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Abstract: It is shown that the discrete-time version of the neoclassical one-sector optimal growth model with endogenous labor supply and standard assumptions on technology and preferences admits periodic solutions of any period as well as chaotic solutions. Solutions with period 2 are possible for any time-preference factor between 0 and 1, whereas the existence of periodic solutions with other periods and the existence of chaotic solutions are only demonstrated by means of a specific example involving strong time-preference. The results are derived via constructive proofs that use Cobb-Douglas production functions.

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Key words: Optimal growth; endogenous labor supply; periodic solutions; chaotic dynamics

1 Introduction

The neoclassical one-sector growth model with infinitely-lived households and endogenous labor supply combines two of the most fundamental macroeconomic tradeoffs in a simple dynamic general equilibrium setting: the division of output between consumption and investment and the division of time between productive activities and leisure. It is therefore not surprising that this model forms the non-stochastic backbone of real business cycle theories, which have been developed to simulate the reaction of output and employment to various types of exogenous shocks. What *is* surprising, though, is that the labor-leisure tradeoff is typically disregarded in deterministic models of economic growth and that rather little is known about the structure of the solutions of these models when the labor supply is endogenous.[1]

Nevertheless, there exist a couple of papers that point to interesting properties of the solutions of the one-sector growth model with endogenous labor supply. Eriksson (1996) shows that the long-run growth rate in such an environment typically depends on the specification of the preferences (both when growth is due to exogenous technological progress and when it is generated endogenously). De Hek (1998) shows by means of a numerical example that there can be multiple (i.e., finitely many) steady states when consumption and leisure are substitutes. Kamihigashi (2015) addresses multiplicity of steady states in a more systematic way and proves that the model can have any finite number of steady states or even a continuum of steady states. Moreover, multiplicity of steady states can occur for all values of the time-preference parameter between 0 and 1 and for all production functions satisfying standard assumptions. All three of the above mentioned papers use a social planner version of the model. Sorger (2000), on the other hand, studies the model as a decentralized market economy and allows that the households differ from each other with respect to their initial capital holdings (they are assumed to be identical in all other respects). He finds that even with the standard parameterizations used in real business cycle models, there exists a continuum of steady states that differ from

[1] Eriksson (1996) writes that "The choice between work and leisure has been remarkably neglected in the theory of economic growth" [Eriksson (1996), p. 533] and even the very comprehensive and more recent survey of economic growth theory provided by Acemoglu (2009) does not discuss the case of elastic labor supply except for briefly mentioning real business cycle models in section 17.3.

each other not only with respect to the distribution of capital among households but also with respect to the level of aggregate output.

The only paper that we are aware of which discusses the emergence of more complicated dynamics is De Hek (1998), who provides an example with an asymptotically stable solution that is periodic with period 2. He concludes his paper by posing the question of "whether this model with leisure-dependent utility is able to generate more complex dynamics, in particular, chaos" [De Hek (1998), p. 270]. In the present paper we provide an affirmative answer to this question. We first extend the findings by De Hek (1998) by proving that, for any time-preference factor between 0 and 1, there exist a production function and an instantaneous utility function – both satisfying standard assumptions – such that the resulting model admits a periodic optimal solution with period 2. Then we construct an economy for which there exists an optimal solution that has period 3. The existence of a solution with period 3 is known to imply the existence of periodic solutions of all periods [see Sarkovskii (1964)] and it implies the occurrence of topological chaos [see Li and Yorke (1975)].

We use the same approach for the two cases with period 2 and period 3, respectively. The first step is to derive necessary and sufficient conditions on the time-preference factor β and the production function f for the existence of an instantaneous utility function u such that the economy defined by f, u, and β admits a given periodic solution. In a second step we show that this necessary and sufficient condition can be satisfied even if one restricts the technology to be of Cobb-Douglas type. For the case of period 2, this approach works for any value of the time-preference parameter β. For the case of period 3, however, the construction works only for very strong time-preference. This had to be expected because of the results by Mitra (1996) and Nishimura and Yano (1996), which imply that solutions with period 3 can occur in a general class of optimal growth models (including the one-sector optimal growth model with endogenous labor supply studied in the present paper) only if the time-preference factor β is smaller than $(3-\sqrt{5})/2 \approx 0.38$.

The rest of the paper is organized as follows. Section 2 describes the model, section 3 presents and discusses the main results and their implications, and section 4 contains the proofs.

2 The model

We consider an infinite-horizon economy in which capital and labor are used to produce a single output good that can be consumed or invested. A social planner seeks to maximize welfare, which depends on the allocation of output to consumption and investment and on the allocation of time to labor and leisure. In the present section we describe the model, state the assumptions, and define feasible and optimal allocations.

Time evolves in discrete periods $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Let us denote by k_t and ℓ_t the period-t factor inputs of capital and labor, respectively, and by $f(k_t, \ell_t)$ the amount of output that is available in period t. This amount consists of output produced in period t plus non-depreciated capital from the previous period. The production function f satisfies the following assumption.

Assumption 1 (i) The function $f : \mathbb{R}_+ \times [0, 1] \mapsto \mathbb{R}_+$ is continuous, concave, homogeneous of degree 1, and continuously differentiable on $\mathbb{R}_{++} \times (0, 1]$.
(ii) For every $k > 0$ it holds that $f(k, \ell)$ is strictly increasing and strictly concave with respect to ℓ.
(iii) For every $\ell \in (0, 1]$ it holds that $f(k, \ell)$ is strictly increasing and strictly concave with respect to k.
(iv) There exists $\bar{k} > 0$ such that $f(\bar{k}, 1) = \bar{k}$.

Output can be used for consumption and for investment and we denote by c_t the amount that is consumed in period t. A sequence $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ is called a *feasible allocation* if the conditions

$$c_t + k_{t+1} = f(k_t, \ell_t),$$
$$0 \leq \ell_t \leq 1,$$
$$c_t \geq 0,$$
$$k_t \geq 0$$

hold for all $t \in \mathbb{N}_0$.

Throughout the paper we assume without further mentioning that the initial endowment of the economy with capital, k_0, is such that $k_0 \in [0, \bar{k}]$. Together with assumption 1 this implies that every feasible allocation satisfies $k_t \in [0, \bar{k}]$ and $c_t \in [0, \bar{k}]$ for all $t \in \mathbb{N}_0$. When we specify the preferences over allocations, we may therefore restrict the domain of the instantaneous utility function accordingly.

The economy is endowed with a single unit of time per period such that $1 - \ell_t$ denotes the time that is available for leisure. The preferences of the social planner are described by the welfare functional

$$\sum_{t=0}^{+\infty} \beta^t u(c_t, 1 - \ell_t), \tag{1}$$

where u is an instantaneous utility function depending on consumption and leisure and where β is a time-preference factor.

Assumption 2 The function $u : [0, \bar{k}] \times [0, 1] \mapsto \mathbb{R}$ is continuous, strictly increasing, and strictly concave.

Assumption 3 It holds that $\beta \in (0, 1)$.

An economy is a triple (f, u, β). Suppose that an economy (f, u, β) and an initial capital endowment $\kappa \in [0, \bar{k}]$ are given. A feasible allocation $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ is said to be *interior* if $(k, \ell_t, c_t) \in (0, \bar{k}) \times (0, 1) \times (0, \bar{k})$ holds for all $t \in \mathbb{N}_0$, and it is said to be *optimal from* κ, if it maximizes the welfare functional (1) over all feasible allocations with the given initial capital endowment $k_0 = \kappa$. A feasible allocation $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ is called an *optimal allocation of the economy* (f, u, β), if there exists an initial endowment $\kappa \in [0, \bar{k}]$ such that $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ is an optimal allocation from κ.

3 The results and their implications

In this section we investigate whether there exist economies (f, u, β) that admit optimal allocations which are periodic of period 2 or 3. Our approach is as follows. We first derive conditions on the production function f and the time-preference factor β which are necessary and sufficient for the existence of an instantaneous utility function u such that the economy (f, u, β) satisfies assumptions 1-3 and such that a given allocation qualifies as a periodic optimal allocation for

(f, u, β). Then we show that these necessary and sufficient conditions can be satisfied even by a Cobb-Douglas production function.

We start with the case of period 2. A feasible allocation $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ is said to be *periodic of period 2* if there exist real numbers k_a, k_b, ℓ_a, ℓ_b, c_a, and c_b such that $(k_i, \ell_i, c_i) \in [0, \bar{k}] \times [0, 1] \times [0, \bar{k}]$ for $i \in \{a, b\}$, $k_a \neq k_b$, and

$$(k_t, \ell_t, c_t) = \begin{cases} (k_a, \ell_a, c_a) & \text{if } t \equiv 0 \bmod 2, \\ (k_b, \ell_b, c_b) & \text{if } t \equiv 1 \bmod 2 \end{cases} \tag{2}$$

hold. We have the following theorem.[2]

Theorem 1 *Let f and β be given such that assumptions 1 and 3 hold. Suppose furthermore that there exist real numbers k_a, k_b, ℓ_a, and ℓ_b with $k_a \neq k_b$ such that $(k_i, \ell_i) \in (0, \bar{k}) \times (0, 1)$ holds for $i \in \{a, b\}$. The following two statements are equivalent:*

(a) *There exists an instantaneous utility function u satisfying assumption 2 and real numbers $c_a \in (0, \bar{k})$ and $c_b \in (0, \bar{k})$ such that the allocation defined by (2) is an optimal allocation for the economy (f, u, β).*

(b) *It holds that*

$$\beta^2 f_1(k_a, \ell_a) f_1(k_b, \ell_b) = 1 \tag{3}$$

and

$$[1 - \beta f_1(k_a, \ell_a)][f(k_b, \ell_b) + k_b - k_a - f_1(k_a, \ell_a)k_a - f_2(k_a, \ell_a)\ell_b] > 0. \tag{4}$$

The proof of the theorem can be found in section 4.1. Note that the conditions stated in part (b) of the theorem involve only the production function f and the time-preference factor β. We will now show that these conditions can be satisfied for any feasible value of $\beta \in (0, 1)$ even if one restricts the technology to be described by a Cobb-Douglas production function.

Example 1 Let $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ be arbitrary constants and define $A = 1/(\alpha\beta)$. We consider a Cobb-Douglas production function of the form $f(k, \ell) = Ak^\alpha \ell^{1-\alpha}$. It holds that

[2]Throughout the paper we denote partial derivatives by subscripts. For example, $u_1(c_a, 1 - \ell_a)$ is the partial derivative of the instantaneous utility function u with respect to its first argument evaluated at the point $(c_a, 1 - \ell_a)$. Analogously, $f_2(k_a, \ell_a)$ is the partial derivative of the production function f with respect to its second argument evaluated at the point (k_a, ℓ_a).

$f_1(k,\ell) = \alpha A(k/\ell)^{\alpha-1}$ and $f_2(k,\ell) = (1-\alpha)A(k/\ell)^{\alpha}$. Suppose furthermore that $k_a/\ell_a = \gamma$, where $\gamma > 1$ is arbitrary. Because of $A = 1/(\alpha\beta)$, condition (3) is satisfied if and only if $k_b/\ell_b = 1/\gamma < 1$. It follows that $k_a = \gamma\ell_a$ and $k_b = (1/\gamma)\ell_b$. Note furthermore that $\beta f_1(k_a,\ell_a) = (k_a/\ell_a)^{\alpha-1} < 1$. This implies that condition (4) is satisfied if and only if the term in the second bracket is strictly positive. Using the Cobb-Douglas specification and the results mentioned so far, we can express the positivity of the term in the second bracket of (4) as $T_1\ell_b > T_2\ell_a$, where

$$T_1 = A\gamma^{-\alpha} + \gamma^{-1} - (1-\alpha)A\gamma^{\alpha} \text{ and } T_2 = \gamma + \alpha A\gamma^{\alpha}.$$

It is obvious that T_2 is positive. As for T_1, we see that $\lim_{\gamma\to 1} T_1 = 1 + \alpha A > 0$ such that one can always find a value $\gamma > 1$ for which T_1 becomes strictly positive. But if both T_1 and T_2 are positive, then one can find two numbers $\ell_a \in (0,1)$ and $\ell_b \in (0,1)$ such that the above inequality $T_1\ell_b > T_2\ell_a$ holds: just let ℓ_b be an arbitrary element from the interval $(0,1)$ and choose ℓ_a from the non-empty interval $(0, T_1\ell_b/T_2)$. Hence, we have demonstrated that, for every time-preference factor $\beta \in (0,1)$, there exists a Cobb-Douglas production function f and numbers k_a, k_b, ℓ_a, ℓ_b with $k_a \neq k_b$ such that condition (b) of theorem 1 holds.

Together with theorem 1 the above example gives rise to the following corollary.

Corollary 1 *For every $\beta \in (0,1)$ there exists a Cobb-Douglas production function f and an instantaneous utility function u satisfying assumption 2 such that the economy (f,u,β) admits an optimal allocation which is periodic of period 2.*

This finding is more general than that of De Hek (1998) in the sense that it guarantees the existence of optimal allocations of period 2 for all time-preference factors $\beta \in (0,1)$. On the other hand, we do not say anything about the stability of these optimal allocations. The stability properties would depend on the second-order derivatives of both the production function and the instantaneous utility function. Since these derivatives are not restricted except for being negative, we believe that optimal allocations of period 2 would exist for all time-preference factors $\beta \in (0,1)$ even if we were to impose their stability as an additional requirement.

Let us now turn to the case of period 3. A feasible allocation $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ is *periodic of period 3* if there exist real numbers k_a, k_b, k_c, ℓ_a, ℓ_b, ℓ_c, c_a, c_b, and c_c such that k_a, k_b, and k_c are mutually different from each other and such that

$$(k_t, \ell_t, c_t) = \begin{cases} (k_a, \ell_a, c_a) & \text{if } t \equiv 0 \text{ mod } 3, \\ (k_b, \ell_b, c_b) & \text{if } t \equiv 1 \text{ mod } 3, \\ (k_c, \ell_c, c_c) & \text{if } t \equiv 2 \text{ mod } 3 \end{cases} \tag{5}$$

holds. To state our main result regarding optimal allocations of period 3, we need to introduce some notation. Let us consider an allocation of the form (5) and define

$$\begin{aligned}
F_{acb} &= f(k_a, \ell_a) + k_c - k_b - f_1(k_b, \ell_b)k_b - f_2(k_b, \ell_b)\ell_a, \\
F_{bac} &= f(k_b, \ell_b) + k_a - k_c - f_1(k_c, \ell_c)k_c - f_2(k_c, \ell_c)\ell_b, \\
F_{cba} &= f(k_c, \ell_c) + k_b - k_a - f_1(k_a, \ell_a)k_a - f_2(k_a, \ell_a)\ell_c, \\
G_{aab} &= f(k_a, \ell_a) + k_a - k_b - f_1(k_c, \ell_c)k_c - f_2(k_c, \ell_c)\ell_a, \\
G_{bbc} &= f(k_b, \ell_b) + k_b - k_c - f_1(k_a, \ell_a)k_a - f_2(k_a, \ell_a)\ell_b, \\
G_{cca} &= f(k_c, \ell_c) + k_c - k_a - f_1(k_b, \ell_b)k_b - f_2(k_b, \ell_b)\ell_c.
\end{aligned}$$

The proof of the following theorem is presented in section 4.2.

Theorem 2 *Let f and β be given such that assumptions 1 and 3 hold. Suppose furthermore that there exist real numbers k_a, k_b, k_c, ℓ_a, ℓ_b, and ℓ_c such that $k_a \neq k_b$, $k_b \neq k_c$, $k_a \neq k_c$, and $(k_i, \ell_i) \in (0, \bar{k}) \times (0, 1)$ hold for all $i \in \{a, b, c\}$. The following two statements are equivalent:*

(a) *There exists an instantaneous utility function u satisfying assumption 2 and real numbers $c_a \in (0, \bar{k})$, $c_b \in (0, \bar{k})$, and $c_c \in (0, \bar{k})$ such that the allocation defined by (5) is an optimal allocation for the economy (f, u, β).*

(b) *It holds that*

$$\beta^3 f_1(k_a, \ell_a) f_1(k_b, \ell_b) f_1(k_c, \ell_c) = 1, \tag{6}$$

$$\beta^2 f_1(k_a, \ell_a) f_1(k_c, \ell_c) F_{acb} + G_{bbc} > 0, \tag{7}$$

$$\beta f_1(k_c, \ell_c) G_{cca} + F_{bac} > 0, \tag{8}$$

$$\beta f_1(k_a, \ell_a) G_{aab} + F_{cba} > 0, \tag{9}$$

$$\beta^2 f_1(k_a,\ell_a)f_1(k_c,\ell_c)G_{cca} + \beta f_1(k_a,\ell_a)G_{aab} + G_{bbc} > 0, \tag{10}$$

$$\beta^2 f_1(k_a,\ell_a)f_1(k_c,\ell_c)F_{acb} + \beta f_1(k_a,\ell_a)F_{bac} + F_{cba} > 0. \tag{11}$$

Having derived the necessary and sufficient conditions on f and β for the existence of an economy (f,u,β) that admits an optimal allocation of period 3, we now show that these conditions can be satisfied even if we restrict the technology to be of the Cobb-Douglas variety. This is the purpose of the following example.[3]

Example 2 Let us specify the production function by $f(k,\ell) = Ak^\alpha \ell^{1-\alpha}$ with $\alpha = 1/3$ and the time-preference factor by $\beta = 7/20$. Furthermore, we choose the allocation variables $k_a = 1507$, $k_b = 2143$, $k_c = 3200$, $\ell_a = 2/5$, $\ell_b = 57/100$, and $\ell_c = 21/25$. The productivity parameter A is determined in such a way that condition (6) holds, which yields a unique value $A \approx 2079$. With these specifications it is straightforward to verify that conditions (7)-(11) hold.[4]

From theorem 2 and example 2 we obtain the following corollary.

Corollary 2 *There exists a Cobb-Douglas production function f, an instantaneous utility function u satisfying assumption 2, and a time-preference factor β satisfying assumption 3 such that the economy (f,u,β) admits an optimal allocation which is periodic of period 3.*

To explain the implications of the above results let an economy (f,u,β) be given such that assumptions 1-3 hold. It follows from standard results on dynamic programming[5] that, for every initial capital endowment $\kappa \in [0,\bar{k}]$, there exists a unique optimal allocation from κ. Moreover, there exists a continuous function $h : [0,\bar{k}] \mapsto [0,\bar{k}]$ such that the set of all optimal allocations of the economy (f,u,β) coincides with the set of all trajectories of the difference equation

$$k_{t+1} = h(k_t) \text{ for all } t \in \mathbb{N}_0 \tag{12}$$

which start in initial states $k_0 \in [0,\bar{k}]$. The function h is called the optimal policy function for (f,u,β). Equation (12) says that the capital stocks in every optimal allocation form a

[3]This is just one of several examples that we have found.

[4]The verification has been executed with the software Mathematica®.

[5]See, e.g., Stokey and Lucas (1989), Miao (2014), or Sorger (2015).

trajectory of a continuous dynamical system defined on the one-dimensional compact state space $[0, \bar{k}]$. When the economy (f, u, β) admits an optimal allocation of period 3, it follows that the corresponding difference equation (12) has a periodic trajectory with period 3. Continuous dynamical systems which are defined on a one-dimensional state space, such as (12), are very well-studied and it is known that the existence of periodic solutions with period 3 has strong implications.[6] First, according to Sarkovskii (1964) it follows that a difference equation that admits a periodic solution of period 3 admits periodic solutions of all periods $p \in \mathbb{N}$. Second, according to Li and Yorke (1975), the existence of a periodic solution of period 3 implies that the dynamical system (12) exhibits topological chaos. This means in particular that there exists an uncountable set S of initial capital endowments such that the unique optimal allocation emanating from any $\kappa \in S$ is neither periodic nor asymptotically periodic and that, for any pair $\{\kappa, \kappa'\} \subseteq S$, the two optimal allocations starting in κ and κ', respectively, become arbitrarily close to each other without converging to each other. Hence, theorem 2 and its corollary prove that the neoclassical one-sector growth model with endogenous labor supply can generate very complicated dynamics and that its optimal allocations can display sensitive dependence on initial conditions.

Whereas we could establish the existence of optimal allocations of period 2 for any feasible time-preference factor $\beta \in (0, 1)$, this is not possible in the case of optimal allocations of period 3. It is known from Mitra (1996) and Nishimura and Yano (1996) that an optimal allocation with period 3 can only exist if $\beta < (3 - \sqrt{5})/2 \approx 0.38$. The economy presented in example 2 features $\beta = 0.35$. We do not know whether the bound $(3 - \sqrt{5})/2$ is sharp for the class of models under consideration.

Both theorems 1 and 2 are established by a constructive proof. The crucial step in this construction is the specification of the instantaneous utility function u. We define it as the minimum of two (in the case of theorem 1) or three (in the case of theorem 2) strictly concave quadratic polynomials in consumption and leisure. The construction is non-trivial because we have to ensure that the first-order optimality conditions for the economy (f, u, β) are satisfied along the given periodic allocation. This gives rise to the conditions stated in part (b) of the theorem

[6] See, e.g., Collet and Eckmann (1980), De Melo and Van Strien (1993), or Sorger (2015).

1 or 2, respectively. Due to the boundedness and convexity imposed in assumptions 1-2 and the strict time-preference resulting from assumption 3, the first-order conditions are sufficient for optimality.

Finally, the reader may be wondering why we impose smoothness of the production function f in assumption 1 but do not make an analogous assumption on the instantaneous utility function u. The reason is that the function u employed in the constructive proofs of theorems 1 and 2, respectively, is the minimum of finitely many smooth polynomials. This function is not differentiable along the one-dimensional manifolds at which the graphs of the polynomials intersect. It is possible, however, to replace the function u in these proofs by a smooth function satisfying assumption 2 without violating any of the first-order conditions. This is the case because the first order conditions involve the partial derivatives of the function u only at the (two or three) points along the allocation, and because these points are separated from the manifolds along which the non-differentiability occurs. Thus, the two theorems would remain valid if we were to strengthen assumption 2 by imposing smoothness of the instantaneous utility function.

4 The proofs

4.1 Proof of theorem 1

The following lemma presents necessary and sufficient first-order optimality conditions for an interior feasible allocation of the form (2).

Lemma 1 *Let (f, u, β) be an economy satisfying assumptions 1-3 and suppose that there exist real numbers k_a, k_b, ℓ_a, ℓ_b, c_a, and c_b such that $k_a \neq k_b$ and $(k_i, \ell_i, c_i) \in (0, \bar{k}) \times (0, 1) \times (0, \bar{k})$ for $i \in \{a, b\}$ hold and such that the utility function u is continuously differentiable locally around both of the two points $(c_a, 1 - \ell_a)$ and $(c_b, 1 - \ell_b)$. The sequence $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ defined by (2) is an optimal allocation for the economy (f, u, β) if and only if the following conditions hold:*

$$c_a + k_b = f(k_a, \ell_a), \tag{13}$$

$$c_b + k_a = f(k_b, \ell_b), \tag{14}$$

$$\beta^2 f_1(k_a,\ell_a)f_1(k_b,\ell_b) = 1, \tag{15}$$

$$u_1(c_b, 1-\ell_b) = \beta f_1(k_a,\ell_a)u_1(c_a, 1-\ell_a), \tag{16}$$

$$u_1(c_a, 1-\ell_a)f_2(k_a,\ell_a) = u_2(c_a, 1-\ell_a), \tag{17}$$

$$u_1(c_b, 1-\ell_b)f_2(k_b,\ell_b) = u_2(c_b, 1-\ell_b). \tag{18}$$

PROOF: Consider the allocation specified by (2). Because of $(k_i, \ell_i, c_i) \in (0, \bar{k}) \times (0, 1) \times (0, \bar{k})$ and (13)-(14) this allocation is feasible and interior. It is known that, under assumptions 1-3, an interior feasible allocation is an optimal allocation if and only if the first-order optimality conditions

$$u_1(c_t, 1-\ell_t) = \beta f_1(k_{t+1},\ell_{t+1})u_1(c_{t+1}, 1-\ell_{t+1}) \text{ for all } t \in \mathbb{N}_0, \tag{19}$$

$$u_1(c_t, 1-\ell_t)f_2(k_t,\ell_t) = u_2(c_t, 1-\ell_t) \text{ for all } t \in \mathbb{N}_0 \tag{20}$$

as well as the transversality condition

$$\lim_{t\to+\infty} \beta^t u_1(c_t, 1-\ell_t)k_{t+1} = 0$$

are satisfied. These conditions require differentiability of the production function f and the instantaneous utility function u locally around the allocation, which has been assumed for f in assumption 1 and for u directly in the lemma. The first-order condition (20) holds along the given allocation if and only if (17)-(18) are satisfied. The Euler equation (19) holds along the given allocation if and only if (16) as well as the corresponding equation

$$u_1(c_a, 1-\ell_a) = \beta f_1(k_b,\ell_b)u_1(c_b, 1-\ell_b) \tag{21}$$

are satisfied. Multiplying the left-hand sides and the right-hand sides of (16) and (21) we obtain (15). Conversely, if (15) holds, then one of the two Euler equations (16) and (21) is redundant. In other words, equations (16) and (21) together are equivalent to equations (15)-(16). Finally, the transversality condition holds because of the boundedness of the allocation, the interiority of $(c_i, 1-\ell_i)$ for $i \in \{a, b\}$, and assumption 3. This completes the proof of the lemma. □

Note that conditions (15)-(16) together with $k_a \neq k_b$ imply that $(c_a, \ell_a) \neq (c_b, \ell_b)$. Indeed, if $(c_a, \ell_a) = (c_b, \ell_b) = (c, \ell)$ holds, then it follows from (15)-(16) that $f_1(k_a, \ell) = f_1(k_b, \ell) = 1/\beta$, which contradicts $k_a \neq k_b$ due to the strict concavity of the mapping $k \mapsto f(k, \ell)$.

Equations (16)-(18) are the only conditions stated in lemma 1 which involve the instantaneous utility function. Since these equations are homogeneous in the partial derivatives $u_1(c_i, 1-\ell_i)$ and $u_2(c_i, 1-\ell_i)$ for $i \in \{a, b\}$, we may normalize these partial derivatives by setting $u_1(c_a, 1-\ell_a) = 1$. Solving (16)-(18) under this normalization yields

$$u_1(c_a, 1-\ell_a) = 1, \tag{22}$$

$$u_1(c_b, 1-\ell_b) = \beta f_1(k_a, \ell_a), \tag{23}$$

$$u_2(c_a, 1-\ell_a) = f_2(k_a, \ell_a), \tag{24}$$

$$u_2(c_b, 1-\ell_b) = \beta f_1(k_a, \ell_a) f_2(k_b, \ell_b). \tag{25}$$

It remains to find out under which conditions there exists an instantaneous utility function satisfying assumption 2 and conditions (22)-(25). To this end we prove the following auxiliary lemma.

Lemma 2 *Let X and Y be non-empty and compact intervals on the real line and let x_a, x_b, y_a, and y_b be real numbers such that $x_i \in$ int (X) and $y_i \in$ int Y hold for all $i \in \{a, b\}$ and such that $(x_a, y_a) \neq (x_b, y_b)$. Furthermore, let w_{1a}, w_{1b}, w_{2a}, and w_{2b} be positive real numbers. The following two statements are equivalent:*

(a) *There exists a function $w : X \times Y \mapsto \mathbb{R}$ which is continuous, strictly increasing, and strictly concave and which is continuously differentiable locally at the points (x_a, y_a) and (x_b, y_b) with partial derivatives*

$$w_1(x_i, y_i) = w_{1i} \text{ and } w_2(x_i, y_i) = w_{2i} \text{ for } i \in \{a, b\}.$$

(b) *The inequality*

$$(w_{1a} - w_{1b})(x_b - x_a) + (w_{2a} - w_{2b})(y_b - y_a) > 0 \tag{26}$$

holds.

Proof: We first prove that (a) implies (b). By strict concavity of w and $(x_a, y_a) \neq (x_b, y_b)$ it follows that

$$w(x_a, y_a) < w(x_b, y_b) + w_{1b}(x_a - x_b) + w_{2b}(y_a - y_b)$$

and

$$w(x_b, y_b) < w(x_a, y_a) + w_{1a}(x_b - x_a) + w_{2a}(y_b - y_a).$$

Combining these two inequalities, we obtain

$$w_{1b}(x_b - x_a) + w_{2b}(y_b - y_a) < w(x_b, y_b) - w(x_a, y_a) < w_{1a}(x_b - x_a) + w_{2a}(y_b - y_a).$$

Obviously, this implies (26).

The proof that (b) implies (a) is divided into three steps.

Step 1: Inequality (26) is equivalent to

$$w_{1b}(x_b - x_a) + w_{2b}(y_b - y_a) < w_{1a}(x_b - x_a) + w_{2a}(y_b - y_a).$$

Hence, there exist real numbers w_a and w_b such that

$$w_{1b}(x_b - x_a) + w_{2b}(y_b - y_a) < w_b - w_a < w_{1a}(x_b - x_a) + w_{2a}(y_b - y_a). \tag{27}$$

Step 2: We define for all $i \in \{a, b\}$ and all $\varepsilon \in \mathbb{R}_+$ the quadratic polynomial $g(\cdot, \cdot \,|\, i, \varepsilon) : X \times Y \mapsto \mathbb{R}$ by

$$g(x, y \,|\, i, \varepsilon) = w_i + w_{1i}(x - x_i) + w_{2i}(y - y_i) - \varepsilon \left[(x - x_i)^2 + (y - y_i)^2\right].$$

Since the numbers w_{1i} and w_{2i} are strictly positive for all $i \in \{a, b\}$ it follows that $g(x, y \,|\, i, 0)$ is strictly increasing with respect to (x, y) for all $i \in \{a, b\}$. Since $X \times Y$ is compact, this property is robust to small perturbations of ε. Hence, $g(x, y \,|\, i, \varepsilon)$ is strictly increasing for all sufficiently small positive numbers ε and all $i \in \{a, b\}$. It is also clear that $g(x, y \,|\, i, \varepsilon)$ is strictly concave for any positive ε and all $i \in \{a, b\}$. The inequalities in (27) can be expressed as $g(x_a, y_a \,|\, a, 0) < g(x_a, y_a \,|\, b, 0)$ and $g(x_b, y_b \,|\, b, 0) < g(x_b, y_b \,|\, a, 0)$. Due to continuity of g, these strict inequalities remain true if ε is a sufficiently small positive number instead of 0. It is therefore possible to find a positive number $\bar{\varepsilon}$ such that $g(x, y \,|\, i, \bar{\varepsilon})$ is strictly increasing and strictly concave with respect to (x, y) for all $i \in \{a, b\}$ and such that

$$g(x_a, y_a \,|\, a, \bar{\varepsilon}) < g(x_a, y_a \,|\, b, \bar{\varepsilon}) \quad \text{and} \quad g(x_b, y_b \,|\, b, \bar{\varepsilon}) < g(x_b, y_b \,|\, a, \bar{\varepsilon}) \tag{28}$$

hold.

STEP 3: Finally, we define the function $w : X \times Y \mapsto \mathbb{R}$ by

$$w(x, y) = \min\{g(x, y \,|\, i, \bar{\varepsilon}) \,|\, i \in \{a, b\}\}.$$

As a minimum of continuous, strictly increasing, and strictly concave functions, the function w itself is also continuous, strictly increasing, and strictly concave. The inequalities stated in (28) imply furthermore that w is continuously differentiable locally around the points (x_a, y_a) and (x_b, y_b) and that its partial derivatives at these points are given by $w_1(x_i, y_i) = g_1(x_i, y_i \,|\, i, \bar{\varepsilon}) = w_{1i}$ and $w_2(x_i, y_i) = g_2(x_i, y_i \,|\, i, \bar{\varepsilon}) = w_{2i}$ for all $i \in \{a, b\}$. This completes the proof of the lemma. □

To conclude the proof of theorem 1 we apply the above lemma with $X = [0, \bar{k}]$, $Y = [0, 1]$, $w(x, y) = u(x, 1 - y)$, $x_i = c_i$, and $y_i = 1 - \ell_i$ for all $i \in \{a, b\}$. This shows that there exists a function $u : [0, \bar{k}] \times [0, 1] \mapsto \mathbb{R}$ such that assumption 2 is satisfied and such that the partial derivatives of u at the points $(c_a, 1 - \ell_a)$ and $(c_b, 1 - \ell_b)$ exist and are given by (22)-(25) if and only if the inequality

$$[1 - \beta f_1(k_a, \ell_a)](c_b - c_a) + f_2(k_a, \ell_a)[1 - \beta f_1(k_a, \ell_a)](\ell_a - \ell_b) > 0$$

holds. We can use (13)-(14) to eliminate c_a and c_b from this condition, which yields

$$[1 - \beta f_1(k_a, \ell_a)][f(k_b, \ell_b) - k_a - f(k_a, \ell_a) + k_b + f_2(k_a, \ell_a)\ell_a - f_2(k_a, \ell_a)\ell_b] > 0.$$

Linear homogeneity of f implies that $f_2(k_a, \ell_a)\ell_a - f(k_a, \ell_a) = -f_1(k_a, \ell_a)k_a$. Substituting this into the above formula, we obtain (4). This completes the proof of theorem 1.

4.2 Proof of theorem 2

The general strategy of the proof is the same as in the case of theorem 1. Some of the details, however, are more complicated. We begin by stating the first-order optimality conditions for an interior feasible allocation of the form (5). Since the proof of the following lemma 3 is completely analogous to that of lemma 1 it is omitted.

Lemma 3 *Let (f, u, β) be an economy satisfying assumptions 1-3 and suppose that there exist real numbers k_a, k_b, k_c, ℓ_a, ℓ_b, ℓ_c, c_a, c_b, and c_c such that $k_a \neq k_b$, $k_b \neq k_c$, $k_a \neq k_c$, and*

$(k_i, \ell_i, c_i) \in (0, \bar{k}) \times (0, 1) \times (0, \bar{c})$ *for all* $i \in \{a, b, c\}$ *hold and such that the utility function* u *is continuously differentiable locally around each of the three points* $(c_a, 1 - \ell_a)$*,* $(c_b, 1 - \ell_b)$*, and* $(c_c, 1 - \ell_c)$*. The sequence* $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ *defined by (5) is an optimal allocation for the economy* (f, u, β) *if and only if the following conditions hold:*

$$c_a + k_b = f(k_a, \ell_a)\,,\ c_b + k_c = f(k_b, \ell_b)\,,\ c_c + k_a = f(k_c, \ell_c), \tag{29}$$

$$\beta^3 f_1(k_a, \ell_a) f_1(k_b, \ell_b) f_1(k_c, \ell_c) = 1, \tag{30}$$

$$u_1(c_b, 1 - \ell_b) = \beta f_1(k_c, \ell_c) u_1(c_c, 1 - \ell_c), \tag{31}$$

$$u_1(c_c, 1 - \ell_c) = \beta f_1(k_a, \ell_a) u_1(c_a, 1 - \ell_a), \tag{32}$$

$$u_1(c_i, 1 - \ell_i) f_2(k_i, \ell_i) = u_2(c_i, 1 - \ell_i) \ \textit{ for all } i \in \{a, b, c\}. \tag{33}$$

As in the case of optimal allocations of period 2, one can see that conditions (30)-(32) imply that not all three points $(c_a, 1 - \ell_a)$, $(c_b, 1 - \ell_b)$, and $(c_c, 1 - \ell_c)$ can coincide. However, we need the stronger result that these three points are mutually different from each other. The following lemma establishes this property.

Lemma 4 *Let* (f, u, β) *be an economy satisfying assumptions 1-3 and suppose that there exist real numbers* k_a*,* k_b*,* k_c*,* ℓ_a*,* ℓ_b*,* ℓ_c*,* c_a*,* c_b*, and* c_c *such that* $k_a \neq k_b$*,* $k_b \neq k_c$*,* $k_a \neq k_c$*, and* $(k_i, \ell_i, c_i) \in (0, \bar{k}) \times (0, 1) \times (0, \bar{c})$ *for all* $i \in \{a, b, c\}$ *hold and such that the utility function* u *is continuously differentiable locally around each of the three points* $(c_a, 1 - \ell_a)$*,* $(c_b, 1 - \ell_b)$*, and* $(c_c, 1 - \ell_c)$*. If the sequence* $(k_t, \ell_t, c_t)_{t=0}^{+\infty}$ *defined by (5) is an optimal allocation for the economy* (f, u, β) *then it follows that the three points* $(c_a, 1 - \ell_a)$*,* $(c_b, 1 - \ell_b)$ *and* $(c_c, 1 - \ell_c)$ *are mutually different from each other.*

PROOF: Since the economy satisfies assumptions 1-3, standard arguments from dynamic programming imply that optimal allocations exist and that the optimal value function $V : [0, \bar{k}] \mapsto \mathbb{R}$ is bounded, continuous, and strictly concave. Moreover, it holds for all $i \in \{a, b, c\}$ that

$$(c_i, \ell_i) = \operatorname{argmax}\{u(c, 1 - \ell) + \beta V(f(k_i, \ell) - c) \,|\, c \in [0, f(k_i, \ell)], \ell \in [0, 1]\}.$$

Since $(k_i, \ell_i, c_i) \in (0, \bar{k}) \times (0, 1) \times (0, \bar{k})$ holds for all $i \in \{a, b, c\}$ it follows that

$$0 \in \{u_1(c_i, 1 - \ell_i) + \beta p \,|\, p \in \partial V(f(k_i, \ell_i) - c_i)\},$$

where $\partial V(k)$ denotes the subdifferential of the strictly concave function V at k.[7] Now suppose that $(c_a, 1-\ell_a) = (c_b, 1-\ell_b) = (\bar{c}, 1-\bar{\ell})$. The above condition implies that

$$-\frac{u_1(\bar{c}, 1-\bar{\ell})}{\beta} \in \partial V(k_b) \cap \partial V(k_c).$$

Because V is strictly concave and $k_b \neq k_c$ by assumption, the right-hand side of this formula is the empty set. Hence, we obtain a contradiction to the assumption $(c_a, 1-\ell_a) = (c_b, 1-\ell_b)$, which proves the lemma. □

Equations (31)-(33) are the only conditions stated in lemma 3 which involve the instantaneous utility function. Since the equations are homogeneous in the partial derivatives $u_1(c_i, 1-\ell_i)$ and $u_2(c_i, 1-\ell_i)$ for $i \in \{a, b, c\}$, we may normalize these partial derivatives by setting $u_1(c_a, 1-\ell_a) = 1$. Solving (31)-(33) under this normalization yields

$$u_1(c_a, 1-\ell_a) = u_{1a} := 1, \tag{34}$$

$$u_1(c_b, 1-\ell_b) = u_{1b} := \beta^2 f_1(k_a, \ell_a) f_1(k_c, \ell_c), \tag{35}$$

$$u_1(c_c, 1-\ell_c) = u_{1c} := \beta f_1(k_a, \ell_a), \tag{36}$$

$$u_2(c_a, 1-\ell_a) = u_{2a} := f_2(k_a, \ell_a), \tag{37}$$

$$u_2(c_b, 1-\ell_b) = u_{2b} := \beta^2 f_1(k_a, \ell_a) f_1(k_c, \ell_c) f_2(k_b, \ell_b), \tag{38}$$

$$u_2(c_c, 1-\ell_c) = u_{2c} := \beta f_1(k_a, \ell_a) f_2(k_c, \ell_c). \tag{39}$$

We can now state the analogue of lemma 2 for the case of period 3.

Lemma 5 *Let X and Y be non-empty and compact intervals on the real line and let x_a, x_b, x_c, y_a, y_b, and y_c be real numbers such that $x_i \in int\ (X)$ and $y_i \in int\ Y$ hold for all $i \in \{a, b, c\}$ and such that the three points (x_a, y_a), (x_b, y_b), and (x_c, y_c) are mutually different from each other. Furthermore, let w_{1a}, w_{1b}, w_{1c}, w_{2a}, w_{2b}, and w_{2c} be positive real numbers. The following two statements are equivalent:*

(a) *There exists a function $w : X \times Y \mapsto \mathbb{R}$ which is continuous, strictly increasing, and*

[7] In parts of the literature, the terminology 'subdifferential' is only used for convex functions, whereas it is replaced by 'superdifferential' in the case of concave functions.

strictly concave and which is continuously differentiable locally at the points (x_a, y_a), (x_b, y_b), *and* (x_c, y_c) *with partial derivatives*

$$w_1(x_i, y_i) = w_{1i} \text{ and } w_2(x_i, y_i) = w_{2i} \text{ for } i \in \{a, b, c\}.$$

(b) *It holds that*

$$(w_{1b} - w_{1a})(x_a - x_b) + (w_{2b} - w_{2a})(y_a - y_b) > 0, \tag{40}$$

$$(w_{1c} - w_{1b})(x_b - x_c) + (w_{2c} - w_{2b})(y_b - y_c) > 0, \tag{41}$$

$$(w_{1c} - w_{1a})(x_a - x_c) + (w_{2c} - w_{2a})(y_a - y_c) > 0, \tag{42}$$

$$\begin{aligned} &w_{1a}(x_b - x_a) + w_{1b}(x_c - x_b) + w_{1c}(x_a - x_c) \\ &\qquad > w_{2a}(y_a - y_b) + w_{2b}(y_b - y_c) + w_{2c}(y_c - y_a), \end{aligned} \tag{43}$$

$$\begin{aligned} &w_{1a}(x_c - x_a) + w_{1b}(x_a - x_b) + w_{1c}(x_b - x_c) \\ &\qquad > w_{2a}(y_a - y_c) + w_{2b}(y_b - y_a) + w_{2c}(y_c - y_b). \end{aligned} \tag{44}$$

PROOF: We first prove that (a) implies (b). Since the three points (x_a, y_a), (x_b, y_b), and (x_c, y_c) are mutually different, it follows from strict concavity of w that

$$w(x_a, y_a) < w(x_b, y_b) + w_{1b}(x_a - x_b) + w_{2b}(y_a - y_b), \tag{45}$$

$$w(x_a, y_a) < w(x_c, y_c) + w_{1c}(x_a - x_c) + w_{2c}(y_a - y_c), \tag{46}$$

$$w(x_b, y_b) < w(x_a, y_a) + w_{1a}(x_b - x_a) + w_{2a}(y_b - y_a), \tag{47}$$

$$w(x_b, y_b) < w(x_c, y_c) + w_{1c}(x_b - x_c) + w_{2c}(y_b - y_c), \tag{48}$$

$$w(x_c, y_c) < w(x_a, y_a) + w_{1a}(x_c - x_a) + w_{2a}(y_c - y_a), \tag{49}$$

$$w(x_c, y_c) < w(x_b, y_b) + w_{1b}(x_c - x_b) + w_{2b}(y_c - y_b). \tag{50}$$

Combining (45) and (47), one obtains (40). In the same way, one gets (41) from (48) and (50) and one gets (42) from (46) and (49). Adding (46), (47), and (50) yields (44), and adding (45), (48), and (49) one obtains (43).

The proof that (b) implies (a) is split up in the same three steps as the proof of the corresponding lemma 2.

STEP 1: We first prove that there exist real numbers w_a, w_b, and w_c such that the inequalities

$$w_{1b}(x_b - x_a) + w_{2b}(y_b - y_a) < w_b - w_a < w_{1a}(x_b - x_a) + w_{2a}(y_b - y_a), \tag{51}$$

$$w_{1c}(x_c - x_b) + w_{2c}(y_c - y_b) < w_c - w_b < w_{1b}(x_c - x_b) + w_{2b}(y_c - y_b), \tag{52}$$

$$w_{1c}(x_c - x_a) + w_{2c}(y_c - y_a) < w_c - w_a < w_{1a}(x_c - x_a) + w_{2a}(y_c - y_a) \tag{53}$$

hold. To this end, we define the real numbers

$$A_{ba} = w_{1b}(x_b - x_a) + w_{2b}(y_b - y_a)\,,\; B_{ba} = w_{1a}(x_b - x_a) + w_{2a}(y_b - y_a),$$

$$A_{cb} = w_{1c}(x_c - x_b) + w_{2c}(y_c - y_b)\,,\; B_{cb} = w_{1b}(x_c - x_b) + w_{2b}(y_c - y_b),$$

$$A_{ca} = w_{1c}(x_c - x_a) + w_{2c}(y_c - y_a)\,,\; B_{ca} = w_{1a}(x_c - x_a) + w_{2a}(y_c - y_a),$$

and the open intervals $I_{ba} = (A_{ba}, B_{ba})$, $I_{cb} = (A_{cb}, B_{cb})$, and $I_{ca} = (A_{ca}, B_{ca})$. It follows from (40)-(42) that all three of these intervals are non-empty. Furthermore, it follows from (43)-(44) that $A_{ba} + A_{cb} < B_{ca}$ and $A_{ca} < B_{ba} + B_{cb}$. This, in turn, implies that

$$\{\delta_{ba} + \delta_{cb} \,|\, \delta_{ba} \in I_{ba}, \delta_{cb} \in I_{cb}\} \cap I_{ca} \neq \emptyset.$$

Consequently, there exist real numbers $\delta_{ba} \in I_{ba}$, $\delta_{cb} \in I_{cb}$, and $\delta_{ca} \in I_{ca}$ such that $\delta_{ca} = \delta_{ba} + \delta_{cb}$. Let w_a be an arbitrary real number and define $w_b = w_a + \delta_{ba}$ and $w_c = w_a + \delta_{ca}$. Then it follows that $w_b - w_a = \delta_{ba} \in I_{ba}$, $w_c - w_b = \delta_{ca} - \delta_{ba} = \delta_{cb} \in I_{cb}$, and $w_c - w_a = \delta_{ca} \in I_{ca}$. Obviously, this is equivalent to (51)-(53).

Steps 2 and 3: Since these steps are completely analogous to the corresponding steps in the proof of lemma 2, we omit many details. One starts by defining for all $i \in \{a, b, c\}$ and all $\varepsilon \in \mathbb{R}_+$ the quadratic polynomial $g(\cdot, \cdot \,|\, i, \varepsilon) : X \times Y \mapsto \mathbb{R}$ by

$$g(x, y \,|\, i, \varepsilon) = w_i + w_{1i}(x - x_i) + w_{2i}(y - y_i) - \varepsilon \left[(x - x_i)^2 + (y - y_i)^2\right].$$

If ε is positive but sufficiently small, $g(\cdot, \cdot \,|\, i, \varepsilon)$ is strictly increasing and strictly concave for all $i \in \{a, b, c\}$. Then, one defines the function $w : X \times Y \mapsto \mathbb{R}$ by

$$w(x, y) = \min\{g(x, y \,|\, i, \varepsilon) \,|\, i \in \{a, b, c\}\}.$$

If ε is positive but sufficiently small, then it follows that w is a continuous, strictly increasing, and strictly concave function. Furthermore, the inequalities in (51)-(53) ensure that w is continuously differentiable locally around the three points (x_a, y_a), (x_b, y_b), and (x_c, y_c) and that its partial derivatives at these three points are given by $w_1(x_i, y_i) = g_1(x_i, y_i \,|\, i, \varepsilon) = w_{1i}$

and $w_2(x_i, y_i) = g_2(x_i, y_i \,|\, i, \varepsilon) = w_{2i}$ for $i \in \{a, b, c\}$. This completes the proof of the lemma. □

To conclude the proof of theorem 2 we apply the above lemma with $X = [0, \bar{k}]$, $Y = [0, 1]$, $w(x, y) = u(x, 1 - y)$, $x_i = c_i$, and $y_i = 1 - \ell_i$ for all $i \in \{a, b, c\}$. This shows that there exists a function $u : [0, \bar{k}] \times [0, 1] \mapsto \mathbb{R}$ such that assumption 2 is satisfied and such that the partial derivatives of u at the points $(c_a, 1 - \ell_a)$, $(c_b, 1 - \ell_b)$, and $(c_c, 1 - \ell_c)$ exist and are given by (34)-(39) if and only if the inequalities

$$
\begin{aligned}
&(u_{1b} - u_{1a})(c_a - c_b) + (u_{2b} - u_{2a})(\ell_b - \ell_a) > 0, \\
&(u_{1c} - u_{1b})(c_b - c_c) + (u_{2c} - u_{2b})(\ell_c - \ell_b) > 0, \\
&(u_{1c} - u_{1a})(c_a - c_c) + (u_{2c} - u_{2a})(\ell_c - \ell_a) > 0, \\
&u_{1a}(c_b - c_a) + u_{1b}(c_c - c_b) + u_{1c}(c_a - c_c) > u_{2a}(\ell_b - \ell_a) + u_{2b}(\ell_c - \ell_b) + u_{2c}(\ell_a - \ell_c), \\
&u_{1a}(c_c - c_a) + u_{1b}(c_a - c_b) + u_{1c}(c_b - c_c) > u_{2a}(\ell_c - \ell_a) + u_{2b}(\ell_a - \ell_b) + u_{2c}(\ell_b - \ell_c)
\end{aligned}
$$

hold, where the numbers u_{1i} and u_{2i} for $i \in \{a, b, c\}$ are defined in (34)-(39). We can use (29) to eliminate c_a, c_b, and c_c from these conditions, and we can use linear homogeneity of the production function to replace $f_2(k_i, \ell_i)\ell_i - f(k_i, \ell_i)$ by $-f_1(k_i, \ell_i)k_i$ for all $i \in \{a, b, c\}$. This leads to conditions (6)-(11) and the proof of theorem 2 is complete.

References

[1] D. Acemoglu, *Introduction to Modern Economic Growth*, Princeton University Press, 2009.

[2] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Birkhäuser, 1980.

[3] P. De Hek, "An aggregative model of capital accumulation with leisure-dependent utility", *Journal of Economic Dynamics and Control* **23** (1998), 255-276.

[4] W. De Melo and S. Van Strien, *One-Dimensional Dynamics*, Springer-Verlag, 1993.

[5] C. Eriksson, "Economic growth with endogenous labor supply", *European Journal of Political Economy* **12** (1996), 533-544.

[6] T. Kamihigashi, "Multiple interior steady states in the Ramsey model with elastic labor supply", *International Journal of Economic Theory* **11** (2015), 25-37.

[7] T. Y. Li and J. A. Yorke, "Period three implies chaos", *American Mathematical Monthly* **82** (1975), 985-992.

[8] J. Miao, *Economic Dynamics in Discrete Time*, MIT Press, 2014.

[9] T. Mitra, "An exact discount factor restriction for period-three cycles in dynamic optimization models", *Journal of Economic Theory* **69** (1996), 281-305.

[10] K. Nishimura and M. Yano, "On the least upper bound of discount factors that are compatible with optimal period-three cycles", *Journal of Economic Theory* **69** (1996), 306-333.

[11] A. N. Sarkovskii, "Coexistence of cycles of a continuous map of a line into itself", *Ukrains'kyi Matematychnyi Zhurnal* **16** (1964), 61-71.

[12] G. Sorger, "Income and wealth distribution in a simple model of growth", *Economic Theory* **16** (2000), 23-42.

[13] G. Sorger, *Dynamic Economic Analysis: Deterministic Models in Discrete Time*, Cambridge University Press, 2015.

[14] N. Stokey and R. E. Lucas Jr, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.