a} - b/2$.*

Proof. In any efficient equilibrium the clock cannot end before the clock price reaches $\underline{a} - b/2$, because otherwise bidders are restricted by the relative cap to bid true marginal bids on efficient shares. In any efficient equilibrium bidders demand 1 up to the lowest price at which the clock can end. Suppose an equilibrium prescribes a bidder to drop demand below 1 before the clock can possibly end. The bidder can still demand 1 without the other bidder noticing this deviation. At the lowest price at which the clock can end he does whatever the equilibrium requires him to do. The clock will consequently stop at the same price as without the deviation. In the supplementary phase it might be necessary to raise bids on interior shares above what equilibrium would prescribe, but that is of no concern because the other bidder does not know of the deviation. Thus, the equilibrium allocation can be implemented. The gain from the deviation is a looser constraint on $S_i(1)$ which can be used to raise the final price of the other bidder. Therefore, bidders demand the full supply at least until $p = \underline{a} - b/2$ in any efficient equilibrium. □

Proposition 5. *If $b(\sqrt{2} - 1) < \bar{a} - \underline{a} < \frac{b}{2}$, then there exists an efficient two-step equilibrium where the clock demand functions are specified above and the corresponding supplementary bidding functions are (8), (9), (10) and (11).*

Proof. The only thing that remains to check is whether the activity rule is satisfied for strong bidders. Recall that they demand 1 for $p < \tilde{p}^2$ and $\tilde{x}_i^2$ at $\tilde{p}^2$. It must be true that $U_i(\bar{x}_i) + \underline{U}(1 - \bar{x}_i) \leq U_i(\tilde{x}_i^2) + \tilde{p}^2(1 - \tilde{x}_i^2)$. We have to check whether $\bar{a}^1 < \bar{a} \leq \hat{a}(\underline{a}) = \underline{a} + b$. This is true by the present assumption on the type distribution $\bar{a} - \underline{a} < b/2$. □

Proposition 6. *If $b/2 < \bar{a} - \underline{a} < b(2 - \sqrt{2})$, then there exists an inefficient two-step equilibrium.*

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Proof. We will now demonstrate that the above can indeed be an equilibrium. First, consider the stipulated behavior in the supplementary round. Given the marginal bids of all the different types at different ending of the clock phase, it is clear that the weak and strong bidders do not want to change the allocation, and as their payment is determined by the behavior of their opponent, the only thing they can do is to raise rival's cost as much as possible given that they do not want to change the allocation and given the constraints imposed by the clock behavior. This is exactly what the proposed strategies do.

Second, let us consider deviating in the clock phase. The only deviations we seriously have to consider are, first, when super-strong types demand truthfully at $\tilde{p}^2$ and thereafter, and second, any type demands the full supply at $p > \tilde{p}^2$. These are the only deviations we have to consider, because in the other cases the deviation has either no effect compared to equilibrium play or a very similar effect as the two deviations. For example, if a super-strong bidder demands more than truthful demand, but less than the full supply at $\tilde{p}^2$, he decreases the likelihood of the clock ending at $\tilde{p}^2$ even more compared to demanding truthfully, but in addition that the optimal subsequent supplementary behavior after $\tilde{p}^2$ is the same. Similarly, demanding less than $\tilde{x}_i^2$ at $\tilde{p}^2$ is not better than demanding $\tilde{x}_i^2$, since it still involves demand reduction and it does not allow the same, possibly beneficial, subsequent supplementary bidding behavior. Moreover, if a bidder has an incentive to deviate it is the super-strong bidder.

We first argue that no super-strong bidder wants to demand truthfully at $\tilde{p}^2$. If a super-strong bidder i faces a strong bidder j his equilibrium pay-off equals

$$U_i(1 - \tilde{x}_j^2) - U_j(\overline{x}_j) - \underline{U}(1 - \overline{x}_j) + U_j(\tilde{x}_j^2).$$

Given the continuation strategy of the competitor, the best possible deviation pay-off equals $U_i(x_i^*) - U_j(\tilde{x}_j^2) - \tilde{p}^2(1 - \tilde{x}_j^2) + U_j(1 - x_i^*)$. To make deviating not profitable, we will now show that the necessary inequality is equivalent to a simpler inequality which is always true. Note that $\overline{x}_j = 1/2 + \tilde{x}_j^2/2$. Thus, the inequality is equivalent to

$$U_i(1 - \tilde{x}_j^2) + 2U_j\left(\tilde{x}_j^2\right) - U_j\left(\frac{1}{2} + \frac{\tilde{x}_j^2}{2}\right) - \underline{U}\left(\frac{1}{2} - \frac{\tilde{x}_j^2}{2}\right) + \underline{a}(1 - \tilde{x}_j^2) >$$

$$U_i(x_i^*) + U_j(1 - x_i^*) \Leftrightarrow$$

$$- (1 - \tilde{x}_j^2)\left(\frac{3}{2}a_j - \frac{1}{2}\underline{a} - a_i\right) + \frac{b}{4} - \frac{5}{4}b\tilde{x}_j^{2^2} + b\tilde{x}_j^2 > \frac{(a_i - a_j + b)^2}{4b} \Leftrightarrow$$

$$- 4a_i a_j + a_j^2 + 4a_i\underline{a} + 2a_j\underline{a} - 3\underline{a}^2 + 4a_i b - 2a_j b - 2\underline{a}b + b^2 >$$

$$a_i^2 - 2a_i a_j + a_j^2 + 2a_i b - 2a_j b + b^2 \Leftrightarrow$$

$$- 2a_i a_j + 4a_i\underline{a} + 2a_j\underline{a} - 3\underline{a}^2 + 2a_i b - 2\underline{a}b > a_i^2 \Leftrightarrow$$

$$(a_i - \underline{a})(a_i + 2a_j - 3\underline{a} - 2b) < 0.$$

Since $a_i > \underline{a}$, the first term is positive. The maximum of the second term is achieved by taking the maximum admissible type a_j and as j is a strong (but not super-strong) bidder, we have $a_j - b/2 \le \underline{a}$, Substituting $a_j = \underline{a} + b/2$ yields $a_i - \underline{a} - b$, which is always negative, even if $a_i = \bar{a}$.

We now consider the case when a super-strong bidder i meets another super-strong bidder j. The equilibrium pay-off in this case equals

$$U_i\left(\frac{1}{2}\right) - U_j(\bar{x}_j) - \underline{U}(1 - \bar{x}_j) + U_j(\tilde{x}_j^2) + \tilde{p}^2\left(\frac{1}{2} - \tilde{x}_j^2\right) \tag{16}$$

and we will show that it is larger than the best possible deviation pay-off $U_i(x_i^*) + U_j(1 - x_i^*) - U_j(\tilde{x}_j^2) - \tilde{p}^2(1 - \tilde{x}_j^2)$. The inequality we have to show to hold is equivalent to

$$4a_i a_j + 4a_j^2 + 6\underline{a}^2 + 8\underline{a}b + 3b^2 > 2(a_i^2 + 6a_j\underline{a} + 4a_j b). \tag{17}$$

Taking the derivative of the left-hand side minus the right-hand side with respect to a_j gives $4a_i + 8a_j - 2(6\underline{a} + 4b)$ and this is negative if, and only if, $4a_i + 8a_j - 12\underline{a} < 8b$. This condition can only be violated if a_i and a_j are as large as possible. But if $a_i = a_j = \bar{a}$, then this condition reduces to $\bar{a} - \underline{a} < \frac{2}{3}b$, which is implied by the condition $\bar{a} - \underline{a} < b(2 - \sqrt{2})$. Taking the same derivative with respect to a_i and substituting $a_j = \bar{a}$ yields $-4a_i + 4\bar{a}$, which is non-negative. Thus, if (17) holds true for $a_i = a_j = \bar{a}$, it holds true for all type combinations. Substituting, yields

$$6(\overline{a} - \underline{a})^2 - 8b(\overline{a} - \underline{a}) + 3b^2 > 0,$$

which is always true.

As under the condition $\overline{a} - \underline{a} < b(2 - \sqrt{2})$, the clock is over at $\tilde{p}^2$ if the competitor is a weak type and the final allocation (and payment) is not affected, it follows that super-strong bidders do not have an incentive to bid truthfully at $\tilde{p}^2$.

The other deviation to consider is such that the deviating bidder wins the full supply. If the non-deviating bidder is weak, then he is able to make the winning bidder pay at least $S_j^{\tilde{p}^1}(1) = U_j(\tilde{x}_j^1) + \tilde{p}^1(1 - \tilde{x}_j^1)$. Bidder i does not want to win 1 if

$$U_i(x_i^*) - S_j^{\tilde{p}^2}(1) + U_j(1 - x_i^*) > U_i(1) - S_j^{\tilde{p}^1}(1).$$

This inequality is true since $S_j^{\tilde{p}^2}(1) \leq S_j^{\tilde{p}^1}$ and by definition of the efficient allocation. If the competitor is strong or super-strong, they are able to make the deviating bidder to pay $\tilde{p}^2 = \underline{a}$, which is larger than what the weak bidder can make him pay. In addition, the equilibrium pay-off is higher in case the competitor is weak compared to the cases where the competitor is strong or super-strong. Thus, as we have shown the super-strong bidder does not want to win the full supply against a weak bidder, he certainly does not want to win the full supply against a strong or super-strong bidder.

$\square$

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