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Implications of Stochastic Singularity in Linear Multivariate Rational Expectations Models

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Abstract

In general, linear multivariate rational expectations models do not have a unique solution. This paper reviews some procedures for determining whether there exists a solution, whether it is unique, and infers on the dimension of indeterminacy and the number of free parameters in a parametrization thereof. A particular emphasis is given to stochastic singularity, i.e. the case in which the number of outputs is strictly larger than the number of (stochastic) inputs. First, it is shown that assuming stochastic singularity of the exogenous driving process has the same effects as (but is more natural than) assuming that some variables are predetermined, i.e have trivial one-step-ahead prediction error. Second, the dimension of the solution set is in general different from the one derived in the case where the number of outputs and inputs coincide. We derive this result in both the framework of [37, 34] (which impose non-explosiveness conditions) and [9, 11] (which do not impose non-explosiveness conditions). In this context, the results of [34] and [11] are corrected and extended. Last, we note that the framework of [11] can be adjusted to incorporate non-explosiveness conditions and lends itself to an identifiability analysis of dynamic stochastic general equilibrium (DSGE) models.

JEL classification: C62, C63, E00

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Contents

1 Introduction

This paper deals with linear multivariate rational expectations models where the number of endogenous variables may be larger than the number of uncorrelated white noise innovations of the exogenous process driving the economy. After reviewing some approaches and clarifying their relation to the problem of stochastic singularity, we generalize an approach by [34] dealing with the influence of so-called sunspot shocks on endogenous variables. Moreover, we describe the set of all solutions of a linear multivariate rational expectations model following [11], correct an important error in Theorem 4 on page 248 in [11], and derive the dimension of the solution set in the stochastically singular case. Last, the analysis of [11] is, in addition to permitting for stochastic singularity, extended to allowing for more general parameter restrictions in order to render the developed theory useful for the analysis of macroeconomic models as, e.g., [38]. This will enable us to conduct an identifiability analysis of dynamic stochastic general equilibrium (DSGE) models without imposing a (for structural models very restrictive) minimality assumption as was done in [33].

We heavily draw on the methods developed in [12, 9, 10, 13, 11]. In particular, we consider the rational expectations model

$$\begin{pmatrix} I_s & I_s z & \cdots & I_s z^k & \cdots & I_s z^K \end{pmatrix} \begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0h} & \cdots & A_{0H} \\ A_{10} & \ddots & & & & \\ \vdots & & & & & \\ A_{k0} & & & A_{kh} & & A_{kH} \\ \vdots & & & & & \vdots \\ A_{K0} & A_{K1} & \cdots & & & A_{KH} \end{pmatrix} \begin{pmatrix} y_t \\ \mathbb{E}_t\left(y_{t+1}\right) \\ \vdots \\ \mathbb{E}_t\left(y_{t+h}\right) \\ \vdots \\ \mathbb{E}_t\left(y_{t+H}\right) \end{pmatrix} = -u_t \tag{1}$$

where z denotes the backward shift operator, i.e. $z\left(y_t\right)_{t\in\mathbb{Z}} = \left(y_{t-1}\right)_{t\in\mathbb{Z}}$, $\mathbb{E}_t\left(y_{t+h}\right)$ denotes the projection[2] of y_{t+h} on closure of the linear[3] space spanned by the present and the past of the components of $\{u_t, u_{t-1}, \ldots\}$ of the exogenous process $\left(u_t\right)_{t\in\mathbb{Z}}$, denoted by $H_u(t) = \overline{span\left\{u_{t-s}^{(i)} \,|\, s \in \mathbb{N},\ i \in \{1,\ldots,s\}\right\}}$. To avoid confusion, we will sometimes write more explicitly $\mathbb{E}\left(y_{t+h}|H_u(t)\right)$ for the same object[4]. Furthermore, we assume that there are no redundant equations, i.e. the matrix polynomial $\pi(z)$ defined in equation (61) on page 42 depending on the matrices A_{kh}, $k \in \{0,\ldots,K\}$ and $h \in \{0,\ldots,H\}$, and that there exists an $h \in \{0,\ldots,H\}$ such that $A_{Kh} \neq 0$ and a $k \in \{0,\ldots,K\}$ such that $A_{kH} \neq 0$. In this way K and H are well defined. The indices k and h in A_{kh} refer to the h-period-ahead forecast of the endogenous variables, k periods ago, i.e. y_{t-k} is forecast h periods ahead with the information[5] available in period $t-k$.

We assume that the stationary s-dimensional exogenous process $\left(u_t\right)_{t\in\mathbb{Z}}$ has a (finite) covariance matrix $\mathbb{E}\left(u_t u_t^T\right)$, where the superscript T denotes transposition, of rank r smaller than or equal to s and a rational spectral density $f_u\left(\lambda\right)$ of rank $q \leq r \leq s$.

A solution in the wide sense[6] of the rational expectations model (1) is a stochastic process $\left(y_t\right)_{t\in\mathbb{Z}}$ such that for given exogenous driving process $\left(u_t\right)_{t\in\mathbb{Z}}$ and given parameters $A_{kh},\ k \in \{0,\ldots,K\},\ h \in \{0,\ldots,H\}$, $\left(y_t\right)_{t\in\mathbb{Z}}$ satisfies equation (1) for all $t \in \mathbb{Z}$. Note that $\left(y_t\right)_{t\in\mathbb{Z}}$ is a deterministic function of $\left(u_t\right)_{t\in\mathbb{Z}}$, i.e. there are no additional error terms involved.

Remark 1. Note that some authors, e.g. [8, 31, 37], study only solutions on the natural numbers $\mathbb{N}$. There are at least two reasons in favor of examining solutions on $\mathbb{Z}$. First, there is an asymmetry between past and future when only solutions on $\mathbb{N}$ are considered. Second, a solution starting from the infinite past can be interpreted as a solution approximating unknown initial values in a reasonable sense. In the stationary state, we do not know initial values, hence a solution starting from the infinite past should be preferred to a solution on $\mathbb{N}$.

Moreover, theorems on spectral representations and spectral decomposition of stationary processes do not always hold true for stochastic processes with $\mathbb{N}$ as index set, compare [19] page 481 and 486. Also, [2] use the term "covariance factorization" (page 233) for processes with finite initial time and reserve "spectral factorization" for stationary processes

[2] Compare [19] page 155, where he defines the conditional in the wide sense as the projection on a linear manifold.

[3] Note that if all random variables in the conditioning set are Gaussian, the conditional expectation coincides with the linear projection outlined here. For more details on conditional expectations see [3] page 445ff.

[4] Some authors, e.g. [25] on page 410, condition on a larger set of variables comprising variables which are independent to the exogenous process. These variables are called "sunspots" by the authors.

[5] We will refer to the space $H_u(t)$ on which the endogenous variables are projected as "the information at time t".

[6] A solution of the rational expectations model (without the addendum "in the wide sense") is a solution in the wide sense of the rational expectations model for which additionally firstly $y_t \in H_u(t)$ (or any other specified linear space generated by components of stochastic processes) holds and which secondly does not violate a non-explosiveness condition to be specified.

with index set $\mathbb{Z}$. Moreover, they show on pages 242-243 that their so-called "innovations model" (which is closely related to the spectral factorization) is not time invariant (but only asymptotically time invariant) for stationary processes with finite initial time, compare also [28] pages 19-20.

2 Zeros of a polynomial at infinity

We follow [29] page 370ff and consider the polynomial

$$p\left(z,c\right)=c_{n}z^{n}+c_{n-1}z^{n-1}+\cdots+c_{1}z+c_{0},$$

where z is a complex variable and $c=(c_{0},\ldots,c_{n})\in\mathbb{C}^{n+1}$ a coefficient vector. The set of polynomials we consider is thus defined by the set of coefficient vectors $c\in\mathbb{C}^{n+1}$. Furthermore, we denote the degree of $p(z,c)$ by $\deg\left(p(z,c)\right)=n(c)$. We say that the polynomial $p(z,c)$ has $(n-n(c))$ zeros at infinity.

The notion of a zero at infinity is motivated by Theorem 4.1.2 on page 371 in [29]. It states that every polynomial $p(z,c)$ which is "sufficiently close" to a non-constant polynomial $p\left(z,\tilde{c}\right)$ with degree $n\left(\tilde{c}\right)$, i.e. $\|c-\tilde{c}\|<\varepsilon$ holds for the corresponding coefficient vectors $c,\tilde{c}\in\mathbb{C}^{n+1}$, a sufficiently small $\varepsilon>0$, and an arbitrary norm $\|\cdot\|$ on $\mathbb{C}^{n+1}$, has exactly $n(c)-n\left(\tilde{c}\right)$ roots outside the set $\left\{z\in\mathbb{C}|\,|z|>\frac{1}{\varepsilon}\right\}$. Thus intuitively, if the degree of $p(z,c)$ is higher than the one of $p\left(z,\tilde{c}\right)$, the "new roots" are "far away" from zero.

Example 2. As an example consider the polynomial

$$p(z)=az^{2}+bz+c,\quad a,b,c\in\mathbb{C},\ b\neq0,\ a\neq0$$

whose roots are $z_{\pm}=\frac{-b\pm\sqrt{b^{2}-4ac}}{2a}$. For $a\to0$, $z_{+}=-\frac{c}{b}$ whereas z_{-} diverges.

2.1 Zeros of a square polynomial matrix at infinity

The zeros of a square polynomial matrix $A(z)=A^{(0)}+A^{(1)}z+\cdots+A^{(r)}z^{r}$ are defined as the zeros of its determinant $\det\left(A(z)\right)=c_{n}z^{n}+\cdots+c_{1}z+c_{0}$ where the coefficients $(c_{0},\ldots,c_{n})$ are (multivariate) polynomials in the elements $\left\{A_{ij}^{(k)}|i,j\in\{1,\ldots,s\},\,k\in\{0,\ldots,r\}\right\}$ of $A(z)$.

Example 3. [31] considers a so-called regular linear pencil (compare [20] page 25-28) $M(z)=Az-B,\ z\in\mathbb{C}$, where A and B are square matrices and $\det\left(Az-B\right)$ is not identically zero. The elements of A are assumed to be unrestricted complex numbers, in particular A may be singular. The matrix pencil has a zero at infinity if and only if $\det\left(A\right)=0$.

Example 4. Let the multi-index $\alpha=(r_{1},\ldots,r_{s})$ prescribe the maximal degrees of the columns $\left(A_{[\bullet,1]}(z),\ldots,A_{[\bullet,s]}(z)\right)$ of the $(s\times s)$-dimensional polynomial matrix $A(z)$ where $A(0)=I_{s}$ and denote the coefficients of $A(z)$ again with a superscript, i.e. $A_{[\bullet,i]}^{(k)}$ is the $(s\times1)$-dimensional coefficient vector of the i-th column of $A(z)$ pertaining to z^{k}. The parameter space $L\subseteq\mathbb{R}^{n}$, where $n=s\left(p_{1}+\cdots+p_{s}\right)$, describing all matrices of the form above consists of all (free) parameters in $A(z)$, i.e.

$$l=vec_{row}\left(A_{[\bullet,1]}^{(1)},\ldots,A_{[\bullet,1]}^{(k)}\ldots,A_{[\bullet,1]}^{(r_{!})}|\cdots|A_{[\bullet,i]}^{(1)},\ldots,A_{[\bullet,i]}^{(k)}\ldots,A_{[\bullet,i]}^{(r_{i})}|\cdots|A_{[\bullet,s]}^{(1)},\ldots,A_{[\bullet,s]}^{(k)}\ldots,A_{[\bullet,s]}^{(r_{s})}\right),$$

where vec_{row} denotes row-wise vectorization. If the column-end matrix, i.e. the matrix $\left(A_{[\bullet,1]}^{(r_{1})},\ldots,A_{[\bullet,s]}^{(r_{s})}\right)$ consisting of the coefficient vectors pertaining to the highest degree in the respective column of $A(z)$, is of full rank, the degree of the determinant is equal to n. The polynomial matrix $A(z)$ has a zero at infinity if and only if the column-end matrix is not of full rank.

Example 5. Let $A_{\theta}(z)=A_{\theta}^{(0)}+A_{\theta}^{(1)}z+\cdots+A_{\theta}^{(p)}z^{r}$ be a polynomial matrix of dimension $(s\times s)$ whose parameter space $L_{r}\subseteq\mathbb{R}^{N}$, $N=\left[(r_{1}+\cdots+r_{s})+s\right]s$, consists of all

$$l=vec_{row}\left(A_{\theta,[\bullet,1]}^{(0)},\ldots,A_{\theta,[\bullet,1]}^{(k)}\ldots,A_{\theta,[\bullet,1]}^{(r_{!})}|\cdots|A_{\theta,[\bullet,i]}^{(0)},\ldots,A_{\theta,[\bullet,i]}^{(k)}\ldots,A_{\theta,[\bullet,i]}^{(r_{i})}|\cdots|A_{\theta,[\bullet,s]}^{(0)},\ldots,A_{\theta,[\bullet,s]}^{(k)}\ldots,A_{\theta,[\bullet,s]}^{(r_{s})}\right)\tag{2}$$

such that additionally $\det\left(A_{\theta}(z)\right)\not\equiv0$ is satisfied.

Furthermore, let $L_{R}\subseteq L_{r}$ be the parameter space for which all coefficients of $A_{\theta}(z)$ are multivariate rational functions of p "deep" parameters $\theta=(\theta_{1},\ldots,\theta_{p})\in\Theta\subseteq\mathbb{R}^{p}$, i.e.

$$A_{\theta,ij}^{(k)}=\frac{p_{ij}^{(k)}\left(\theta_{1},\ldots,\theta_{p}\right)}{q_{ij}^{(k)}\left(\theta_{1},\ldots,\theta_{p}\right)}$$

where $p_{ij}^{(k)}(\theta_1,\ldots,\theta_p)$ and $q_{ij}^{(k)}(\theta_1,\ldots,\theta_p)$ are multivariate polynomials in θ, and $q_{ij}^{(k)}(\theta_1,\ldots,\theta_p)$ is not identically zero. Note that the coefficients of $\det(A_\theta(z))$ are again multivariate rational functions of θ. Thus, it follows that the degree $n_{generic} = \max_{\theta\in\Theta}[\deg\{\det(A_\theta(z))\}]$ is constant almost everywhere in Θ and equivalently that the leading coefficient $c_{n_{generic}}$ is almost everywhere not equal to zero. On points θ^0 for which $\deg\{\det(A_{\theta^0}(z))\} < n_{generic}$, $A_\theta(z)$ has a zero at infinity for given parameter space L_R.

If we were to consider L_r as the parameter space, then $A_\theta(z)$ has generically $(r_1+\cdots+r_s)-n_{generic}$ zeros at infinity. This emphasizes that zeros at infinity are a concept that is closely related to the parameter space describing a set of polynomials.

Remark 6. It is often the case that a priori restrictions are described through the kernel of a (linear) map, see e.g. [17]. The parameter vector $l\in\mathbb{R}^N$ describing the "linear parameters" (2), i.e. the parameters in $A_\theta(z)$, in example 5 may be restricted by $Dl+d=0$ where $D\in\mathbb{R}^{p\times N}$ and $d\in\mathbb{R}^p$ are a priori given. In example 5 above, however, the a priori restrictions are given through the image of a rational function.

Let us assume that the multivariate rational function $\Lambda(\cdot)$ attaching the "linear" parameters $l\in L_r$ (describing the polynomial matrix in example 5) to the "deep" parameters $\theta\in\Theta$ is affine, i.e.

$$\Lambda:\begin{cases}\Theta & \to L_r\\ \theta & \mapsto C\theta+c=l,\end{cases}$$

where $C\in\mathbb{R}^{N\times p}$, $N>p$, and $rk(C)=p$. Thus all "linear" parameters $l\in L_r\subseteq\mathbb{R}^N$ in the model are described as the image of $\Lambda(\cdot)$, i.e. restricted to be contained in the space spanned by the columns of C, translated by c.

The parameters $l\in L_r$ satisfying these restrictions can equivalently be described by the kernel of C^T. Indeed, note first that all vectors $l\in L_r$ which are contained in $im(C)$ are orthogonal to $(im(C))^\perp$. Since $(im(C))^\perp = ker\left(C^T\right)$, the vectors $l\in L_r$ which satisfy $C^Tl=0$ span $im(C)$. Hence, the parameters $l\in L_r$ which are given by the image of $\Lambda(\cdot)$ satisfy $C^T(l-c)=0$ and can thus be equivalently described by the kernel of a map.

3 Literature on linear multivariate rational expectations models

In this section we give an overview of some solution methods for multivariate linear rational expectations models. Rather than describing all solution methods, we point out how the literature on solution methods developed with respect to allowing for zeros at infinity, stochastic singularity, and parameter restrictions.

3.1 Blanchard and Kahn, and predetermined and non-predetermined variables

An early influential paper is [8]. They consider the model

$$\begin{pmatrix} \mathbb{E}_t\left(y_{t+1}^{(pre)}\right) \\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right) \end{pmatrix} = B \begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix} + Cz_t, \quad t \in \mathbb{N} \tag{3}$$

where $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$, $\mathbb{E}_t\left(y_{t+1}^{(pre)}\right) = y_{t+1}^{(pre)}$, the predetermined variables $y_t^{(pre)}$ have initial value $y_0^{(pre)}$, and $(z_t)_{t\in\mathbb{N}}$ is an m-dimensional exogenously given stochastic process which is bounded in the sense that

$$\forall t \in \mathbb{N} : \exists \bar{Z}_t \in \mathbb{R}^m \wedge \theta_t \in \mathbb{R} \text{ such that } -(1+i)^{\theta_t}\bar{Z}_t \leq \mathbb{E}_t\left(z_{t+i}\right) \leq (1+i)^{\theta_t}\bar{Z}_t \quad \forall i \geq 0. \tag{4}$$

They ask as to when a stochastic process $(y_t)_{t\in\mathbb{N}}$ which satisfies equation (3) of the rational expectations model above for every $t \in \mathbb{N}$, which satisfies the non-explosiveness condition

$$\begin{aligned} &\forall t \in \mathbb{N} : \exists \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \in \mathbb{R}^{n(pre)+n(\neg pre)} \wedge \sigma_t \in \mathbb{R} \text{ such that} \\ -(1+i)^{\sigma_t} &\begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \leq \mathbb{E}_t \begin{pmatrix} y_{t+i}^{(pre)} \\ y_{t+i}^{(\neg pre)} \end{pmatrix} \leq (1+i)^{\sigma_t} \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \quad \forall i \geq 0, \end{aligned} \tag{5}$$

and which is contained in[7] $H_z(t)$ exists, and (if it exists) whether it is unique. A process $(y_t)_{t\in\mathbb{N}}$ with these properties is called a solution of the rational expectations model (3). Note that the set of all solution is thus restricted in two ways. First, some processes for which equation (3) holds for every $t \in \mathbb{N}$ are excluded because they do not satisfy the non-explosiveness condition; second, some processes are excluded because they are not contained in $H_z(t)$ at time t. The latter fact excludes in particular processes orthogonal to $H_z(t)$.

Some remarks on the structure of the model are in order.

Remark 7 (Consequences of a larger conditioning set). Following [25] page 411, we consider the model (3) with the only difference that the solution $(y_t)_{t\in\mathbb{N}}$ is not required to be contained in $H_z(t)$ but in (with obvious notation) $H_{z,\zeta}(t)$ where $(\zeta_t)_{t\in\mathbb{N}}$ is a p-dimensional stochastic process (satisfying the non-explosiveness condition (4) on exogenous processes outlined above) orthogonal to $(z_t)_{t\in\mathbb{N}}$. Obviously, if the conditional expectation is taken with respect to $H_{z,\zeta}(t)$ a larger solution set might be obtained. Otherwise, the terms $\mathbb{E}\left(y_{t+1}|H_z(t)\right)$ and Cz_t only depend on elements in $H_z(t)$, implying that also y_t is a function of elements in $H_z(t)$.

The following superposition principle holds: If $\left(y_t^1\right)_{t\in\mathbb{N}}$ is a particular solution of

$$\mathbb{E}\left(y_{t+1}^1|H_z(t)\right) = By_t^1 + C_1z_t$$

in the sense that it solves the equation above for given $(z_t)_{t\in\mathbb{N}}$ for every time point and $\left(y_t^2\right)_{t\in\mathbb{N}}$ is a particular solution of

$$\mathbb{E}\left(y_{t+1}^2|H_\zeta(t)\right) = By_t^2 + C_2\zeta_t$$

then (by independence of $(z_t)_{t\in\mathbb{N}}$ and $(\zeta_t)_{t\in\mathbb{N}}$) we obtain that

$$\mathbb{E}\left(\left(y_{t+1}^1 + y_{t+1}^2\right)|H_{z,\zeta}(t)\right) = B\left(y_{t+1}^1 + y_{t+1}^2\right) + C_1z_t + C_2\zeta_t.$$

Thus, if we allow in the Blanchard and Kahn model for a larger conditioning set than $H_z(t)$, the solution set of the Blanchard and Kahn model is enlarged by the solutions of the homogenous equation

$$\mathbb{E}\left(y_{t+1}|H_\zeta(t)\right) = By_t.$$

[7] [8] define the conditioning set differently. For ease and continuity of presentation we deviate from their definition.

Remark 8 ((Non)-predetermined variables). Blanchard and Kahn distinguish between predetermined variables $y_t^{(pre)}$ (intended to capture, e.g., the notion of capital in the economy), i.e. $\mathbb{E}_t\left(y_{t+1}^{(pre)}\right) = y_{t+1}^{(pre)}$ holds, and non-predetermined variables which are sometimes called "jump-variables". Many authors consider this distinction to be unnatural. E.g. [37] argues that the parametric structure of the model determines endogenously which linear combinations of endogenous variables have no expectational error term, compare section 3.3 starting on page 20. Also, one might think that it should be possible to give an interpretation in terms of backward and forward looking behavior for the solution corresponding to predetermined and non-predetermined variables respectively. However, both predetermined and non-predetermined variables depend on expectations at time t of future exogenous variables.

We consider the distinction in predetermined and non-predetermined variables as a clever, although ad hoc, way to obtain the "right" number of degrees of freedom. This enables Blanchard and Kahn to analyze existence and uniqueness of solutions with regard to the number of non-predetermined variables without getting very technical.

Remark 9 (Conditional expectations in every equation). Every single equation in the system of equations (3) involves expectational terms which cannot be canceled out by elementary row operations, i.e. there is an identity matrix on the left hand side of equation (3) instead of a potentially singular matrix A as in [31] discussed in section 3.2 starting on page 14. This is a serious drawback since, in general, one does not obtain an identity matrix (or even a non-singular matrix) on the left hand side of the rational expectations model (3) if it is derived from agents' utility optimizing behavior.

Remark 10 (Non-explosiveness condition). The non-explosiveness condition (5) means that the projections on the linear space spanned by the components of $\{z_t, z_{t-1}, \ldots\}$ may not grow faster than polynomial when the forecasting horizon increases unboundedly. [31] note on page 1020 in footnote 20 that "the stability condition can be written as the requirement that if $|\mathbb{E}_t(x_{t+k})| < \bar{x}$ a.s. for some finite $\bar{x}$ and all t and k, then $|\mathbb{E}_t(y_{t+k})| < \bar{y}$ for some finite $\bar{y}$ and all t and k." There are many similar boundedness conditions (not all of them equivalent) which capture the notion that "small inputs imply small outputs" and thus the existence of a bounded linear (and hence continuous) mapping between linear spaces, compare [28] page 11.

[7] discusses the role of this non-explosiveness condition and whether imposing such a condition is justified. He notes, among other things, that in some cases "the implications of the explosion of an endogenous variable are inconsistent with some assumption of the model" and refers to [27] "where some prices become negative in finite time" if a solution not satisfying the non-explosiveness condition is chosen. Also, [6] mentions on page 116 that "in certain models non-stationarity may violate the assumption of market clearing and of rationality of expectations". Furthermore, [1] argues on page 187 that "we are not interested in optimization problems in which households or firms achieve infinite value, [...], since the essence of economics, trade-offs in the face of scarcity, would be absent in these cases".

Decoupling of unstable and stable roots. The authors use the Jordan decomposition of $B = T^{-1}JT$, where the rows of T are a basis for the left-invariant subspaces of B, to decouple the unstable from the stable part of the system, i.e. the Jordan blocks are ordered with respect to weakly increasing absolute values of eigenvalues. Eigenvalues with absolute value smaller than or equal to unity are stable[8] and otherwise they are unstable. Left-multiplying equation (3) with T leads to

$$\begin{pmatrix}\mathbb{E}_t(s_{t+1})\\ \mathbb{E}_t(u_{t+1})\end{pmatrix} = \begin{pmatrix}J_s & \\ & J_u\end{pmatrix}\begin{pmatrix}s_t\\ u_t\end{pmatrix} + TCz_t, \quad t \in \mathbb{N} \tag{6}$$

where $\begin{pmatrix}s_t\\ u_t\end{pmatrix} = T\begin{pmatrix}y_t^{(pre)}\\ y_t^{(\neg pre)}\end{pmatrix}$, and J_s and J_u contain the Jordan blocks corresponding to the stable and unstable eigenvalues.

Obtaining a solution for the unstable part of the system. The unstable part of the system is solved forward and thus the solution $(u_t)_{t\in\mathbb{N}}$ depends on conditional expectations at time t of future values of the (given) exogenous process $(z_t)_{t\in\mathbb{N}}$. The solution

$$u_t = -\sum_{i=0}^{\infty} J_u^{-(i+1)} T_{u\bullet} C \mathbb{E}_t(z_{t+i})$$

is obtained by successive forward substitution in the second block of rows in (6), i.e. $u_t = J_u^{-1}\mathbb{E}_t(u_{t+1}) - T_{u,\bullet}Cz_t$, where $T_{u,\bullet} \in \mathbb{R}^{n(pre)\times n}$ denotes the rows of T corresponding to unstable Jordan blocks.

[8] Compare page 1307 line -4 in [8]. [32] note on page 72 below their formula (19) that unit roots are considered stable because they do not violate the non-explosiveness condition. A root λ is treated as unstable if $\beta\lambda > 1$, where $\beta \in (0,1)$ is a discount factor.

Existence and uniqueness of the solution of the stable part of the system: Initialization and induction step. A solution $(s_t)_{t\in\mathbb{N}}$ of the stable part is obtained in the following way. For $t=0$, we have from the first block of rows in (6) that

$$\begin{aligned} y_0^{(pre)} &= \left(T^{-1}\right)_{pre,s} s_0 + \left(T^{-1}\right)_{pre,u} u_0 \\ \iff \left(T^{-1}\right)_{pre,s} s_0 &= y_0^{(pre)} - \left(T^{-1}\right)_{pre,u} u_0 \end{aligned}$$

where the subscripts in, e.g., $\left(T^{-1}\right)_{pre,s}$ denote the row indices corresponding to the predetermined variables and the column indices corresponding to the variables pertaining to the stable eigenvalues. Note that both $y_0^{(pre)}$ (as part of the model formulation) and u_0 (as solution at time $t=0$ of the unstable part of the system as described above) are known. Thus, if the given vector $\left(y_0^{(pre)} - \left(T^{-1}\right)_{pre,u} u_0\right)$ is contained in the column space of $\left(T^{-1}\right)_{pre,s}$, there exists an s_0 such that the initial value of the predetermined variables $y_0^{(pre)}$ does not contradict the initial model specification.

For $t \mapsto t+1$, i.e. obtaining s_{t+1} for given s_t, we proceed as follows. We obtain s_{t+1} by subtracting its expectation at time t from the first block of rows of the system (6), i.e. we subtract

$$\mathbb{E}_t\left(y_{t+1}^{(pre)}\right) = \left(T^{-1}\right)_{pre,s} \mathbb{E}_t\left(s_{t+1}\right) + \left(T^{-1}\right)_{pre,u} \mathbb{E}_t\left(u_{t+1}\right)$$

from

$$y_{t+1}^{(pre)} = \left(T^{-1}\right)_{pre,s} s_{t+1} + \left(T^{-1}\right)_{pre,u} u_{t+1}$$

and note that $y_{t+1}^{(pre)}$ is predetermined. Thus, we decompose $\left(T^{-1}\right)_{pre,s}\left(s_{t+1}\right)$ in its projection[9] on the space $H_z(t)$ and the innovation $\left(T^{-1}\right)_{pre,u}\left(u_{t+1} - \mathbb{E}_t\left(u_{t+1}\right)\right)$, i.e.

$$\begin{aligned} 0 = y_{t+1}^{(pre)} - \mathbb{E}_t\left(y_{t+1}^{(pre)}\right) &= \left(T^{-1}\right)_{pre,s}\left(s_{t+1} - \mathbb{E}_t\left(s_{t+1}\right)\right) + \left(T^{-1}\right)_{pre,u}\left(u_{t+1} - \mathbb{E}_t\left(u_{t+1}\right)\right) \\ \iff \left(T^{-1}\right)_{pre,s}\left(s_{t+1}\right) &= \left(T^{-1}\right)_{pre,s} \mathbb{E}_t\left(s_{t+1}\right) - \left(T^{-1}\right)_{pre,u}\left(u_{t+1} - \mathbb{E}_t\left(u_{t+1}\right)\right). \end{aligned} \tag{7}$$

Hence, if the right hand side of (7) is contained in the column space of $\left(T^{-1}\right)_{pre,s}$, there exists an s_{t+1} for given s_t which solves (6), is contained in $H_z(t)$, and satisfies the non-explosiveness condition.

Remark 11. It is important to realize that $(s_t)_{t\in\mathbb{N}}$, assuming that such a solution exists, has the same innovations as $(u_t)_{t\in\mathbb{N}}$. It follows that neither $(u_t)_{t\in\mathbb{N}}$ nor $(s_t)_{t\in\mathbb{N}}$ are predetermined and that the process $\begin{pmatrix} s_t \\ u_t \end{pmatrix}_{t\in\mathbb{N}}$ has a singular innovation covariance matrix. Note that the rank of the innovation covariance matrix is bounded from above by the number of unstable roots and that there might be multiple solutions $(s_t)_{t\in\mathbb{N}}$ (such that $s_t \in H_z(t)$ and that it satisfies the non-explosiveness condition) if there are strictly more non-predetermined variables than unstable roots.

Remark 12. Equation (7) is of paramount importance for understanding existence and (non)-uniqueness of a solution of Blanchard and Kahn's model. We want to obtain an s_{t+1} such that the predetermined variables $y_{t+1}^{(pre)}$ satisfy the initial model specification, i.e. we ask whether there is an s_{t+1} such that its one-step-ahead forecast error $\left(T^{-1}\right)_{pre,s}\left(s_{t+1} - \mathbb{E}_t\left(s_{t+1}\right)\right)$ offsets the one-step-ahead forecast error $\left(T^{-1}\right)_{pre,u}\left(u_{t+1} - \mathbb{E}_t\left(u_{t+1}\right)\right)$ of the solution of the unstable part of the system.

If there is more than one way to do that, i.e. $\left(T^{-1}\right)_{pre,u}\left(u_{t+1} - \mathbb{E}_t\left(u_{t+1}\right)\right)$ is contained in the column space of $\left(T^{-1}\right)_{pre,s}$ and the kernel of $\left(T^{-1}\right)_{pre,s}$ is non-trivial, the solution is not unique. Note that this non-uniqueness affects, in general, both predetermined and non-predetermined variables after transformation to original variables.

If there is no such s_{t+1} which ensures that $y_{t+1}^{(pre)} - \mathbb{E}_t\left(y_{t+1}^{(pre)}\right) = 0$ holds, there is no solution which satisfies the initial model specification. In other words, there is no way that the one-step-ahead prediction errors of the solution of the unstable part of the system get offset by the one-step-ahead prediction errors of the solution of the stable part of the system. We can still calculate a solution which however contradicts the original specification. Adding an additional non-predetermined variable amounts to including an additional component in the vector of endogenous forecast errors η_t in Sims notation, which in turn makes satisfying the existence condition easier. In [37], this situation is described by the fact that the endogenous forecast error cannot offset the influence of the exogenous variables, and hence there is no solution in $H_z(t)$ satisfying the non-explosiveness condition.

[9] Of course, $\mathbb{E}_t\left(s_{t+1}\right) = J_s s_t + T_{s,\bullet} C z_t$ is known at time t in equation (7).

Blanchard and Kahn's full rank assumption on $\left(T^{-1}\right)_{pre,s}$ and their counting rule. [8] state in their Proposition 1, 2, and 3 that under the assumption that $\left(T^{-1}\right)_{pre,s}$ is of full rank,

- there exists a unique solution $(y_t)_{t\in\mathbb{N}}$ if $n(u) = n(\neg pre)$,
- there does not exist a solution $(y_t)_{t\in\mathbb{N}}$ if $n(u) > n(\neg pre)$, and finally that
- there is an infinity of solutions if $n(u) < n(\neg pre)$.

This shaped the (oversimplifying) understanding that rational expectations models have a unique solution if there are as many "jump-variables", i.e. non-predetermined variables, as there are unstable roots of B. However, it is obvious from the derivation above that the following proposition holds.

Proposition 13. *A solution $(y_t)_{t\in\mathbb{N}}$ to the rational expectations model*

$$\begin{pmatrix} \mathbb{E}_t\left(y_{t+1}^{(pre)}\right) \\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right) \end{pmatrix} = B \begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix} + Cz_t, \quad t \in \mathbb{N}$$

satisfying firstly $y_t \in H_z(t),\ t \in \mathbb{N}$ and secondly the non-explosiveness condition

$$\forall t \in \mathbb{N} : \exists \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \in \mathbb{R}^{n(pre)+n(\neg pre)} \wedge \sigma_t \in \mathbb{R} \text{ such that}$$
$$-(1+i)^{\sigma_t} \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \leq \mathbb{E}_t \begin{pmatrix} y_{t+i}^{(pre)} \\ y_{t+i}^{(\neg pre)} \end{pmatrix} \leq (1+i)^{\sigma_t} \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \quad \forall i \geq 0$$

for bounded inputs $(z_t)_{t\in\mathbb{N}}$ satisfying

$$\forall t \in \mathbb{N} : \exists \bar{Z}_t \in \mathbb{R}^m \wedge \theta_t \in \mathbb{R} \text{ such that}$$
$$-(1+i)^{\theta_t} \bar{Z}_t \leq \mathbb{E}_t(z_{t+i}) \leq (1+i)^{\theta_t} \bar{Z}_t \quad \forall i \geq 0,$$

exists if $y_0^{(pre)} - \left(T^{-1}\right)_{pre,u} u_0$ (initial value) and $\left(T^{-1}\right)_{pre,u}(u_{t+1} - \mathbb{E}_t(u_{t+1}))$ (innovations) are contained in the column space of $\left(T^{-1}\right)_{pre,s}$ for all $t \in \mathbb{N}$.

Furthermore, the solution (if it exists) is unique if the kernel of $\left(T^{-1}\right)_{pre,s}$ is trivial.

3.1.1 Analysis of Blanchard and Kahn's model with the martingale difference method

[13] analyze the model of Blanchard and Kahn without restricting the processes satisfying the rational expectations equation (3) to be non-explosive which they consider potentially unjustified ([13] pages 129-133). However, their analysis is in error because they do not take into account that some endogenous variables are predetermined. They start by transforming

$$\begin{pmatrix} \mathbb{E}_t\left(y_{t+1}^{(pre)}\right) \\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right) \end{pmatrix} = B \begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix} + Cz_t, \quad t \in \mathbb{N}$$

to

$$\begin{pmatrix} \mathbb{E}_t(s_{t+1}) \\ \mathbb{E}_t(u_{t+1}) \end{pmatrix} = \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} + TCz_t, \quad t \in \mathbb{N}$$

and subsequently introduce (on page 131 below formula (4.93)) revision processes (compare section 4 starting on page 42)

$$\nu_t = s_t - \mathbb{E}_{t-1}(s_t)$$
$$\eta_t = u_t - \mathbb{E}_{t-1}(u_t).$$

By introducing these revision processes, they suggest that the components of ν_t and η_t are linearly independent of each other. However, this is obviously wrong because only $n(\neg pre)$ revision processes appear in the model.

Writing

$$\begin{pmatrix} y_{t+1}^{(pre)} \\ y_{t+1}^{(\neg pre)} - \varepsilon_{t+1} \end{pmatrix} = B \begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix} + Cz_t, \quad t \in \mathbb{N},$$

where $\varepsilon_t = y_t^{(\neg pre)} - \mathbb{E}_{t-1}\left(y_t^{(\neg pre)}\right)$, and premultiplying T, we obtain after rearranging

$$\begin{pmatrix} \left(I_{n(s)} - J_s z\right) s_t \\ \left(I_{n(u)} - J_u z\right) u_t \end{pmatrix} = T_{\bullet,\neg pre}\varepsilon_t + \begin{pmatrix} T_{s\bullet} \\ T_{u\bullet} \end{pmatrix} Cz_{t-1}. \tag{8}$$

We proceed to analyze existence and uniqueness of a solution of (8) with the methods put forward in [37] and [11] in order to develop an understanding as to how these methods are connected. For ease of presentation we assume that the exogenous process $(z_t)_{t\in\mathbb{N}}$ is white noise ξ_t (with zero mean and the identity matrix as covariance matrix).

Analyzing solutions of equation (8) with the method in [37]. For a more detailed derivation we refer to section 3.3.2 and 3.3.3. Since the backward solution does not satisfy the non-explosiveness condition (which in Blanchard and Kahn's model applies to all components of the endogenous variables), we focus on the forward solution

$$u_t = \left[-\sum_{i=0}^{\infty} \left(J_u z\right)^{-i-1}\right] \left(T_{u,\neg pre}\varepsilon_t + T_{u\bullet} C z_{t-1}\right) \tag{9}$$

and assume that a solution of the rational expectations model exists, i.e. $u_t \in H_z(t)$, $t \in \mathbb{N}$, and thus $\mathbb{E}_t(u_t) = u_t$ holds. The existence condition derived in Corollary 25 on page 28 in the notation of this example[10] is

$$span\left(J_u^{-1} T_{u\bullet} C\right) \subseteq span\left(T_{u,\neg pre}\right). \tag{10}$$

Since $u_t = \mathbb{E}_t(u_t)$, we obtain that the endogenous forecast errors ε_t (compare section 3.3 on page 20) can be expressed as a function of expectations at time t of future exogenous variables. Thus, in a solution (9) for the unstable part of system (8), we have found a representation of the endogenous forecast errors ε_t in terms of quantities known at time t.

The solution is unique, if we can express the endogenous forecast error $T_{s,\neg pre}\varepsilon_t$ influencing the "stable" part of the system as a function of the endogenous forecast error $T_{u,\neg pre}\varepsilon_t$ affecting the "unstable" part of the system (which in turn is a function of expectations at time t of future exogenous variables). The condition that the solution be unique is equivalent to the existence of a matrix Φ of dimension $(n(s) \times n(u))$ such that

$$T_{s,\neg pre} = \Phi T_{u,\neg pre},$$

compare section 3.3.4 starting on page 29.

Analyzing solutions of equation (8) with the method in [11]. First, note that the endogenous forecast error $\varepsilon_t = y_t^{(\neg pre)} - \mathbb{E}_{t-1}\left(y_t^{(\neg pre)}\right)$ is a linear function of the m-dimensional exogenous process ξ_t, i.e. we may write $\varepsilon_t = K\xi_t$, where K is of dimension $(n(\neg pre) \times m)$. Furthermore, we assume for expositional convenience that $n(u) = n(\neg pre) = m$.

[11] claim in their Theorem 4 on page 249 and 250 that, under the assumption that the exogenous process admits a stationary (finite or infinite) moving-average representation, there exists a unique solution to the rational expectations model if the number of unstable roots equals the number of free parameters. This is incorrect because one may only cancel as many unstable roots as there are linearly independent martingale difference sequences (as is also the case in [8] under their full rank assumption). Moreover, in the derivation in [11] it is implicitly assumed that the rank of the innovation covariance matrix coincides with the number of endogenous variables even though this excludes a great many state of the art models.

Indeed, they argue that their result holds by claiming that one can cancel an unstable root of a certain polynomial matrix by fixing one of the free parameters. However, this is in general not correct since the roots to be canceled out have to lie in the same space, taking account of which requires additional free parameters. The example below shows that in general $(n(\neg pre))^2$ free parameters are needed in order to cancel $n(u)$ unstable roots, we want to find a matrix K in equation (11) below such that $(T_{u,\neg pre}K + T_{u\bullet}Cz)$ can be factored as $\left(I_{n(u)} - J_u z\right) A(z)$, where $A(z)$ is a polynomial matrix of appropriate dimensions.

[10] Note that the timing convention in Blanchard and Kahn's model is different to the one Sims is using. For this reason, J_u^{-1} appears in the existence condition adjusted to the notation in this example.

Example 14. Consider the "unstable" part of the system (8), i.e.

$$\begin{aligned}\left(I_{n(u)} - J_u z\right) u_t &= \left(T_{u,\neg pre} K + T_{u\bullet} C z\right) \xi_t \\ &= \left(T_{u,\neg pre} K + J_u \left(J_u\right)^{-1} T_{u\bullet} C z\right) \xi_t,\end{aligned} \tag{11}$$

and note that (in accordance to (10)) the matrix polynomial can be factorized in the desired way if $span\left(J_u^{-1} T_{u\bullet} C\right) \subseteq span\left(T_{u,\neg pre}\right)$. In order to fix ideas, we assume $T_{u,\neg pre}$ to be invertible, take

$$K = -\left(T_{u,\neg pre}\right)^{-1} \left(J_u\right)^{-1} T_{u\bullet} C,$$

obtain

$$\begin{aligned}\left(I_{n(u)} - J_u z\right) u_t &= \left(-T_{u,\neg pre} \left(T_{u,\neg pre}\right)^{-1} \left(J_u\right)^{-1} T_{u\bullet} C + J_u \left(J_u\right)^{-1} T_{u\bullet} C z\right) \xi_t \\ &= \left(I_{n(u)} - J_u z\right) \left[-J_u^{-1} T_{u\bullet} C\right] \xi_t,\end{aligned}$$

and thus it follows that

$$\begin{aligned}u_t &= \frac{\left[adj\left(I_{n(u)} - J_u z\right)\right]\left(I_{n(u)} - J_u z\right)}{\det\left(I_{n(u)} - J_u z\right)} \left[-J_u^{-1} T_{u\bullet} C\right] \xi_t \\ &= \frac{\det\left(I_{n(u)} - J_u z\right) I_{n(u)}}{\det\left(I_{n(u)} - J_u z\right)} \left[-J_u^{-1} T_{u\bullet} C\right] \xi_t = -J_u^{-1} T_{u\bullet} C \xi_t.\end{aligned}$$

[39] proves (starting on his page 91) a similar result (using similar methods as above) to [11] under more restrictive assumptions[11]. Whiteman's result is correct and therefore consistent with the result in [8] (compare Proposition 13) but not consistent with the one in Theorem 4 on page 249 and 250 in [11] mentioned above.

3.1.2 Stochastic singularity as a different way of obtaining the right number of degrees of freedom.

As already mentioned in remark 8 on page 8, prescribing some variables to be predetermined is considered ad hoc by, e.g., [37] and [9, 11], and the corresponding solution process does not have a nice economic interpretation. A more natural way for obtaining a model in which some linear combinations of endogenous variables have trivial one-step-ahead prediction errors is to require that the stationary exogenous process $(z_t)_{t\in\mathbb{Z}}$ have a singular innovation covariance matrix. This insight into the structure of rational expectations model is illustrated by analyzing the Blanchard and Kahn model (3) with the martingale difference method developed in [9, 11] and by subsequently showing that the distinction in predetermined and non-predetermined is both unnecessary and restrictive.

Distinction between predetermined and non-predetermined variables. First, note that the endogenous forecast error $\varepsilon_t = y_t^{(\neg pre)} - \mathbb{E}_{t-1}\left(y_t^{(\neg pre)}\right)$ is a linear function of the innovations $\nu_{t+1} = z_{t+1} - \mathbb{E}_t\left(z_{t+1}\right)$ of the exogenous process, i.e. there is a matrix K of appropriate dimensions such that $\varepsilon_t = K\nu_t$. Second, since the covariance matrix of the innovations ν_t of the exogenous process is singular with rank q, we may write[12] $v_t = b\xi_t$, where the covariance matrix of ξ_t is non-singular. Thus, in the model (3), we obtain that

$$\varepsilon_t = Kb\xi_t$$

where $K \in \mathbb{R}^{n(\neg pre)\times m}$ and $b \in \mathbb{R}^{m\times q}$. Hence, we obtain that

$$\begin{aligned}\begin{pmatrix}\mathbb{E}_t\left(y_{t+1}^{(pre)}\right) \\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right)\end{pmatrix} &= \begin{pmatrix}y_{t+1}^{(pre)} \\ y_{t+1}^{(\neg pre)} - \varepsilon_{t+1}\end{pmatrix} \\ &= \begin{pmatrix}y_{t+1}^{(pre)} \\ y_{t+1}^{(\neg pre)}\end{pmatrix} - \begin{pmatrix}0 \\ I\end{pmatrix} Kb\xi_{t+1}.\end{aligned} \tag{12}$$

[11] Whiteman does not allow, e.g., for zeros at infinity.

[12] Note that b is unique up to orthogonal post-multiplication.

No distinction between predetermined and non-predetermined variables. Without imposing that some variables are predetermined, we obtain in the same way as above that

$$\begin{aligned}\mathbb{E}_t\left(y_{t+1}\right) &= y_{t+1} - \varepsilon_{t+1} \\ &= y_{t+1} - Rb\xi_{t+1}\end{aligned} \tag{13}$$

where $R \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^{m\times q}$. It follows that the number of linearly independent linear combinations of endogenous variables can be gauged by the rank of the innovation covariance matrix of the exogenous process. Also note that equation (13) is more general than (12). Thus, the distinction between predetermined and non-predetermined variables is both unnecessary and restrictive.

A more useful rule of thumb than $n\left(\neg pre\right) = n(u)$ for the uniqueness of a solution of the Blanchard and Kahn model would thus be that the number of unstable roots has to be equal to the rank of the innovation covariance matrix of the exogenous process.

3.2 King and Watson: Allowing for Zeros at Infinity and Zeros at Zero

[31] generalizes the model in [8] to models of the form

$$A\begin{pmatrix}\mathbb{E}_t\left(y_{t+1}^{(pre)}\right)\\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right)\end{pmatrix}=B\begin{pmatrix}y_t^{(pre)}\\ y_t^{(\neg pre)}\end{pmatrix}+C\mathbb{E}_t\left(z_t\right),\quad t\in\mathbb{N} \tag{14}$$

where $A\in\mathbb{R}^{n\times n}$, $n=n\left(pre\right)+n\left(\neg pre\right)$, $n(pre)$ and $n\left(\neg pre\right)$ are the dimensions of the predetermined and non-predetermined variables respectively, is allowed to be singular, but the determinant $\det\left(Az-B\right)$ of the matrix pencils $Az-B$, $z\in\mathbb{C}$, must not be identically zero. All other assumption of Blanchard and Kahn's model remain the same.

Forward "shift". King and Watson introduce, following [36] (Chapter XI, Section 21, page 307), the forward shift[13] operator F which operates on the time index of the endogenous process but not on the information set, i.e.

$$F\mathbb{E}_t\left(y_t\right)=\mathbb{E}_t\left(y_{t+1}\right),\ t\in\mathbb{N}.$$

Thus, they write equation (14) above as

$$\left(AF-B\right)\begin{pmatrix}\mathbb{E}_t\left(y_t^{(pre)}\right)\\ \mathbb{E}_t\left(y_t^{(\neg pre)}\right)\end{pmatrix}=C\mathbb{E}_t\left(z_t\right),\quad t\in\mathbb{N}. \tag{15}$$

Decoupling. Similar to the approach in [8], King and Watson transform the equations by premultiplying equation (15) with a non-singular matrix V, and transform the endogenous variables $\begin{pmatrix}y_{t+1}^{(pre)}\\ y_{t+1}^{(\neg pre)}\end{pmatrix}$ by premultiplying them with a non-singular matrix W, i.e. we obtain

$$\left(VAW^{-1}F-VBW^{-1}\right)W\begin{pmatrix}\mathbb{E}_t\left(y_t^{(pre)}\right)\\ \mathbb{E}_t\left(y_t^{(\neg pre)}\right)\end{pmatrix}=VC\mathbb{E}_t\left(z_t\right),\quad t\in\mathbb{N}.$$

$$\iff\left(A^*F-B^*\right)\mathbb{E}_t\begin{pmatrix}s_t\\ u_t\\ i_t\end{pmatrix}=C^*\mathbb{E}_t\left(z_t\right),\quad t\in\mathbb{N}.$$

where the separation of the canonical variables s_t, u_t, and i_t is determined by the location of the zeros of the pencil (A^*z-B^*).

If the matrix A is singular, King and Watson use the theory developed in [20] (Chapter 12, pages 24-28) on regular matrix pencils $Az-B$, where z is a complex variable, A and B are square matrices and $\det\left(Az-B\right)$ is not identically

[13] Note that this operator does not correspond to an isometric or unitary transformation, in the case of a stationary process $(y_t)_{t\in\mathbb{N}}$ or $(y_t)_{t\in\mathbb{Z}}$ respectively, generating the process, compare [19] page 461 and 462. [31] remark in footnote 4 on page 1017 that Fy_t is not defined because the conditioning set is not specified in this case.

zero[14]. They obtain matrices V and W of a more complex[15] nature and finally obtain

$$\left[\begin{pmatrix} I & \\ & N \end{pmatrix} F - \begin{pmatrix} \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} & \\ & I \end{pmatrix}\right] \mathbb{E}_t \begin{pmatrix} s_t \\ u_t \\ i_t \end{pmatrix} = C^* z_t, \quad t \in \mathbb{N}, \tag{16}$$

where N is a nilpotent matrix, i.e. there exists a positive integer l such that $N^l = 0$, the matrices J_s and J_u contain Jordan blocks with eigenvalues of absolute value smaller than or equal to unity[16] and larger than unity respectively. The dimensions of the canonical variables (s_t) and $(u_t)_{t\in\mathbb{N}}$ correspond to the respective dimensions of J_s and J_u.

If the matrix A in (15) is non-singular, the equation is premultiplied by A^{-1}, and subsequently the Jordan decomposition of $A^{-1}B = T^{-1}JT$, where T are a basis of the left-invariant subspaces of B^*, is considered. Thus, we obtain for equation (16) with $V = TA^{-1}$ and $W = T$ that

$$\begin{aligned} &\left(\underbrace{(TA^{-1})}_{=V} A \underbrace{T^{-1}}_{=W^{-1}} F - TA^{-1}BT^{-1}\right) T \begin{pmatrix} \mathbb{E}_t \begin{pmatrix} y_{t+1}^{(pre)} \end{pmatrix} \\ \mathbb{E}_t \begin{pmatrix} y_{t+1}^{(\neg pre)} \end{pmatrix} \end{pmatrix} = VCz_t, \quad t \in \mathbb{N}. \\ &\iff \left(I_n F - \begin{pmatrix} J_s & \\ & J_u \end{pmatrix}\right) \mathbb{E}_t \begin{pmatrix} s_t \\ u_t \end{pmatrix} = C^* z_t, \quad t \in \mathbb{N}. \end{aligned}$$

3.2.1 Obtaining a solution of the model (14).

King and Watson do not state a theorem[17] in [31], but only note that they "...show that initial conditions on..." $y^{(\neg pre)}$ "...can be determined and a unique solution obtained if (i) the solution is restricted to be stable and (ii) a certain sub-matrix of..." W "...has full rank", on page 1020, line 13 in [31]. Their conditions are slightly stronger than the conditions given in our Proposition 16 below.

[14] Note that the rank deficiencies of A and B correspond to zeros at infinity and zeros at zero respectively.

[15]

$$\begin{aligned}
(Az - B) &= [A(z-\alpha) - (B - A\alpha)] \\
&= \left[A(B-A\alpha)^{-1}(z-\alpha) - I\right](B - A\alpha) \\
&= Q^{-1}\left[\begin{pmatrix} \mu_+ & \\ & \mu_0 \end{pmatrix}(z-\alpha) - I\right] Q(B - A\alpha) \\
&= Q^{-1}\left[\begin{pmatrix} \mu_+ & \\ & \mu_0 \end{pmatrix} z - \begin{pmatrix} I + \alpha\mu_+ & \\ & I + \alpha\mu_0 \end{pmatrix}\right] Q(B - A\alpha) \\
&= Q^{-1}\underbrace{\begin{pmatrix} \mu_+ & \\ & I + \alpha\mu_0 \end{pmatrix}}_{=S^{-1}}\left\{\begin{pmatrix} \mu_+ & \\ & I + \alpha\mu_0 \end{pmatrix}^{-1}\left[\begin{pmatrix} \mu_+ & \\ & \mu_0 \end{pmatrix} z - \begin{pmatrix} I + \alpha\mu_+ & \\ & I + \alpha\mu_0 \end{pmatrix}\right]\right\} Q(B - A\alpha) \\
&= Q^{-1}\underbrace{\begin{pmatrix} \mu_+ & \\ & I + \alpha\mu_0 \end{pmatrix}}_{=S^{-1}}\left\{\begin{pmatrix} I & \\ & (I + \alpha\mu_0)^{-1}\mu_0 \end{pmatrix} z - \begin{pmatrix} \mu_+^{-1}(I + \alpha\mu_+) & \\ & I \end{pmatrix}\right\} Q(B - A\alpha) \\
&= Q^{-1}S^{-1}\left\{\begin{pmatrix} I & \\ & (I + \alpha\mu_0)^{-1}\mu_0 \end{pmatrix} z - \begin{pmatrix} G_+ & \\ & I \end{pmatrix}^{-1}\begin{pmatrix} \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} & \\ & I \end{pmatrix}\begin{pmatrix} G_+ & \\ & I \end{pmatrix}\right\} Q(B - A\alpha) \\
&= Q^{-1}S^{-1}\underbrace{\begin{pmatrix} G_+ & \\ & I \end{pmatrix}^{-1}}_{=G^{-1}}\left\{\begin{pmatrix} I & \\ & (I + \alpha\mu_0)^{-1}\mu_0 \end{pmatrix} z - \begin{pmatrix} \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} & \\ & I \end{pmatrix}\right\}\underbrace{\begin{pmatrix} G_+ & \\ & I \end{pmatrix}}_{=G} Q(B - A\alpha) \\
&= \underbrace{[GSQ]^{-1}}_{=V^{-1}}\left\{\begin{pmatrix} I & \\ & (I + \alpha\mu_0)^{-1}\mu_0 \end{pmatrix} z - \begin{pmatrix} \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} & \\ & I \end{pmatrix}\right\}\underbrace{[GQ(B - A\alpha)]}_{=W}
\end{aligned}$$

[16] We remind that [32] note on page 72 below their formula (19) that unit roots are considered to be stable because they do not violate the non-explosiveness condition. A root λ is treated as unstable if $\beta\lambda > 1$, where $\beta \in (0,1)$ is a discount factor.

[17] They only refer to [8].

Solution for the unstable[18] part of the system (16). We start by deriving a solution $(U_t)_{t\in\mathbb{N}} = \begin{pmatrix} u_t \\ i_t \end{pmatrix}_{t\in\mathbb{N}}$ for the unstable (including the zeros at infinity of the determinantal equation $\det(Az - B)$) part using the fact that we restrict the solution space to non-explosive (in the sense of Proposition 13) solutions[19]. As solution for the unstable part of the system, we thus obtain[20] as solution pertaining to the subsystem corresponding to finite, unstable roots

$$\begin{aligned} u_t &= (IF - J_u)^{-1} C^*_{u,\bullet} \mathbb{E}_t(z_t) \\ &= -J_u^{-1} \left(I - J_u^{-1}F\right)^{-1} C^*_{u,\bullet} \mathbb{E}_t(z_t) \\ &= -J_u^{-1} \sum_{k=0}^{\infty} \left(J_u^{-1}\right)^k F^k C^*_{u,\bullet} \mathbb{E}_t(z_t) \\ &= -\sum_{k=0}^{\infty} J_u^{-k-1} C^*_{u,\bullet} \mathbb{E}_t(z_{t+k}) \end{aligned}$$

and as solution pertaining to the subsystem corresponding to infinite roots

$$\begin{aligned} i_t &= (NF - I)^{-1} C^*_{i,\bullet} \mathbb{E}_t(z_t) \\ &= \frac{adj(NF - I)}{\det(NF - I)} C^*_{i,\bullet} \mathbb{E}_t(z_t) \\ &= (-1)^{n(i)} adj(NF - I) C^*_{i,\bullet} \mathbb{E}_t(z_t) \end{aligned}$$

where $n(i)$ denotes the dimension of the square nilpotent matrix N, or (compare [11] page 232 and [4] page 154, formula (3.57))

$$\begin{aligned} i_t &= -(I - NF)^{-1} C^*_{i,\bullet} \mathbb{E}_t(z_t) \\ &= -\sum_{k=0}^{n(i)-1} N^k C^*_{i,\bullet} \mathbb{E}_t(z_{t+k}). \end{aligned} \tag{17}$$

Remark 15. In both derivations for the solution $(i_t)_{t\in\mathbb{N}}$ pertaining to the subsystem corresponding to infinite roots we see that there are only finitely many expectations at time t of future exogenous variables. Moreover, there is no ambiguity as to whether one should consider the forward or backward solution, i.e. whether one considers the power series development of $[\det(NF - I)]^{-1}$ in terms of non-negative or non-positive powers of F, because the determinant of $(NF - I)$ is either $+1$ or -1 and hence constant. The reason why the canonical variable i_t is considered unstable is that $i_t = -\sum_{k=0}^{n(i)-1} N^k C^*_{i,\bullet} z_{t+k}$ is in general not contained in $H_z(t)$. It follows that, regarding the solution method, there is formally no difference to the case considered by Blanchard and Kahn.

Obtaining a solution for the original variables. In contrast to [8], King and Watson do not first derive a solution $(s_t)_{t\in\mathbb{N}}$ for the stable part of the decoupled system but obtain $\left(y_t^{(pre)}\right)_{t\in\mathbb{N}}$ directly by using the first block of rows in the equation $\begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix} = W^{-1} \begin{pmatrix} s_t \\ U_t \end{pmatrix}$. In order to obtain (if it exists) a solution $\left(y_t^{(\neg pre)}\right)_{t\in\mathbb{N}}$ for the non-predetermined variables, they use the second block of rows in the equation $\begin{pmatrix} s_t \\ U_t \end{pmatrix} = W \begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix}$.

[18] The word "unstable" refers to the location of the zeros of $Az - B$, $z \in \mathbb{C}$.

[19] Note that a solution $(i_t)_{t\in\mathbb{N}}$ of the subsystem of (16) pertaining to infinite roots never violates the non-explosiveness condition. However, it is always considered to be "explosive" because the solution $(i_t)_{t\in\mathbb{N}}$ may involve expectations at time t of future exogenous variables (but no past exogenous variables) as will become clear below.

[20] Note that the formula (8) in [31] on page 1020 is not correct. Their J_u^{-h} should be J_u^{-h-1}.

Initialization: Obtaining $y_0^{(\neg pre)}$ for given U_0 and $y_0^{(pre)}$. Initial conditions for $y_0^{(pre)}$ are given, for $y_0^{(\neg pre)}$ a solution is obtained using the variable transformation $\begin{pmatrix} s_0 \\ U_0 \end{pmatrix} = W \begin{pmatrix} y_0^{(pre)} \\ y_0^{(\neg pre)} \end{pmatrix}$, i.e.

$$\begin{aligned} U_0 &= W_{U,pre} y_0^{(pre)} + W_{U,\neg pre} y_0^{(\neg pre)} \\ \iff W_{U,\neg pre} y_0^{(\neg pre)} &= U_0 - W_{U,pre} y_0^{(pre)}. \end{aligned}$$

If $U_0 - W_{U,pre} y_0^{(pre)}$ is contained in the column space of $W_{U,\neg pre}$, we obtain a $y_0^{(\neg pre)}$ satisfying the equation above. If the kernel of $W_{U,\neg pre}$ is trivial, such a $y_0^{(\neg pre)}$ is unique.

Induction step $t \mapsto t+1$: Obtaining $y_{t+1}^{(pre)}$ and $y_{t+1}^{(\neg pre)}$ for given $y_t^{(pre)}$, $y_t^{(\neg pre)}$, and U_t. For the predetermined variables, we first use the inverse of the variable transformation to represent $y_{t+1}^{(pre)}$ as a function of the stable and unstable part, i.e.

$$y_{t+1}^{(pre)} = \left(W^{-1}\right)_{pre,s} s_{t+1} + \left(W^{-1}\right)_{pre,U} U_{t+1},$$

take expectations at time t and use the predeterminedness of $y_{t+1}^{(pre)}$, i.e.

$$\begin{aligned} \mathbb{E}_t\left(y_{t+1}^{(pre)}\right) &= \left(W^{-1}\right)_{pre,s} \mathbb{E}_t\left(s_{t+1}\right) + \left(W^{-1}\right)_{pre,U} \mathbb{E}_t\left(U_{t+1}\right) \\ &= y_{t+1}^{(pre)} \end{aligned}$$

and finally replace $\mathbb{E}_t\left(s_{t+1}\right)$ and s_t by known quantities, i.e.

$$\begin{aligned} y_{t+1}^{(pre)} &= \left(W^{-1}\right)_{pre,s} \mathbb{E}_t\left(s_{t+1}\right) + \left(W^{-1}\right)_{pre,U} \mathbb{E}_t\left(U_{t+1}\right) \\ &= \left(W^{-1}\right)_{pre,s} \left[J_s s_t + V_{s,\bullet} C z_t\right] + \left(W^{-1}\right)_{pre,U} \mathbb{E}_t\left(U_{t+1}\right) \\ &= \left(W^{-1}\right)_{pre,s} \left[J_s \left\{W_{s,pre} y_t^{(pre)} + W_{s,\neg pre} y_t^{(\neg pre)}\right\} + V_{s,\bullet} C z_t\right] + \left(W^{-1}\right)_{pre,U} \mathbb{E}_t\left(U_{t+1}\right). \end{aligned} \tag{18}$$

Regarding the non-predetermined variables, we proceed analogously to the initialization step, i.e. we use the second block of rows in the variable transformation $\begin{pmatrix} s_{t+1} \\ U_{t+1} \end{pmatrix} = W \begin{pmatrix} y_{t+1}^{(pre)} \\ y_{t+1}^{(\neg pre)} \end{pmatrix}$ and obtain

$$\begin{aligned} U_{t+1} &= W_{U,pre} y_{t+1}^{(pre)} + W_{U,\neg pre} y_{t+1}^{(\neg pre)} \\ \iff W_{U,\neg pre} y_{t+1}^{(\neg pre)} &= U_{t+1} - W_{U,pre} y_{t+1}^{(pre)}. \end{aligned} \tag{19}$$

Thus, if $U_{t+1} - W_{U,pre} y_{t+1}^{(pre)}$ is contained in the column space of $W_{U,\neg pre}$, there exists a $y_{t+1}^{(\neg pre)}$ satisfying all requirements of a solution.

Blanchard and Kahn result generalized by King and Watson. We may thus generalize Proposition (13) in the following way.

Proposition 16. *A solution $(y_t)_{t\in\mathbb{N}}$ to the rational expectations model*

$$A \begin{pmatrix} \mathbb{E}_t\left(y_{t+1}^{(pre)}\right) \\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right) \end{pmatrix} = B \begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix} + C z_t, \quad t \in \mathbb{N}$$

satisfying firstly $y_t \in H_z(t),\ t \in \mathbb{N}$ and secondly the non-explosiveness condition

$$\forall t \in \mathbb{N} : \exists \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \in \mathbb{R}^{n(pre)+n(\neg pre)} \wedge \sigma_t \in \mathbb{R} \text{ such that } -(1+i)^{\sigma_t} \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \leq \mathbb{E}_t \begin{pmatrix} y_{t+i}^{(pre)} \\ y_{t+i}^{(\neg pre)} \end{pmatrix} \leq (1+i)^{\sigma_t} \begin{pmatrix} \bar{y}_t^{(pre)} \\ \bar{y}_t^{(\neg pre)} \end{pmatrix} \quad \forall i \geq 0,$$

for bounded inputs $(z_t)_{t\in\mathbb{N}}$, i.e.

$$\forall t \in \mathbb{N} : \exists \bar{Z}_t \in \mathbb{R}^m \wedge \theta_t \in \mathbb{R} \text{ such that } -(1+i)^{\theta_t} \bar{Z}_t \leq \mathbb{E}_t\left(z_{t+i}\right) \leq (1+i)^{\theta_t} \bar{Z}_t \quad \forall i \geq 0,$$

exists if $U_t - W_{U,pre} y_t^{(pre)}$ is contained in the column space of $W_{U,\neg pre}$ for all $t \in \mathbb{N}$.

Furthermore, the solution (if it exists) is unique if the kernel of $W_{U,\neg pre}$ is trivial.

Comparison with the method used in Blanchard and Kahn. Assuming that $A = I$ and $n(pre) = n(s)$, note that the matrix W in the derivation of [31] is identical to the matrix T in the derivation of [8]; both relate the original variables $\begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix}$ to the canonical variables $\begin{pmatrix} s_t \\ u_t \end{pmatrix}$. While Blanchard and Kahn require that $y_0^{(pre)} - \left(T^{-1}\right)_{pre,u} u_0$ and $\left(T^{-1}\right)_{pre,u} \{u_{t+1} - \mathbb{E}_t(u_{t+1})\}$ (for all $t \in \mathbb{N}$) be contained in the column space spanned by $\left(T^{-1}\right)_{pre,s}$, King and Watson require that $U_t - W_{U,pre} y_t^{(pre)}$ be (for all $t \in \mathbb{N}$) contained in the column space spanned by $W_{u,\neg pre}$. As can be easily seen[21], if one of these matrices is non-singular (assuming it is a square matrix) the same holds for the other matrix.

In order to interpret the condition that $U_t - W_{U,pre} y_t^{(pre)}$ be (for all $t \in \mathbb{N}$) contained in the column space spanned by $W_{u,\neg pre}$, we take conditional expectations at time t of equation (19) and subtract it from equation (19), i.e. we obtain

$$W_{U,\neg pre}\left[y_{t+1}^{(\neg pre)} - \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right)\right] = U_{t+1} - \mathbb{E}_t(U_{t+1}).$$

We ask whether there exist innovations of the non-predetermined variables such that $W_{u,\neg pre}$ times these innovations coincides with the innovations of the solution of the unstable part of the system. This ensures the existence of a solution of the rational expectations model satisfying the original model specification, in particular the term $\left(y_{t+1}^{(pre)} - \mathbb{E}_t\left(y_{t+1}^{(pre)}\right)\right)$ is indeed zero. In the Blanchard and Kahn model, the same effect is achieved by finding an s_t such that the innovations of the solution $(s_t)_{t\in\mathbb{N}}$ of the stable part of the system offset the innovations of the solution $(u_t)_{t\in\mathbb{N}}$ of the unstable part of the system. Note that if the kernel of $W_{u,\neg pre}$ is non-trivial, the possible indeterminacy also affects the predetermined variables through equation (18).

3.2.2 System reduction

This section shows that zeros at infinity of the pencil $Az - B$, $z \in \mathbb{C}$, are not interesting for describing the dynamics of the system. In their subsequent paper [32], King and Watson transform system (16) further and link (under suitable conditions outlined below) the variables pertaining to the zeros at infinity of the pencil $Az - B$ to dynamic identities in the following way. Starting from equation (16), i.e.

$$\left[\begin{pmatrix} I & \\ & N \end{pmatrix} F - \begin{pmatrix} \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} & \\ & I \end{pmatrix}\right] \mathbb{E}_t \begin{pmatrix} s_t \\ u_t \\ i_t \end{pmatrix} = C^* \mathbb{E}_t(z_t), \quad t \in \mathbb{N},$$

they obtain by first left-multiplying

$$\eta(F) = \begin{pmatrix} I & \\ & (NF - I)^{-1} \end{pmatrix}$$

and then left-multiplying[22]

$$T(F) = \begin{pmatrix} \hat{W}^{-1} & -\hat{W}^{-1}\left(FI - \begin{pmatrix} J_s & \\ & J_u \end{pmatrix}\right)\begin{pmatrix} W_{sf} \\ W_{uf} \end{pmatrix} W_{if}^{-1} \\ 0 & -W_{if}^{-1} \end{pmatrix}$$

[21] Consider the partitioned non-singular matrix $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and assume that T_{11} is non-singular. Left-multiplying $\begin{pmatrix} I & \\ -T_{21}T_{11}^{-1} & I \end{pmatrix}$ on T we see that the determinant of $\left(T_{22} - T_{21}T_{11}^{-1}T_{12}\right)$ is necessarily unequal from zero, i.e.

$$\det\left(\begin{pmatrix} I & \\ -T_{21}T_{11}^{-1} & I \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\right) = \det\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} - T_{21}T_{11}^{-1}T_{12} \end{pmatrix} = \det(T_{11}) \det\left(T_{22} - T_{21}T_{11}^{-1}T_{12}\right).$$

The inverse of A takes the form

$$T^{-1} = \begin{pmatrix} T_{11}^{-1} + T_{11}^{-1}T_{12}\left(T_{22} - T_{21}T_{11}^{-1}T_{12}\right)^{-1} T_{21}T_{11}^{-1} & -T_{11}^{-1}T_{12}\left(T_{22} - T_{21}T_{11}^{-1}T_{12}\right)^{-1} \\ -\left(T_{22} - T_{21}T_{11}^{-1}T_{12}\right)^{-1} T_{21}T_{11}^{-1} & \left(T_{22} - T_{21}T_{11}^{-1}T_{12}\right)^{-1} \end{pmatrix},$$

compare e.g. [26] pages 417-420. Hence, the element $\left(T^{-1}\right)_{22} = \left(T_{22} - T_{21}T_{11}^{-1}T_{12}\right)^{-1}$ is non-singular.

[22] Note that W_{if} can be chosen to be invertible under the assumptions outlined in [32], i.e. a rank condition which implies a unique solution of the original system. If follows from the invertibility of W_{if} that also $\hat{W}$ is invertible.

where $\hat{W} = \left[\begin{pmatrix} W_{s,pre} & W_{s,\neg p\tilde{r}e} \\ W_{s,pre} & W_{u,\neg p\tilde{r}e} \end{pmatrix} - \begin{pmatrix} W_{sf} \\ W_{uf} \end{pmatrix} W_{if}^{-1} \begin{pmatrix} W_{i,pre} & W_{i,\neg p\tilde{r}e} \end{pmatrix} \right]$ in which the subscripts $\neg p\tilde{r}e$ and f refer to the new vector of non-predetermined variables $\tilde{y}_t^{(\neg pre)}$ (a subvector of $y_t^{(\neg pre)}$ which gets linked to the canonical variables u_t) and a subvector f_t of $y_t^{(\neg pre)}$ which gets linked to the canonical variables i_t, a system of the form[23]

$$\begin{pmatrix} \left(\begin{pmatrix} I_{n(pre)} & \\ & I_{n(\neg pre)} \end{pmatrix} F - \hat{W}^{-1} J \hat{W} \right) & 0 \\ -W_{if}^{-1} \begin{pmatrix} W_{i,pre} & W_{i,\neg p\tilde{r}e} \end{pmatrix} & I \end{pmatrix} \mathbb{E}_t \begin{pmatrix} y_t^{(pre)} \\ \tilde{y}_t^{(\neg pre)} \\ f_t \end{pmatrix} = T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right). \tag{20}$$

This equation is obviously equivalent to the system

$$f_t = W_{if}^{-1} \begin{pmatrix} W_{i,pre} & W_{i,\neg p\tilde{r}e} \end{pmatrix} \begin{pmatrix} y_t^{(pre)} \\ \tilde{y}_t^{(\neg pre)} \end{pmatrix} + T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right).$$

$$\mathbb{E}_t \begin{pmatrix} y_{t+1}^{(pre)} \\ \tilde{y}_{t+1}^{(\neg pre)} \end{pmatrix} = \hat{W}^{-1} J \hat{W} \begin{pmatrix} y_t^{(pre)} \\ \tilde{y}_t^{(\neg pre)} \end{pmatrix} + T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right).$$

The first equation does not involve conditional expectations and the second one involves a pencil without zeros at infinity. King and Watson prove in their Theorem 1 that under the assumption of a unique solution of the rational expectations model (14), i.e.

$$A \begin{pmatrix} \mathbb{E}_t \left(y_{t+1}^{(pre)} \right) \\ \mathbb{E}_t \left(y_{t+1}^{(\neg pre)} \right) \end{pmatrix} = B \begin{pmatrix} y_t^{(pre)} \\ y_t^{(\neg pre)} \end{pmatrix} + C \mathbb{E}_t \left(z_t \right), \quad t \in \mathbb{N},$$

the reduction described above, i.e. finding a subvector f_t of $y_t^{(\neg pre)}$ such that equation (20) holds, is possible. In their Theorem 2, they show that if the original system has no redundant equations, i.e. $\det(Az - B) \not\equiv 0$, and if a process that satisfies system (14) exists from all initial conditions $y_0^{(pre)}$ (which means that the column space of $W_{U,\neg pre}$ must contain the space spanned by the columns of $W_{U,pre}$ and U_0), then the solutions of the original and the reduced system are the same. This is proved in [32] in a constructive way by providing an algorithm.

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[23]

$$\begin{pmatrix} \hat{W}^{-1} & -\hat{W}^{-1} \left(FI - \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} \right) \begin{pmatrix} W_{sf} \\ W_{uf} \end{pmatrix} W_{if}^{-1} \\ 0 & W_{if}^{-1} \end{pmatrix} \begin{pmatrix} I & \\ & (NF - I)^{-1} \end{pmatrix} \times \cdots$$

$$\cdots \times \left[\begin{pmatrix} I & \\ & N \end{pmatrix} F - \begin{pmatrix} \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} & \\ & I \end{pmatrix} \right] \mathbb{E}_t \begin{pmatrix} s_t \\ u_t \\ i_t \end{pmatrix} = T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right)$$

$$\iff \begin{pmatrix} \hat{W}^{-1} & -\hat{W}^{-1} \left(FI - \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} \right) \begin{pmatrix} W_{sf} \\ W_{uf} \end{pmatrix} W_{if}^{-1} \\ 0 & W_{if}^{-1} \end{pmatrix} \begin{pmatrix} \left[\begin{pmatrix} I_{n(s)} & \\ & I_{n(u)} \end{pmatrix} F - \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} \right] & \\ & I \end{pmatrix} \times \cdots$$

$$\cdots \times \begin{pmatrix} W_{s,pre} & W_{s,\neg p\tilde{r}e} & W_{sf} \\ W_{u,pre} & W_{u,\neg p\tilde{r}e} & W_{uf} \\ W_{i,pre} & W_{i,\neg p\tilde{r}e} & W_{if} \end{pmatrix} \mathbb{E}_t \begin{pmatrix} y_t^{(pre)} \\ \tilde{y}_t^{(\neg pre)} \\ f_t \end{pmatrix} = T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right)$$

$$\iff \begin{pmatrix} \hat{W}^{-1} & -\hat{W}^{-1} \left(FI - \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} \right) \begin{pmatrix} W_{sf} \\ W_{uf} \end{pmatrix} W_{if}^{-1} \\ 0 & W_{if}^{-1} \end{pmatrix} \times \cdots$$

$$\iff \cdots \times \begin{pmatrix} (IF - J) \begin{pmatrix} W_{s,pre} & W_{s,\neg p\tilde{r}e} \\ W_{u,pre} & W_{u,\neg p\tilde{r}e} \end{pmatrix} & (IF - J) \begin{pmatrix} W_{sf} \\ W_{uf} \end{pmatrix} \\ \begin{pmatrix} W_{i,pre} & W_{i,\neg p\tilde{r}e} \end{pmatrix} & W_{if} \end{pmatrix} \mathbb{E}_t \begin{pmatrix} y_t^{(pre)} \\ \tilde{y}_t^{(\neg pre)} \\ f_t \end{pmatrix} = T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right)$$

$$\iff \begin{pmatrix} \hat{W}^{-1} (IF - J) \underbrace{\left[\begin{pmatrix} W_{s,pre} & W_{s,\neg p\tilde{r}e} \\ W_{s,pre} & W_{u,\neg p\tilde{r}e} \end{pmatrix} - \begin{pmatrix} W_{sf} \\ W_{uf} \end{pmatrix} W_{if}^{-1} \begin{pmatrix} W_{i,pre} & W_{i,\neg p\tilde{r}e} \end{pmatrix} \right]}_{=\hat{W}} & 0 \\ -W_{if}^{-1} \begin{pmatrix} W_{i,pre} & W_{i,\neg p\tilde{r}e} \end{pmatrix} & I \end{pmatrix} \mathbb{E}_t \begin{pmatrix} y_t^{(pre)} \\ \tilde{y}_t^{(\neg pre)} \\ f_t \end{pmatrix} = T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right)$$

$$\iff \begin{pmatrix} \left(IF - \hat{W}^{-1} J \hat{W} \right) & 0 \\ -W_{if}^{-1} \begin{pmatrix} W_{i,pre} & W_{i,\neg p\tilde{r}e} \end{pmatrix} & I \end{pmatrix} \mathbb{E}_t \begin{pmatrix} y_t^{(pre)} \\ \tilde{y}_t^{(\neg pre)} \\ f_t \end{pmatrix} = T(F)\eta(F)C^* \mathbb{E}_t \left(z_t \right).$$

3.3 Sims: No distinction between predetermined and non-predetermined variables

[37] generalizes [8] and [31] in various ways. First, there is no distinction between predetermined and non-predetermined variables. The structure of the model, i.e. the parameter matrices $(\Gamma_0, \Gamma_1, C, \Psi, \Pi)$ below, implies that certain linear combinations of the endogenous variables are predetermined, i.e. have no endogenous forecast error. Thus, the researcher does not need to specify in advance which variables are predetermined. Second, the non-explosiveness condition (to be specified precisely below) does not apply to every component individually, but only to certain linear combinations of the endogenous variables. Last, "it covers all linear models with expectational error terms" (page 1 in [37]), which means that after having obtained a system of the form described below one can consider the problem of obtaining "a solution of the rational expectation model" to be solved.

On a fundamental level, it should be noted that Sims uses a different solution method than [8] and [31]. He uses the martingale difference method introduced by [9, 11]. However, Sims only shows through an example[24] which omits intricacies shown to be important in [9, 11], how to obtain a system in his canonical form from a rational expectations model[25]. The discussion of [9, 11] will also make clear that this should not be considered a trivial task.

Sims' canonical form. Sims considers the model[26]

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + C + \Psi z_t + \Pi \eta_t, \quad t \in \{1, \ldots, T\} \tag{21}$$

where y_t are the n-dimensional endogenous variables, z_t are the m-dimensional exogenous variables, η_t are the k-dimensional so-called endogenous forecast errors satisfying[27] $\mathbb{E}_t(\eta_{t+1}) = 0$, Γ_0 and Γ_1 are (complex) matrices of dimension $(n \times n)$ which may be singular, C is a vector of constants, $\Psi \in \mathbb{R}^{n \times m}$, and $\Pi \in \mathbb{R}^{n \times k}$.

Note that Sims does not put any growth restriction on the exogenous process $(z_t)_{t \in \mathbb{N}}$. We assume[28] here, that the exogenous process is (weakly) stationary.

A stochastic processes $(y_t)_{t \in \mathbb{N}}$ which satisfies (21) at every point in time, which satisfies the non-explosiveness condition

$$\mathbb{E}_t\left(\xi_i^{-h} \phi_i y_{t+h}\right) \xrightarrow{h \to \infty} 0, \ \xi_i > 1, \phi_i \in \mathbb{R}^{1 \times n} \ i \in \{1, \ldots, m\}, \tag{22}$$

where convergence is understood in mean square sense, and for which $y_t \in H_z(t), \ t \in \mathbb{N}$, holds is called a solution of the rational expectations model.

Remark 17 (Homogenous solution). Sims does not consider homogenous solutions of the rational expectations model. As an example and assuming that the problem is well behaved, we consider the generalized eigenvalue μ_j with modulus larger than one and the corresponding right eigenvector z_j of the pair (Γ_0, Γ_1). Substituting the process $y_t = (\mu_j)^{-t} z_j$ in the difference equation $(\Gamma_0 - \Gamma_1 z) y_t = 0$, i.e.

$$\Gamma_0 (\mu_j)^{-t} z_j - \Gamma_1 (\mu_j)^{-(t-1)} z_j = \underbrace{\left[(\Gamma_0 - \Gamma_1 \mu_j) z_j\right]}_{=0} (\mu_j)^{-t} = 0$$

verifies that the deterministic process $y_t = (\mu_j)^{-t} z_j$ is indeed a solution of the rational expectations model.

Regarding stationary solutions of the homogenous difference equation $(\Gamma_0 - \Gamma_1 z) y_t^h = 0$, note that as long as the solutions of this equation are orthogonal to the particular stationary solution of $(\Gamma_0 - \Gamma_1 z) y_t^p = 0$, the sum $\left(y_t^p + y_t^h\right)_{t \in \mathbb{N}}$ is stationary as well.

Outline of Sims' method. Since [37] is not written in a linear way, we want to give an outline of the steps in his method. For more details we refer to the subsequent sections. The goal of his analysis is obtaining a "system in a form that can be simulated from arbitrary initial conditions, delivering a solution path that does not violate the stability conditions", compare [37] page 7 below formula (19).

[24] This example contains some errors that are corrected in subsection 3.3.5.

[25] Instead of writing the conditional expectations as the endogenous variable minus forecast error, Sims defines new variables for conditional expectations and adds definitional equations for them.

[26] Note that Sims uses the index set $t \in \{1, \ldots, T\}$. However, he considers "stability of a solution" in the sense that e.g. for a scalar process (y_t), y_{t+h} satisfies $\mathbb{E}_t\left(\xi^{-h} y_{t+h}\right) \xrightarrow{h \to \infty} 0, \ \xi > 1$. Thus, we set $T = \infty$.

[27] Sims does not specify the information set. We, again, assume that the conditioning set is $H_z(t)$.

[28] [5], e.g., require on page 879 only that it be adapted to the information set, and that the conditional expectations $\mathbb{E}_t(z_{t+h}), \ h \in \mathbb{N}$, exist.

1. We apply the QZ transformation to the matrix pair (Γ_0, Γ_1) such that the generalized eigenvalues are in non-descending order, i.e. $Q\Gamma_0 Z = \Lambda$, $Q\Gamma_1 Z = \Omega$, where Q and Z are orthogonal matrices, Λ and Ω are upper triangular, and the ratios $\frac{\omega_{ii}}{\lambda_{ii}}$ of the diagonal elements of Ω and Λ are ordered with respect to non-descending absolute value.

2. We check whether the k-th non-explosiveness condition (ξ_k, ϕ_k) applies to the backward solution of the j-th canonical variable $w_j(t)$. If $(w_j(t))_{t\in\mathbb{N}}$ is contained in $H_z(t)$ at time t, satisfies equation (23) below but violates the non-explosiveness condition for a $k \in \{1, \ldots, m\}$, then it belongs to the "unstable part" of the system. The canonical variables contained in w_t^U have a forward solution which does not violate the non-explosiveness condition; however, this solution is not necessarily contained in $H_z(t)$ at time t.

3. We obtain the existence condition (for a solution $(y_t)_{t\in\mathbb{N}}$ of the rational expectations model) by using the fact that for a solution contained in $H_z(t)$, the equation $w_t^U = \mathbb{E}_t\left(w_t^U\right)$ must hold. The vector w_t^U contains all variables whose backward solution violates a non-explosiveness condition. It follows that the existence equation holds if and only if the equation

$$-Q_{U\bullet}\Pi\eta(t+1) = \sum_{h=1}^{\infty} \Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^{h-1} Q_{U\bullet}\Psi\left\{\mathbb{E}_{t+1}\left(z(t+h)\right) - \mathbb{E}_t\left(z(t+h)\right)\right\}$$

   which determines η_{t+1} as a function of the exogenous process holds. This condition is called the "decision rule for the various types of agents in the economy", compare [37] page 10 below his equation (37). The equation is equivalent to

$$span\left(\left\{\Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^{i}\Omega_{UU}^{-1}Q_{U\bullet}\Psi\right\}_{i=0}^{n(U)-1}\right) \subseteq span\left(Q_{U\bullet}\Pi\right)$$

   where $n(U)$ is the number of canonical variables whose backward solutions do not satisfy the non-explosiveness condition. In the case $\mathbb{E}_t\left(z_{t+1}\right) = 0$, this simplifies to

$$span\left(Q_{U\bullet}\Psi\right) \subseteq span\left(Q_{U\bullet}\Pi\right).$$

   The matrix $Q_{U\bullet}\Pi$ is of dimension $n(U) \times k$, which suggests that Sims' condition for existence is similar to the one derived in [8]. Indeed, if there are at least as many endogenous forecast errors variables (corresponding to the number of non-predetermined variables in Blanchard and Kahn) as there are backward solutions of canonical variables which do not satisfy a non-explosiveness condition (corresponding to the number of unstable roots in Blanchard and Kahn), i.e. $k \geq n(U)$, the condition above is "likely" to be satisfied.

4. We obtain the uniqueness condition by ensuring that the endogenous forecast errors which enter the stable part, i.e. which influence those variables to which the growth restrictions do not apply, of the equation through $Q_{S\bullet}\Pi$ can be expressed by the endogenous forecast errors which enter the unstable part through $Q_{U\bullet}\Pi$ and thus through the exogenous variables. A solution of the rational expectations model is thus unique if and only if

$$rowspan\left(Q_{S\bullet}\Pi\right) \subseteq rowspan\left(Q_{U\bullet}\Pi\right)$$

   or equivalently if and only if there exists a matrix Φ such that $Q_{S\bullet}\Pi = \Phi Q_{U\bullet}\Pi$.

5. If the existence and uniqueness conditions are satisfied, we obtain a new system in the canonical variables in which no endogenous forecast errors appear.

6. Using the orthogonal basis transformation Z, we transform the system in canonical variables back to original variables.

3.3.1 QZ transformation of Sims' canonical form

Sims applies the QZ transformation (compare [24] page 406, Theorem 7.7.1) to the matrix pair (Γ_0, Γ_1), i.e. $Q\Gamma_0 Z = \Lambda$, $Q\Gamma_1 Z = \Omega$, where Q and Z are orthogonal matrices, Λ and Ω are upper triangular (note that none of the matrices Q, Z, Λ, Ω are assumed to be real), and the ratios $\frac{\omega_{ii}}{\lambda_{ii}}$ corresponding to the diagonal elements $(\lambda_{ii}, \omega_{ii})\,,\ i \in \{1, \ldots, n\}\,,$ of Λ and Ω are ordered with respect to non-descending absolute value (for $\lambda_{ii} = 0$, we define $\frac{\omega_{ii}}{\lambda_{ii}} = \infty$).

Remark 18. Note that the QZ decomposition always exists but is in general not unique, i.e. there are many orthogonal matrices Q, Z and upper diagonal matrices Λ, Ω such that $Q\Gamma_0 Z = \Lambda$, $Q\Gamma_1 Z = \Omega$.

Sims states on page 9 in the paragraph below formula (33) that the set of generalized eigenvalues $\left\{\frac{\omega_{ii}}{\lambda_{ii}}, i \in \{1,\ldots,n\}\right\}$ appearing in the QZ decomposition is unique unless Γ_0 and Γ_1 have zero eigenvalues corresponding to the same eigenvector[29]. Although Sims does not assume that $\det(\Gamma_0\mu - \Gamma_1)$ is not identically zero in the complex variable μ, we will make this assumption here.

Remark 19. The QZ decomposition is a generalization of the QR decomposition and reduces to it if $\Gamma_0 = I_n$, see [35]. The QR decomposition of a non-singular matrix is unique if we require the diagonal elements of R to be positive (otherwise every matrix Q^* whose columns are multiplied by a complex number $c = e^{i\theta},\ \theta \in \mathbb{R}$, satisfies $Q^TQ = I$ as well).

Transformation of Sims' canonical system. Left-multiplying Q on

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + C + \Psi z_t + \Pi\eta_t, \quad t \in \{1,\ldots,T\}$$

and premultiplying y_t and y_{t-1} with $ZZ^T = I_n$ leads to

$$\begin{aligned}[Q\Gamma_0 Z]\,(Z'y_t) &= [Q\Gamma_1 Z]\,(Z'y_{t-1}) + QC + Q\Psi z_t + Q\Pi\eta_t \\ \iff \Lambda w_t &= \Omega w_{t-1} + QC + Q\Psi z_t + Q\Pi\eta_t \end{aligned} \tag{23}$$

where the ratios $\frac{\omega_{ii}}{\lambda_{ii}}$ corresponding to the diagonal elements $(\lambda_{ii}, \omega_{ii}),\ i \in \{1,\ldots,n\}$, of Λ and Ω are ordered with respect to non-descending absolute value (for $\lambda_{ii} = 0$, we define $\frac{\omega_{ii}}{\lambda_{ii}} = \infty$).

3.3.2 Non-explosiveness condition and backward solutions.

The question of existence of a solution of the rational expectations model (21) is closely related to the question as to whether the backward solution (if there is one) of the components of the canonical variable u_t described in system (24) below satisfies the non-explosiveness conditions. The backward solution has an advantage over the forward solution in the sense that the former is obviously contained in $H_z(t)$ at time t. In case the backward solution of such a canonical variable does not satisfy a non-explosiveness condition, the condition that the solution coincide with its projection on $H_z(t)$ at time t has to be imposed on the forward solution. This will eventually lead us to the existence condition for a solution of the rational expectations model described in section 3.3.3 starting on page 26.

Kronecker canonical form. In order to better understand the issues involved in the process of deciding whether the backward solution of a component of w_t in equation (23) satisfies the non-explosiveness condition, we first consider a decoupled version of the system, i.e. we start from the Kronecker canonical form the matrix pencil $(\Gamma_0\mu - \Gamma_1),\ \mu \in \mathbb{C}$, (see [20], Chapter 12, page 35) of system (21), as already described in section 3.2 on page 14, i.e.

$$\begin{aligned}&\left[V\Gamma_0 W^{-1}\right](Wy_t) = \left[V\Gamma_1 W^{-1}\right](Wy_{t-1}) + VC + V\Psi z_t + V\Pi\eta_t \\ \iff &\begin{pmatrix} \begin{pmatrix} I_{n(s)} & \\ & I_{n(u)} \end{pmatrix} & \\ & N \end{pmatrix} \begin{pmatrix} s_t \\ u_t \\ i_t \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} J_s & \\ & J_u \end{pmatrix} & \\ & I \end{pmatrix} \begin{pmatrix} s_{t-1} \\ u_{t-1} \\ i_{t-1} \end{pmatrix} + VC + V\Psi z_t + V\Pi\eta_t \end{aligned} \tag{24}$$

where the partitions are again according to whether the roots are inside or on the unit circle, outside the unit circle, or infinite and $n(s)$ and $n(u)$ denote the dimensions of s_t and u_t respectively.

There are m pairs of growth rates $\xi_i > 1,\ i \in \{1,\ldots,m\}$, and linear combinations $\phi_i \in \mathbb{R}^{1\times n},\ i \in \{1,\ldots,m\}$, of endogenous variables y_t which restrict growth requiring that (22), i.e.

$$\mathbb{E}_t\left(\xi_i^{-h}\phi_i y_{t+h}\right) \xrightarrow{h\to\infty} 0,\ \xi_i > 1, \phi \in \mathbb{R}^{1\times n}\ i \in \{1,\ldots,m\},$$

hold.

[29] This is equivalent to the fact that the determinant of the linear matrix pencil $\Gamma_0\mu - \Gamma_1$, $\mu \in \mathbb{C}$, is identically zero because

$$\det(\Gamma_0\mu - \Gamma_1) = \det[Q(\Gamma_0\mu - \Gamma_1)Z] = \det(\Lambda\mu - \Omega) = \prod_{i=1}^{n}(\lambda_{ii}\mu - \omega_{ii}) \equiv 0$$

if $\lambda_{ii} = \omega_{ii} = 0$ for at least one $i \in \{1,\ldots,n\}$. This corresponds to a redundant equation in (21). Note that [31] assume that the determinant $\det(Az - B)$ of $Az - B$, $z \in \mathbb{C}$, is not identically zero.

Backward solution. First, consider a generic Jordan block for u_t and denote it by

$$u_t^{(j)} = \Lambda_j u_{t-1}^{(j)} + C^{(j)} + \Psi^{(j)} z_t + \Pi^{(j)} \eta_t. \tag{25}$$

It is easy to see[30] that the backward solution[31]

$$u_t^{(j)} = (I - \Lambda_j)^{-1} C^{(j)} + \sum_{k=0}^{t-1} \Lambda_j^k \left(\Psi^{(j)} z_{t-k} + \Pi^{(j)} \eta_{t-k}. \right) + \Lambda_j^t \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) \tag{27}$$

does not satisfy the non-explosiveness condition unless

- $\phi_i \left(W^{-1} \right)_{\bullet,j} = 0$, i.e. the "influence" $\left(W^{-1} \right)_{\bullet,j}$ of $u_t^{(j)}$ on the endogenous variables y_t is orthogonal to the vector ϕ_i specifying the linear combination of endogenous variables y_t which is restricted in growth, or

[30] Indeed, substituting the solution (27) of $u_t^{(j)}$ in the stability condition (22), we obtain by writing the conditional expectations term in a non-explosiveness condition (ξ_i, ϕ_i) as $\mathbb{E}_t \left(\xi_i^{-h} \phi_i W^{-1} W y_t \right)$, noting that $W y_t = \begin{pmatrix} s_t \\ u_t \\ i_t \end{pmatrix}$ and that the blocks

$$u_t^{(j)} = \Lambda_j u_{t-1}^{(j)} + C^{(j)} + \Psi^{(j)} z_t + \Pi^{(j)} \eta_t$$

are decoupled from each other that

$$\begin{aligned}
&\mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ (I - \Lambda_j)^{-1} C^{(j)} + \Lambda_j^{t+h} \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) + \sum_{k=0}^{t+h-1} \Lambda_j^k \left(\Psi^{(j)} z_{t+h-k} + \Pi^{(j)} \eta_{t+h-k} \right) \right\} \right) = \\
&= \mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ (I - \Lambda_j)^{-1} C^{(j)} + \Lambda_j^{t+h} \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) \right\} \right) + \cdots \\
&\quad \cdots + \mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ \sum_{k=h}^{t+h-1} \Lambda_j^k \left(\Psi^{(j)} z_{t+h-k} + \Pi^{(j)} \eta_{t+h-k} \right) + \underbrace{\sum_{k=0}^{h-1} \Lambda_j^k \left(\Psi^{(j)} z_{t+h-k} + \Pi^{(j)} \eta_{t+h-k} \right)}_{\eta_{t+h-k} \text{cancels out when taking expectations}} \right\} \right) \\
&= \mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ (I - \Lambda_j)^{-1} C^{(j)} + \Lambda_j^{t+h} \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) \right\} \right) + \cdots \\
&\quad \cdots + \mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ \Lambda_j^h \sum_{k=0}^{t-1} \Lambda_j^k \left(\Psi^{(j)} z_{t-k} + \Pi^{(j)} \eta_{t-k} \right) + \sum_{k=0}^{h-1} \Lambda_j^k \Psi^{(j)} z_{t+h-k} \right\} \right) \\
&= \mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ (I - \Lambda_j)^{-1} C^{(j)} + \Lambda_j^{t+h} \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) \right\} \right) + \cdots \\
&\quad \cdots + \mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ \Lambda_j^h \left(u_t^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} - \Lambda_j^t \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) \right) + \sum_{k=0}^{h-1} \Lambda_j^k \Psi^{(j)} z_{t+h-k} \right\} \right) \\
&= \mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ (I - \Lambda_j)^{-1} C^{(j)} + \Lambda_j^h \left(u_t^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) + \sum_{k=0}^{h-1} \Lambda_j^k \Psi^{(j)} z_{t+h-k} \right\} \right) \\
&= \underbrace{\mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ (I - \Lambda_j)^{-1} C^{(j)} \right\} \right)}_{\xrightarrow{h \to \infty} 0} + \mathbb{E}_t \left(\phi_i \left(W^{-1} \right)_{\bullet,j} \left\{ \xi_i^{-h} \Lambda_j^h \left(u_t^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) \right\} \right) + \cdots \\
&\quad \cdots + \left[\xi_i^{-h} \phi_i \left(W^{-1} \right)_{\bullet,j} \right] \mathbb{E}_t \left(\sum_{k=0}^{h-1} \Lambda_j^k \Psi^{(j)} z_{t+h-k} \right)
\end{aligned} \tag{26}$$

[31] Substitution of the solution (27) in the difference equation (25) gives

$$\begin{aligned}
&(I - \Lambda_j)^{-1} C^{(j)} + \sum_{k=0}^{t-1} \Lambda_j^k \left(\Psi^{(j)} z_{t-k} + \Pi^{(j)} \eta_{t-k}. \right) + \Lambda^t \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) = \\
&= \Lambda_j \left((I - \Lambda_j)^{-1} C^{(j)} + \sum_{k=0}^{t-2} \Lambda_j^k \left(\Psi^{(j)} z_{t-1-k} + \Pi^{(j)} \eta_{t-1-k} \right) + \Lambda^{t-1} \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right) \right) + C^{(j)} + \Psi^{(j)} z_t + \Pi^{(j)} \eta_t \\
&= \left\{ \Lambda_j (I - \Lambda_j)^{-1} C^{(j)} + C^{(j)} \right\} + \left[\Lambda_j \sum_{k=0}^{t-2} \Lambda_j^k \left(\Psi^{(j)} z_{t-1-k} + \Pi^{(j)} \eta_{t-1-k} \right) + \Psi^{(j)} z_t + \Pi^{(j)} \eta_t \right] + \Lambda_j^t \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)} \right)
\end{aligned}$$

which shows that (27) is indeed a solution.

- $\mathbb{E}_t(z_{t+1}) = 0$ and $u_t^{(j)} = (I - \Lambda_j)^{-1} C^{(j)}$, which is only a solution of (25) if $\Psi^{(j)} z_t + \Pi^{(j)} \eta_t = 0$, compare Corollary 25 on page 28.

It follows that if restriction (ξ_i, ϕ_i) is such that $\xi_i < |\lambda_j|$ and $\phi_i \left(W^{-1}\right)_{\bullet,j} \neq 0$, i.e. the "influence" $\left(W^{-1}\right)_{\bullet,j}$ of $u_t^{(j)}$ on the endogenous variables y_t is not orthogonal to restriction ϕ_i, the terms $\sum_{k=0}^{t-1} \Lambda_j^k \left(\Psi^{(j)} z_{t-k} + \Pi^{(j)} \eta_{t-k}.\right)$ and $\Lambda_j^t \left(u_0^{(j)} - (I - \Lambda_j)^{-1} C^{(j)}\right)$ in the backward solution (27) above have to be zero in order that the non-explosiveness restriction (ξ_i, ϕ_i) be satisfied. In this sense, the exogenous variables (together with the endogenous forecast errors) and the initial values are, in general, sources of explosiveness for the backward solution (27). As already mentioned above, $u_t^{(j)} = (I - \Lambda_j)^{-1} C^{(j)}$ is only a solution of (25) if $\Psi^{(j)} z_t + \Pi^{(j)} \eta_t = 0$, $t \in \mathbb{N}$. Even if this condition holds, $u_t^{(j)} = (I - \Lambda_j)^{-1} C^{(j)}$ does not violate the non-explosiveness conditions only if $\mathbb{E}_t(z_{t+1}) = 0$ (and, of course, $\phi_i \left(W^{-1}\right)_{\bullet,j} \neq 0$), compare Theorem 24 and Corollary 25.[32]

Remark 20 (Different growth rates). Note that if $|\lambda_{j+1}| > |\lambda_j|$ holds for two unstable roots λ_j and λ_{j+1}, then the backward solution pertaining to $u_t^{(j+1)}$ has to satisfy at least as many non-explosiveness conditions (ξ_i, ϕ_i), $i \in \{1, \ldots, m\}$, as the backward solution for $u_t^{(j)}$ because $\xi_i < |\lambda_j|$ implies $\xi_i < |\lambda_{j+1}|$. Then, it has to be checked for all ϕ_i pertaining to a ξ_i with $\xi_i < |\lambda_j|$ whether it is orthogonal to the "influence" $\left(W^{-1}\right)_{\bullet,j}$ of the canonical variable $u_t^{(j)}$ on the endogenous variables y_t. If ϕ_i is not orthogonal to $\left(W^{-1}\right)_{\bullet,j}$ for one such restriction, the backward solution (27) cannot be part of the solution of the rational expectations model. As will be shown below, the fact that the forward solution (which in general is not contained in $H_z(t)$ at time t) of the canonical variable $u_t^{(j)}$ must then be part of the solution of the rational expectations model, is used to deduce an existence condition for a solution of the rational expectations model (21).

Forward solution. Considering the forward solution

$$u_t^{(j)} = (I - \Lambda_j)^{-1} C^{(j)} - \sum_{k=1}^{\infty} \Lambda_j^{-k} \left(\Psi^{(j)} z_{t+k} + \Pi^{(j)} \eta_{t+k}\right)$$

of the difference equation (25), i.e.

$$u_t^{(j)} = \Lambda_j u_{t-1}^{(j)} + C^{(j)} + \Psi^{(j)} z_t + \Pi^{(j)} \eta_t,$$

we see that the non-explosiveness conditions are satisfied. Indeed, we have

$$\begin{aligned}
&\mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1}\right)_{\bullet,j} \left\{(I - \Lambda_j)^{-1} C^{(j)} + \sum_{k=1}^{\infty} \Lambda_j^{-k} \left(\Psi^{(j)} z_{t+h+k} + \Pi^{(j)} \eta_{t+h+k}\right)\right\}\right) \\
&= \underbrace{\mathbb{E}_t \left(\xi_i^{-h} \phi_i \left(W^{-1}\right)_{\bullet,j} \left\{(I - \Lambda_j)^{-1} C^{(j)}\right\}\right)}_{\xrightarrow{h \to \infty} 0} + \left[\xi_i^{-h} \phi_i \left(W^{-1}\right)_{\bullet,j}\right] \mathbb{E}_t \left(\sum_{k=1}^{\infty} \Lambda_j^{-k} \Psi^{(j)} z_{t+h+k}\right)
\end{aligned}$$

from which follows that the last term[33] converges in mean square sense to zero for $h \to \infty$.

[32] Note that under the assumption that (z_t) has Wold representation

$$z_t = \sum_{i=0}^{\infty} k_i \varepsilon_{t-i},$$

it follows that the last term $\mathbb{E}_t \left(\sum_{k=0}^{h-1} \Lambda_j^k \Psi^{(j)} z_{t+h-k}\right)$ does not converge to zero since the last term $\Lambda_j^{h-1} \Psi^{(j)} \mathbb{E}_t(z_{t+1})$ in the sum $\mathbb{E}_t \left(\sum_{k=0}^{h-1} \Lambda_j^k \Psi^{(j)} z_{t+h-k}\right)$, which dominates the other summands asymptotically, diverges in mean square sense faster than ξ^h for $h \to \infty$. Indeed, taking $k = h - 1$, we obtain

$$\xi^{-h} \Lambda_j^{h-1} \Psi^{(j)} \mathbb{E}_t(z_{t+1}) = \xi^{-h} \Lambda_j^{h-1} \Psi^{(j)} \left(\sum_{i=1}^{\infty} k_i \varepsilon_{t+1-i}\right)$$

goes to infinity (in mean square sense) for $h \to \infty$.

[33] Note that we assume that $(z_t)_{t \in \mathbb{N}}$ is (weakly) stationary. Sims does not state any assumption on the exogenous process $(z_t)_{t \in \mathbb{N}}$. A reasonable assumption (which is weaker than ours) on the exogenous process is, e.g., that the conditional expectations $\mathbb{E}_t(z_{t+h})$ exist for $h > 0$, compare [5] page 879.

Remark 21 (Canonical variables corresponding to infinite zeros). The solution $(i_t)_{t\in\mathbb{N}}$ of the "infinite" canonical variables i_t are a function of finitely many expectations at time t of future values of the exogenous process (compare, e.g., [11] page 232 or [4] page 154, formula (3.57) and also the derivation (17) in section 3.2.1), i.e.[34]

$$\begin{aligned}
& Ni_t = i_{t-1} + V^{inf}\left(C + \Psi z_t + \Pi\eta_t\right) \qquad (28)\\
\iff\ & i_t = Ni_{t+1} - V^{inf}\left(C + \Psi z_{t+1} + \Pi\eta_{t+1}\right)\\
\iff\ & i_t = N\left(Ni_{t+2} - V^{inf}\left(C + \Psi z_{t+2} + \Pi\eta_{t+2}\right)\right) - V^{inf}\left(C + \Psi z_{t+1} + \Pi\eta_{t+1}\right)\\
& \quad = N^2 i_{t+2} - NV^{inf}\left(C + \Psi z_{t+2} + \Pi\eta_{t+2}\right) - V^{inf}\left(C + \Psi z_{t+1} + \Pi\eta_{t+1}\right)\\
\iff\ & i_t = \underbrace{N^l}_{=0} i_{t+l} - \sum_{k=0}^{l-1} N^k V^{inf}\left(C + \Psi z_{t+1+k} + \Pi\eta_{t+1+k}\right)\\
& \quad = -\left(I - N\right)^{-1} V^{inf} C - \sum_{k=0}^{l-1} N^k V^{inf}\left(\Psi z_{t+1+k} + \Pi\eta_{t+1+k}\right).
\end{aligned}$$

It follows that solutions of "infinite" canonical variables always satisfy the non-explosiveness conditions. However, it will always be part of the system which is solved forward[35] and thus creates restrictions on $(\eta_t)_{t\in\mathbb{N}}$ ensuring that the solution $(y_t)_{t\in\mathbb{N}}$ is contained in $H_z(t)$ for every $t \in \mathbb{N}$, for more details see section 3.3.3.

Remark 22 (Developing components of s_t forward). Note that there is in general no reason why the forward solutions (which will be explosive) of the canonical variables s_t corresponding to stable roots should not be considered. In the case where the "influence" $\left(W^{-1}\right)_{\bullet,j}$ of the canonical variable $s_t^{(j)}$ pertaining to a Jordan block in J_s on the endogenous variables y_t is orthogonal to all ϕ_i for which $\xi_i < |\lambda_j|^{-1}$, there is another solution to the rational expectations model which is not considered in [37]. By not considering such a solution, one excludes a priori explosive behavior of the endogenous variables y_t along $\left(W^{-1}\right)_{\bullet,j}$ which might be relevant for economic theory, compare [15]. Of course, when the goal of the analysis is characterizing the dimension of the solution set of rational expectations models, a "minimal" existence condition, i.e. developing as few variables as possible in terms of non-negative powers of the forward shift operator $\left(z^{-1}\right)$ is desirable. Be that as it may, including such a solution has the following implications.

First, whenever a forward solution of a variable is considered, the endogenous forecast errors must offset the exogenous disturbances in order that a solution which is contained in $H_z(t)$ for every $t \in \mathbb{N}$ exist. Thus, there would be one more row in the equation system of the existence condition (33) derived below which might imply that such a solution does not exist. Second, after having obtained existence for this solution, the uniqueness condition described below will be more easily satisfied. Thus, by developing a component of s_t forward, we get rid of a source of indeterminacy, given existence.

QZ **decomposition.** The situation is slightly more complicated if we consider instead of the (decoupled) Kronecker canonical form of the pencil $\left(\Gamma_0\mu - \Gamma_1\right),\ \mu \in \mathbb{C}$, the QZ decomposition of the pair (Γ_0, Γ_1), i.e. as in (23)

$$\begin{aligned}
& \left[Q\Gamma_0 Z\right]\left(Z' y_t\right) = \left[Q\Gamma_1 Z\right]\left(Z' y_{t-1}\right) + QC + Q\Psi z_t + Q\Pi\eta_t\\
\iff\ & \begin{pmatrix} \Lambda_{ss} & \Lambda_{su} & \Lambda_{si} \\ & \Lambda_{uu} & \Lambda_{ui} \\ & & \Lambda_{ii} \end{pmatrix} w_t = \begin{pmatrix} \Omega_{ss} & \Omega_{su} & \Omega_{si} \\ & \Omega_{uu} & \Omega_{ui} \\ & & \Omega_{ii} \end{pmatrix} w_{t-1} + QC + Q\Psi z_t + Q\Pi\eta_t,
\end{aligned}$$

where the matrices Λ and Ω from equation (23) are partitioned in an obvious way, together with the non-explosiveness conditions (22)

$$\mathbb{E}_t\left(\xi_i^{-h}\phi_i Z Z' y_{t+h}\right) = \mathbb{E}_t\left(\xi_i^{-h}\phi_i Z w_{t+h}\right) \xrightarrow{h\to\infty} 0,\ \xi_i > 1,\ i \in \{1,\ldots,m\}.$$

[34] Note that for a nilpotent matrix N for which $N^l = 0$, where l is a positive integer smaller than or equal to the dimension $n(i)$ of N, the relation $\sum_{k=0}^{l-1} N^l = \left(I - N\right)^{-1}\left(I + N^l\right) = \left(I - N\right)^{-1}$ holds.

[35] Note that the solution is unique in the sense that there is no ambiguity as to whether the determinant in $\left(N - Iz\right)^{-1} = \frac{adj(N-Iz)}{\det(N-Iz)}$ is to be developed in terms of non-negative powers of z or $\left(z^{-1}\right)$. The solution $(i_t)_{t\in\mathbb{N}}$ of (28) may only depend on finitely many future values of the exogenous variables and the endogenous forecast errors. However, the solution never depends on present or past values of these variables.

Procedure for checking the non-explosiveness conditions. In order to check whether a backward solution $\left(w_t^{(j)}\right)_{t\in\mathbb{N}}$ corresponding to the finite generalized eigenvalue μ_j satisfies the non-explosiveness conditions, we proceed as follows. First, the generalized eigenvalue μ_j has to be brought into the $(1,1)$ position[36] (e.g. by switching subsequently diagonal elements of the pencil $(\Lambda\mu-\Omega)$, $\mu\in\mathbb{C}$,). Second, we consider a solution of the new difference equation where only the first element $w_t^{(1)}$ (which now corresponds to a candidate root $\bar{\mu}$ for which $\bar{\mu}>\xi_i$ holds) is allowed to grow in norm unboundedly for stationary exogenous process $(z_t)_{t\in\mathbb{N}}$ (compare [16]), i.e. we consider the solution of

$$w_t=\frac{1}{\prod_{j=1}^{n}\left(\lambda_{jj}\left(z^{-1}\right)-\omega_{jj}\right)}adj\left(\Lambda\left(z^{-1}\right)-\Omega\right)\left(QC+Q\Psi z_{t+1}+Q\Pi\eta_{t+1}\right)$$

for which the denominator is developed in terms of non-negative powers of the backwardshift operator z for all generalized eigenvalues $\mu_j=\frac{\omega_{jj}}{\lambda_{jj}}$ with absolute value smaller than one, i.e.

$$\frac{1}{\left(\lambda_{jj}\left(z^{-1}\right)-\omega_{jj}\right)}=\frac{1}{\lambda_{jj}\left(z^{-1}\right)\left(1-\frac{\omega_{jj}}{\lambda_{jj}}z\right)}=\frac{1}{\lambda_{jj}\left(z^{-1}\right)}\sum_{i=0}^{\infty}\left(\frac{\omega_{jj}}{\lambda_{jj}}\right)^{i}z^{i},$$

and in terms of non-negative powers of the forwardshift operator $\left(z^{-1}\right)$ for all generalized eigenvalues $\mu_j=\frac{\omega_{jj}}{\lambda_{jj}}$ with absolute value larger than one (except the candidate root $\bar{\mu}$), i.e.

$$\frac{1}{\left(\lambda_{jj}\left(z^{-1}\right)-\omega_{jj}\right)}=\frac{1}{-\omega_{jj}\left(1-\frac{\lambda_{jj}}{\omega_{jj}}\left(z^{-1}\right)\right)}=-\frac{1}{\omega_{jj}}\sum_{i=0}^{\infty}\left(\frac{\lambda_{jj}}{\omega_{jj}}\right)^{i}\left(z^{-1}\right)^{i}.$$

Note that, due to the structure of the matrices, the exploding first component does not influence any other component. In analogy to the analysis conducted with the Kronecker canonical form above, the backward solution pertaining to a candidate unstable root $\bar{\mu}$ violates the i-th non-explosiveness condition (ξ_i,ϕ_i) if $\bar{\mu}>\xi_i$ and $\phi_i Z_{\bullet 1}\neq 0$ (where $Z_{\bullet 1}$ corresponds to the new QZ decomposition) and thus the forward solution has to be considered.

Remark 23. A similar procedure can be used if one is interested in explosive behavior of variables pertaining to stable roots as described in remark 22 on page 25. The only difference is that the canonical variable under investigation is solved forward and pertains to a candidate root with absolute value smaller than or equal to unity.

3.3.3 Existence condition

In this section, we derive an existence condition in terms of $(Q_{U\bullet},\Omega_{UU},\Lambda_{UU},\Psi,\Pi)$ determining whether there is a solution $(y_t)_{t\in\mathbb{N}}$ of the rational expectations model, i.e. a process which satisfies equation (21) for all $t\in\mathbb{N}$, which satisfies the non-explosiveness conditions (as derived in the previous section) and for which $y_t\in H_z(t)$, $t\in\mathbb{N}$. First, the forward solution[37] $\left(w_t^U\right)_{t\in\mathbb{N}}$ satisfying the "unstable" part of the decoupled system (29) is obtained. The corresponding backward solution (in the case of finitely unstable roots) violates the non-explosiveness conditions. The forward solution $\left(w_t^U\right)_{t\in\mathbb{N}}$, however, depends in general on values not contained in $H_z(t)$ at time t. Subsequently, we derive an existence condition involving the exogenous process $(z_t)_{t\in\mathbb{N}}$ and the endogenous forecast errors $(\eta_t)_{t\in\mathbb{N}}$ which ensures that $y_t\in H_z(t)$, $t\in\mathbb{N}$, by requiring that for the forward solution $\left(w_t^U\right)_{t\in\mathbb{N}}$ the equation $\mathbb{E}_t\left(w_t^U\right)=w_t^U$ hold. If the latter condition is satisfied, a solution of the rational expectations model exists. If $\mathbb{E}_t\left(w_t^U\right)=w_t^U$ does not hold, there does not exist a solution which satisfies the non-explosiveness conditions and is contained in $H_z(t)$ at time t for all $t\in\mathbb{N}$.

Forward solution of unstable part. All variables $w_t^{(j)}$ which are solved forward because the backward solutions either violate the non-explosiveness condition or correspond to zeros at infinity are grouped into w_t^U. We consider the system

$$\iff\begin{pmatrix}\Lambda_{SS} & \Lambda_{SU}\\ & \Lambda_{UU}\end{pmatrix}\begin{pmatrix}w_t^S\\ w_t^U\end{pmatrix}=\begin{pmatrix}\Omega_{SS} & \Omega_{SU}\\ & \Omega_{UU}\end{pmatrix}\begin{pmatrix}w_{t-1}^S\\ w_{t-1}^U\end{pmatrix}+\begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}C+\begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi z_t+\begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Pi\eta_t \qquad (29)$$

[36] The corresponding concept for an invariant subspace of a matrix, is a deflating subspace of a matrix pencil, see [24] (Chapter 7.7.8 Generalized Invariant Subspace Computations, page 414). A k-dimensional subspace $S\subseteq\mathbb{C}^n$ is deflating for the pencil $(\Gamma_0\mu-\Gamma_1)$, $\mu\in\mathbb{C}$, if the subspace $\{\Gamma_0 x+\Gamma_1 y\mid x,y\in S\}$ has dimension k or less. The QZ decomposition of the pair (Γ_0,Γ_1) as described in section 3.3.1 on page 21 implies $Q^T(\Lambda\mu-\Omega)Z^T=\Gamma_0\mu-\Gamma_1$. It follows that

$$\{\Gamma_0 x+\Gamma_1 y\mid x,y\in span\left(Z_{\bullet 1},\ldots,Z_{\bullet k}\right)\}\subseteq span\left(\left(Q^T\right)_{\bullet 1},\ldots,\left(Q^T\right)_{\bullet k}\right).$$

Thus, the only component whose solution does not influence any other variable is the solution of the first component of w_t.

[37] It solves the second block of rows in equation (29) for given $Q_{U\bullet}(\Psi z_t+\Pi\eta_t)$.

or more particularly its second block of rows[38], i.e.

$$w_t^U = \left(\Omega_{UU}^{-1}\Lambda_{UU}\right) w_{t+1}^U - \Omega_{UU}^{-1}Q_{U\bullet}\left(C + \Psi z_{t+1} + \Pi\eta_{t+1}\right).$$

The solution[39] not violating the non-explosiveness conditions is

$$\begin{aligned} w_t^U &= -\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\left(C + \Psi z_{t+1+i} + \Pi\eta_{t+1+i}\right) \\ &= -\underbrace{\left(I - \left(\Omega_{UU}^{-1}\Lambda_{UU}\right)\right)^{-1}\Omega_{UU}^{-1}}_{=(\Omega_{UU}-\Lambda_{UU})^{-1}} Q_{U\bullet}C - \sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i} + \Pi\eta_{t+1+i}\right). \end{aligned} \tag{30}$$

Deriving the existence condition and some economic intuition. If the solution 30 is contained in $H_z(t)$ at time t for every $t \in \mathbb{N}$, then $w_t^U = \mathbb{E}_t\left(w_t^U\right)$ holds. It follows that the equation

$$\begin{aligned} &\mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i} + \Pi\eta_{t+1+i}\right)\right) = \sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i} + \Pi\eta_{t+1+i}\right) \\ &\iff \mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right) = \sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i} + \Pi\eta_{t+1+i}\right) \end{aligned} \tag{31}$$

is satisfied in this case. In other words, the endogenously determined forecast errors $(\eta_t)_{t\in\mathbb{N}}$ must offset the expectations of the given exogenous process $(z_t)_{t\in\mathbb{N}}$ in order that a solution of the rational expectations model exists. We will show in the proof of Theorem 24 below that equation (31) is equivalent to

$$-Q_{U\bullet}\Pi\eta_{t+1} = \sum_{i=0}^{\infty}\Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\Psi\left[\mathbb{E}_{t+1}\left(z_{t+1+i}\right) - \mathbb{E}_t\left(z_{t+1+i}\right)\right], \tag{32}$$

i.e. the endogenous forecast errors offset today's expectations today of future changes in the exogenous process[40]. This condition is often interpreted as the decision rule of the agents in the economy.

Theorem 24. *A solution $(y_t)_{t\in\mathbb{N}}$ to (21) satisfying the non-explosiveness condition (22) and for which $y_t \in H_z(t),\ t \in \mathbb{N}$, holds exists, if and only if*

$$span\left(\left\{\Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\Psi\right\}_{i=0}^{n(U)-1}\right) \subseteq span\left(Q_{U\bullet}\Pi\right). \tag{33}$$

Proof. If a solution $(y_t)_{t\in\mathbb{N}}$ to (21) satisfying the non-explosiveness condition (22) and for which $y_t \in H_z(t),\ t \in \mathbb{N}$ holds exists, it follows that equation (31) holds. On the other hand, if equation (31) is satisfied, then the solution satisfying the non-explosiveness conditions described above is contained in $H_z(t)$ at time t for all $t \in \mathbb{N}$.

It remains to show that

$$\begin{aligned} &\mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right) = \sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i} + \Pi\eta_{t+1+i}\right) \\ &\iff -Q_{U\bullet}\Pi\eta_{t+1} = \sum_{i=0}^{\infty}\Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\Psi\left[\mathbb{E}_{t+1}\left(z_{t+1+i}\right) - \mathbb{E}_t\left(z_{t+1+i}\right)\right] \end{aligned}$$

where the last expression is obviously equivalent to $span\left(\left\{\Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i \Omega_{UU}^{-1}Q_{U\bullet}\Psi\right\}_{i=0}^{n(U)-1}\right) \subseteq span\left(Q_{U\bullet}\Pi\right).$

[38] Note that Ω_{UU} is non-singular due to our assumption that $\det\left(\Gamma_0\mu - \Gamma_1\right) \not\equiv 0,\ \mu \in \mathbb{C}$, and that the ratios $\left\{\frac{\omega_{jj}}{\lambda_{jj}} \mid j \in \{1,\ldots,n\}\right\}$ of the diagonal elements of Λ and Ω in equation (29) are ordered with respect to non-descending modulus.

[39] Note that $(\Omega_{UU} - \Lambda_{UU}) = \Omega_{UU}\left(I - \left(\Omega_{UU}^{-1}\Lambda_{UU}\right)\right)$ is invertible because all ratios $\frac{\omega_{jj}}{\lambda_{jj}}$ pertaining to variables w_t^U have modulus strictly larger than one (and thus $\left|\frac{\omega_{jj}}{\lambda_{jj}}\right| < 1$ for all such ratios) .

[40] Note that if (z_t) admits a Wold representation $z_t = \sum_{j=0}^{\infty} k_j\varepsilon_{t-j}$, then

$$\mathbb{E}_{t+1}\left(z_{t+1+i}\right) - \mathbb{E}_t\left(z_{t+1+i}\right) = k_{i-1}\varepsilon_{t+1}.$$

"⇒": Apply $\mathbb{E}_{t+1}(\cdot)$. Taking conditional expectations of equation (31) with respect to information set $H_z(t+1)$ gives

$$\begin{aligned}\mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right) &= \mathbb{E}_{t+1}\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i}+\Pi\eta_{t+1+i}\right)\right)\\ &= \mathbb{E}_{t+1}\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right)+\mathbb{E}_{t+1}\left(\Omega_{UU}^{-1}Q_{U\bullet}\Pi\eta_{t+1}\right)\end{aligned}$$

which is equivalent to

$$-Q_{U\bullet}\Pi\eta_{t+1}=\Omega_{UU}\left\{\mathbb{E}_{t+1}\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right)-\mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right)\right\}.$$

"⇐": Sum over $\eta_{t+1+i},\ i\in\mathbb{N}$, and reorder summation. Premultiplying $-Q_{U\bullet}\Pi\eta_{t+1+i}$ in equation (32) by $\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}$ and summing over $i\in\mathbb{N}$ gives

$$\begin{aligned}&-\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Pi\eta_{t+1+i}=\\ &=\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}\left\{\sum_{k=0}^{\infty}\Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^k\Omega_{UU}^{-1}Q_{U\bullet}\Psi\left[\mathbb{E}_{t+1+i}\left(z_{t+1+i+k}\right)-\mathbb{E}_{t+i}\left(z_{t+1+i+k}\right)\right]\right\}.\end{aligned}$$

Reordering the sum on the right hand side of the equation above gives

$$\begin{aligned}&\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}\left\{\sum_{k=0}^{\infty}\Omega_{UU}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^k\Omega_{UU}^{-1}Q_{U\bullet}\Psi\left[\mathbb{E}_{t+1+i}\left(z_{t+1+i+k}\right)-\mathbb{E}_{t+i}\left(z_{t+1+i+k}\right)\right]\right\}=\\ &=\sum_{r=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^r\Omega_{UU}^{-1}Q_{U\bullet}\Psi\sum_{i=0}^{r}\left[\mathbb{E}_{t+1+i}\left(z_{t+1+r}\right)-\mathbb{E}_{t+i}\left(z_{t+1+r}\right)\right]\\ &=\sum_{r=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^r\Omega_{UU}^{-1}Q_{U\bullet}\Psi\left\{\left[\mathbb{E}_{t+1+r}\left(z_{t+1+r}\right)-\mathbb{E}_{t+r}\left(z_{t+1+r}\right)\right]+\cdots+\left[\mathbb{E}_{t+1}\left(z_{t+1+r}\right)-\mathbb{E}_t\left(z_{t+1+r}\right)\right]\right\}\\ &=\sum_{r=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^r\Omega_{UU}^{-1}Q_{U\bullet}\Psi\left\{\underbrace{\mathbb{E}_{t+1+r}\left(z_{t+1+r}\right)}_{=z_{t+1+r}}-\mathbb{E}_t\left(z_{t+1+r}\right)\right\}.\end{aligned}$$

Thus, we obtain

$$-\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Pi\eta_{t+1+i}=\sum_{r=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^r\Omega_{UU}^{-1}Q_{U\bullet}\Psi\left\{z_{t+1+r}-\mathbb{E}_t\left(z_{t+1+r}\right)\right\}.$$

□

Corollary 25. *Under the assumptions of Theorem 24 and additionally assuming that $\mathbb{E}_t(z_{t+1})=0$ holds, it follows that the existence of a solution of the rational expectations model is equivalent to*

$$span\left(Q_{U\bullet}\Psi\right)\subseteq span\left(Q_{U\bullet}\Pi\right). \tag{34}$$

Proof. Since the left hand side of

$$\mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right)=\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i}+\Pi\eta_{t+1+i}\right)$$

is zero in this case and Ω_{UU}^{-1} is non-singular it follows that a solution exists if and only if $0=Q_{U\bullet}\left(\Psi z_{t+1+i}+\Pi\eta_{t+1+i}\right)$ for all possible exogenous processes $(z_t)_{t\in\mathbb{N}}$. □

Remark 26. The larger the block of variables which have to be solved forward, the harder satisfying the existence condition becomes. From this point of view, it is beneficial to include as few variables as possible in w_t^U because for all of them the exogenous "disturbances" have to be offset by the endogenous forecast errors. Given existence, however, we will see in the next subsection that it should become easier to satisfy the uniqueness condition described below, the more components are contained in w_t^U and the fewer are in w_t^S. Of course, if the goal of the analysis is obtaining the dimension of the solution set of a rational expectations model, it is desirable to solve as few variables as possible forward.

Remark 27. Note that solutions pertaining to infinite generalized eigenvalues always satisfy the non-explosiveness conditions (since there are only finitely many terms involved in their solution). However, since the solution for the block corresponding to infinite roots in (24) on page 22 is unique (compare the remark 21 on page 25) and may involve future values of exogenous variables, the variables pertaining to infinite generalized eigenvalues are always contained in w_t^U and are thus part of the system which has to be solved forward.

Remark 28 (Extended state vector). The condition $\mathbb{E}_t\left(z_{t+1}\right) = 0$ can be justified if we know the structure of the exogenous process, e.g. that $\left(z_t\right)_{t\in\mathbb{N}}$ is an ARMA process. The exogenous variables may then be incorporated into an extended vector of endogenous variables $\tilde{y}_t = \begin{pmatrix} y_t \\ z_t \end{pmatrix}$ such that the new vector of exogenous variables consists only of the one-step-ahead forecast errors of the exogenous process $\left(z_t\right)_{t\in\mathbb{N}}$.

3.3.4 Uniqueness of solution

In order to obtain a unique solution of (21) satisfying the non-explosiveness condition and being contained in $H_z(t)$, $t \in \mathbb{N}$, we need to get rid of the dependence of the solution on the endogenous forecast errors $\left(\eta_t\right)_{t\in\mathbb{N}}$. For a solution $\left(w_t^U\right)_{t\in\mathbb{N}}$ of the unstable part of the system, this is possible in the way described above, i.e. by substituting for η_t using the existence condition

$$\mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i}+\Pi\eta_{t+1+i}\right)\right) = \sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i}+\Pi\eta_{t+1+i}\right)$$

in

$$w_t^U = -\left(\Omega_{UU}-\Lambda_{UU}\right)^{-1}Q_{U\bullet}C - \sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\left(\Psi z_{t+1+i}+\Pi\eta_{t+1+i}\right)$$

such that we obtain

$$w_t^U = -\left(\Omega_{UU}-\Lambda_{UU}\right)^{-1}Q_{U\bullet}C - \mathbb{E}_t\left(\sum_{i=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^i\Omega_{UU}^{-1}Q_{U\bullet}\Psi z_{t+1+i}\right).$$

In order to obtain a solution $\left(w_t^S\right)_{t\in\mathbb{N}}$ of the first block of equations in (29) for a given solution $\left(w_t^U\right)_{t\in\mathbb{N}}$, we need $n(S)$ linear combinations of the whole system

$$\begin{pmatrix}\Lambda_{SS} & \Lambda_{SU} \\ & \Lambda_{UU}\end{pmatrix}\begin{pmatrix}w_t^S \\ w_t^U\end{pmatrix} = \begin{pmatrix}\Omega_{SS} & \Omega_{SU} \\ & \Omega_{UU}\end{pmatrix}\begin{pmatrix}w_{t-1}^S \\ w_{t-1}^U\end{pmatrix} + \begin{pmatrix}Q_{S\bullet} \\ Q_{U\bullet}\end{pmatrix}C + \begin{pmatrix}Q_{S\bullet} \\ Q_{U\bullet}\end{pmatrix}\Psi z_t + \begin{pmatrix}Q_{S\bullet} \\ Q_{U\bullet}\end{pmatrix}\Pi\eta_t$$

such that the resulting system does not involve any endogenous forecast errors η_t. It is possible to find such linear combinations if the influence of the endogenous forecast errors on the first part of the system, i.e. $Q_{S\bullet}\Pi\eta_t$, can be explained by the influence of the endogenous forecast errors on the second part of the system, i.e. $Q_{U\bullet}\Pi\eta_t$ for which we already found an expression in terms of known variables through the existence condition. This condition may be expressed as

$$rowspan\left(Q_{S\bullet}\Pi\right) \subseteq rowspan\left(Q_{U\bullet}\Pi\right)$$

which is equivalent to the fact that there exists a matrix Φ of dimension $\left(n(S)\times n(U)\right)$ such that

$$Q_{S\bullet}\Pi = \Phi Q_{U\bullet}\Pi \tag{35}$$

holds.

Indeed, left-multiplying the system above by $\begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix}$, we obtain (under the assumption that equation (35) hold)

$$\begin{aligned}
&\begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix} \begin{pmatrix} \Lambda_{SS} & \Lambda_{SU} \\ & \Lambda_{UU} \end{pmatrix} \begin{pmatrix} w_t^S \\ w_t^U \end{pmatrix} = \\
&\quad = \begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix} \begin{pmatrix} \Omega_{SS} & \Omega_{SU} \\ & \Omega_{UU} \end{pmatrix} \begin{pmatrix} w_{t-1}^S \\ w_{t-1}^U \end{pmatrix} + \begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} C + \cdots \\
&\qquad \cdots + \begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} \Psi z_t + \underbrace{\left[\begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} \right]}_{=0} \Pi \eta_t \\
&\iff \begin{pmatrix} \Lambda_{SS} & \Lambda_{SU} - \Phi\Lambda_{UU} \end{pmatrix} \begin{pmatrix} w_t^S \\ w_t^U \end{pmatrix} = \qquad (36) \\
&\qquad = \begin{pmatrix} \Omega_{SS} & \Omega_{SU} - \Phi\Omega_{UU} \end{pmatrix} \begin{pmatrix} w_{t-1}^S \\ w_{t-1}^U \end{pmatrix} + \begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} C + \begin{pmatrix} I_{n(S)} & -\Phi \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} \Psi z_t.
\end{aligned}$$

If $\left(w_t^U\right)_{t\in\mathbb{N}}$ is a solution satisfying the non-explosiveness condition and being contained in $H_z(t)$ at time t of the difference equation

$$\Lambda_{UU} w_t^U = \Omega_{UU} w_{t-1}^U + Q_{U\bullet} \left(C + \Psi z_t + \Pi \eta_t \right) \iff w_t^U = \left(\Omega_{UU}^{-1} \Lambda_{UU} \right) w_{t+1}^U - \Omega_{UU}^{-1} Q_{U\bullet} \left(C + \Psi z_{t+1} + \Pi \eta_{t+1} \right),$$

it is also (in a trivial way) a solution satisfying the non-explosiveness condition and being contained in $H_z(t)$ at time t of the system

$$w_t^U = -\left(\Omega_{UU} - \Lambda_{UU} \right)^{-1} Q_{U\bullet} C - \mathbb{E}_t \left(\sum_{i=0}^{\infty} \left(\Omega_{UU}^{-1} \Lambda_{UU} \right)^i \Omega_{UU}^{-1} Q_{U\bullet} \Psi z_{t+1+i} \right). \qquad (37)$$

We may thus combine the above systems (36) and (37) to eventually obtain

$$\begin{aligned}
&\begin{pmatrix} \Lambda_{SS} & \Lambda_{SU} - \Phi\Lambda_{UU} \\ & I_{n(U)} \end{pmatrix} \begin{pmatrix} w_t^S \\ w_t^U \end{pmatrix} = \\
&\quad = \begin{pmatrix} \Omega_{SS} & \Omega_{SU} - \Phi\Omega_{UU} \\ & 0 \end{pmatrix} \begin{pmatrix} w_{t-1}^S \\ w_{t-1}^U \end{pmatrix} + \begin{pmatrix} I_{n(S)} & -\Phi \\ & \left(\Omega_{UU} - \Lambda_{UU} \right)^{-1} \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} C + \cdots \qquad (38) \\
&\qquad \cdots + \begin{pmatrix} I_{n(S)} & -\Phi \\ & 0 \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} \Psi z_t - \begin{pmatrix} 0 \\ \mathbb{E}_t \left(\sum_{i=0}^{\infty} \left(\Omega_{UU}^{-1} \Lambda_{UU} \right)^i \Omega_{UU}^{-1} Q_{U\bullet} \Psi z_{t+1+i} \right) \end{pmatrix}. \qquad (39)
\end{aligned}$$

Transformation to original variables. Left-multiplying system (38) with[41]

$$Z \begin{pmatrix} \Lambda_{SS} & \Lambda_{SU} - \Phi\Lambda_{UU} \\ & I_{n(U)} \end{pmatrix}^{-1} = Z \begin{pmatrix} \Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1} \left(\Lambda_{SU} - \Phi\Lambda_{UU} \right) \\ & I_{n(U)} \end{pmatrix},$$

[41] Note that Λ_{SS} is non-singular due to our assumption that $\det\left(\Gamma_0 \mu - \Gamma_1\right) \not\equiv 0,\ \mu \in \mathbb{C}$, and that the ratios $\left\{ \frac{\omega_{jj}}{\lambda_{jj}} \mid j \in \{1, \ldots, n\} \right\}$ of the diagonal elements of Λ and Ω in equation (29) are ordered with respect to non-descending modulus.

we obtain

$$
\begin{aligned}
Zw_t = y_t = \Bigg[(Z_{\bullet S}, Z_{\bullet U}) \begin{pmatrix} \Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}(\Lambda_{SU} - \Phi\Lambda_{UU}) \\ & I_{n(U)} \end{pmatrix} \begin{pmatrix} \Omega_{SS} & \Omega_{SU} - \Phi\Omega_{UU} \\ & 0 \end{pmatrix} Z^T \Bigg] y_{t-1} + ... \\
\cdots + \underbrace{Z \begin{pmatrix} \Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}(\Lambda_{SU} - \Phi\Lambda_{UU}) \\ & I_{n(U)} \end{pmatrix} \begin{pmatrix} I_{n(S)} & -\Phi \\ & (\Omega_{UU} - \Lambda_{UU})^{-1} \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix}}_{=\Theta_c} C + \cdots \\
... + \underbrace{Z \begin{pmatrix} \Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}(\Lambda_{SU} - \Phi\Lambda_{UU}) \\ & I_{n(U)} \end{pmatrix} \begin{pmatrix} I_{n(S)} & -\Phi \\ & 0 \end{pmatrix} \begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet} \end{pmatrix} \Psi}_{=\Theta_z} z_t - \cdots \\
\cdots - Z \begin{pmatrix} \Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}(\Lambda_{SU} - \Phi\Lambda_{UU}) \\ & I_{n(U)} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbb{E}_t \left(\sum_{i=0}^{\infty} \left(\Omega_{UU}^{-1}\Lambda_{UU} \right)^i \Omega_{UU}^{-1} Q_{U\bullet} \Psi z_{t+1+i} \right) \end{pmatrix} \\
= \underbrace{Z_{\bullet S} \begin{pmatrix} \Lambda_{SS}^{-1}\Omega_{SS} & \Lambda_{SS}^{-1}(\Lambda_{SU} - \Phi\Lambda_{UU}) \end{pmatrix} Z^T}_{=\Theta_y} y_{t-1} + \Theta_c C + \Theta_z z_t - \cdots \\
\cdots - \underbrace{Z \begin{pmatrix} -\Lambda_{SS}^{-1}(\Lambda_{SU} - \Phi\Lambda_{UU}) \\ I_{n(U)} \end{pmatrix}}_{=\Theta_{\eta_1}} \mathbb{E}_t \left(\sum_{i=0}^{\infty} \underbrace{\left(\Omega_{UU}^{-1}\Lambda_{UU} \right)^i}_{=\Theta_{\eta_2}} \underbrace{\Omega_{UU}^{-1} Q_{U\bullet}\Psi}_{=\Theta_{\eta_3}} z_{t+1+i} \right) \\
= \Theta_y y_{t-1} + \Theta_c C + \Theta_z z_t - \Theta_{\eta_1} \mathbb{E}_t \left(\sum_{i=0}^{\infty} \Theta_{\eta_2}^i \Theta_{\eta_3} z_{t+1+i} \right). \qquad (40)
\end{aligned}
$$

Remark 29 (Consequences of restricting growth unnecessarily). If the solutions of all components of the endogenous variables y_t are restricted to satisfy the non-explosiveness conditions by the modeler, even though the model itself implies only that certain linear combination are restricted in growth, this could lead to missing sources of indeterminacy. This can be seen as follows (we restrict ourselves to the case $\mathbb{E}_t(z_{t+1}) = 0$ for expositional reasons). Restricting a component unnecessarily makes the existence condition $span(Q_{U\bullet}\Psi) \subseteq span(Q_{U\bullet}\Pi)$ harder to satisfy, since there is one additional row in $Q_{U\bullet}\Psi$. On the other hand and given existence of a solution, restricting a component unnecessarily makes it easier to satisfy the uniqueness condition $rowspan(Q_{S\bullet}\Pi) \subseteq rowspan(Q_{U\bullet}\Pi)$ because there is one more row in $Q_{U\bullet}\Pi$.

It follows that if a certain component is unnecessarily restricted in growth, uniqueness is obtained too easily and sources of indeterminacy might be missed.

Remark 30 (Initial conditions). Note that Lubik and Schorfheide require for the existence of a solution that the initial values at time $t = 0$ of the solution of w_t^U satisfy a certain condition (compare [34] page 276, line -2, and the associated footnote). Sims imposes such a condition (compare [37] page 8 line 4) only in his first derivation (compare section 3 in [37]) of a solution of the rational expectations model which treats the case where $\mathbb{E}_t(z_{t+1}) = 0$ and $\Gamma_0 = I$ hold and where he starts from the backward solution of w_t^U. In this case, it is indeed necessary to impose such a condition. If we consider the forward solution, however, it is not required in the derivation to impose such a condition.

3.3.5 Example in Sims' paper

The example given in [37] contains some typos which are corrected below. Moreover, we want to show (more explicitly than in Sims' paper) how to obtain the "endogenous forecast errors". Equation (2) on page 2 in [37] is here repeated as

$$
\begin{aligned}
w(t) &= \frac{1}{3}\mathbb{E}_t(W(t) + W(t+1) + W(t+2)) - \alpha(u(t) - u_n) + \nu(t) && (41) \\
W(t) &= \frac{1}{3}(w(t) + w(t-1) + w(t-2)) && (42) \\
u(t) &= \theta u(t-1) + \gamma W(t) + \mu + \varepsilon(t) && (43)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E}_t(\nu(t+1)) &= 0 \\
\mathbb{E}_t(\varepsilon(t+1)) &= 0.
\end{aligned}
$$

Sims defines the expanded state vector as

$$y(t) = \begin{pmatrix} w(t) \\ w(t-1) \\ W(t) \\ u(t) \\ \mathbb{E}_t\left(W(t+1)\right) \end{pmatrix}$$

and writes the vector difference equation as

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi\eta(t)$$

$$\iff \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & -\gamma & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w(t) \\ w(t-1) \\ W(t) \\ u(t) \\ \mathbb{E}_t\left(W(t+1)\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \alpha & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w(t-1) \\ w(t-2) \\ W(t-1) \\ u(t-1) \\ \mathbb{E}_{t-1}\left(W(t)\right) \end{pmatrix} + \cdots$$

$$\cdots + \begin{pmatrix} \alpha \cdot u_n \\ 0 \\ \mu \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon(t) \\ \nu(t-1) \end{pmatrix}$$

$$\cdots + \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_{2,Sims}(t) \end{pmatrix}$$

or

$$\frac{1}{3}W(t) + \frac{1}{3}\mathbb{E}_t\left(W(t+1)\right) = w(t-1) - \frac{1}{3}W(t-1) + \alpha u(t-1) + \alpha u_n + \varepsilon(t) + \eta_{2,Sims}(t) \tag{44}$$

$$-\frac{1}{3}w(t) - \frac{1}{3}w(t-1) + W(t) = \frac{1}{3}w(t-2) \tag{45}$$

$$-\gamma W(t) + u(t) = \theta u(t-1) + \mu + \nu(t-1) \tag{46}$$

$$w(t-1) = w(t-1) \tag{47}$$

$$W(t) = \mathbb{E}_{t-1}\left(W(t)\right) + \eta_1(t) \tag{48}$$

The following errors on page 3 in [37] are obvious:

- In equation (44) and (46) $\varepsilon(t)$ and $\nu(t-1)$ must be exchanged.
- In equation (44), the term $\eta_{2,Sims}(t)$ needs some clarification (it should be linked to the endogenous prediction errors), see remarks below.
- The sign for $\varepsilon(t)$ (which should be $\nu(t-1)$) is wrong (the sign of $\alpha u(t)$ and $\nu(t)$ in equation (41) is not the same).
- The sign in the first element of C , i.e. $\alpha \cdot u_n$, is wrong.
- In Γ_1 the $(3,4)$ element should be θ and the $(3,5)$ element should be 0.

Thus we have to use (without changing notation)

$$z(t) = \begin{pmatrix} \nu(t-1) \\ \varepsilon(t) \end{pmatrix}$$

instead of the $z(t)$ vector above.

The corrected vector difference equation. We now have (with different/correct $z(t)$)

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi\eta(t)$$

$$\iff \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & -\gamma & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} w(t) \\ w(t-1) \\ W(t) \\ u(t) \\ \mathbb{E}_t\left(W(t+1)\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \alpha & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w(t-1) \\ w(t-2) \\ W(t-1) \\ u(t-1) \\ \mathbb{E}_{t-1}\left(W(t)\right) \end{pmatrix} + \cdots$$

$$\cdots + \begin{pmatrix} -\alpha \cdot u_n \\ 0 \\ \mu \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu(t-1) \\ \varepsilon(t) \end{pmatrix} + \cdots$$

$$\cdots + \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_{2,Sims}(t) \end{pmatrix}$$

or

$$\frac{1}{3}W(t) + \frac{1}{3}\mathbb{E}_t\left(W(t+1)\right) = w(t-1) - \frac{1}{3}W(t-1) + \alpha u(t-1) - \alpha u_n - \nu(t-1) + \eta_{2,Sims}(t) \tag{49}$$

$$-\frac{1}{3}w(t) - \frac{1}{3}w(t-1) + W(t) = \frac{1}{3}w(t-2) \tag{50}$$

$$-\gamma W(t) + u(t) = \theta u(t-1) + \mu + \varepsilon(t) \tag{51}$$

$$w(t-1) = w(t-1) \tag{52}$$

$$W(t) = \mathbb{E}_{t-1}\left(W(t)\right) + \eta_1(t) \tag{53}$$

Remark on equation (53). The random variable $\eta_1(t)$ is the one-step-ahead prediction error of $W(t)$ at time t.

Remark on equation (49). Shifting both the variables and the information set in equation (41) leads to

$$w(t-1) = \frac{1}{3}W(t-1) + \frac{1}{3}\underbrace{\mathbb{E}_{t-1}\left(W(t)\right)}_{=W(t)-\eta_1(t)} + \frac{1}{3}\mathbb{E}_{t-1}\left(W(t+1)\right) - \alpha\left(u(t-1) - u_n\right) + \nu(t-1).$$

By defining the two-step-ahead prediction error $\tilde{\eta}_2(t+1)$ of $W(t+1)$ at time $t-1$ as

$$\tilde{\eta}_2(t+1) = W(t+1) - \mathbb{E}_{t-1}\left(W(t+1)\right)$$

and further decomposing it as

$$\eta_1(t+1) + \eta_2(t) = \left\{W(t+1) - \mathbb{E}_t\left(W(t+1)\right)\right\} + \left[\mathbb{E}_t\left(W(t+1)\right) - \mathbb{E}_{t-1}\left(W(t+1)\right)\right]$$

we obtain

$$w(t-1) = \frac{1}{3}W(t-1) + \frac{1}{3}\left(W(t) - \eta_1(t)\right) + \frac{1}{3}\left(W(t+1) - \eta_1(t+1) - \eta_2(t)\right) - \alpha\left(u(t-1) - u_n\right) + \nu(t-1).$$

$$\iff \frac{1}{3}W(t) + \frac{1}{3}W(t+1) = w(t-1) - \frac{1}{3}W(t-1) + \alpha u(t-1) - \alpha u_n - \nu(t-1) + \frac{1}{3}\left(\eta_1(t) + \eta_1(t+1) + \eta_2(t)\right).$$

Applying the conditional expectation operator at time t, i.e. $\mathbb{E}_t\left(\cdot\right)$, on both sides of the equation above gives

$$\frac{1}{3}W(t) + \frac{1}{3}\mathbb{E}_t\left(W(t+1)\right) = w(t-1) - \frac{1}{3}W(t-1) + \alpha u(t-1) - \alpha u_n + \nu(t-1) + \frac{1}{3}\underbrace{\mathbb{E}_t\left(\eta_1(t) + \eta_1(t+1) + \eta_2(t)\right)}_{=(\eta_1(t)+\eta_2(t))}.$$

Thus the term $\eta_{2,Sims}(t)$ in Sims' equation (49) is three times the two-step-ahead prediction error, i.e.

$$\eta_{2,Sims}(t) = \frac{1}{3}\left(\eta_1(t) + \eta_2(t)\right) = \frac{1}{3}\left\{\left[W(t+1) - \mathbb{E}_t\left(W(t+1)\right)\right] + \left[\mathbb{E}_t\left(W(t+1)\right) - \mathbb{E}_{t-1}\left(W(t+1)\right)\right]\right\}.$$

3.3.6 Comparison of the Blanchard and Kahn/King and Watson model using the methods in [37] and [11].

Approach in [37]. First, note that the predetermined variables $y_{t+1}^{(pre)} = \mathbb{E}_t\left(y_{t+1}^{(pre)}\right)$ in

$$A\begin{pmatrix}\mathbb{E}_t\left(y_{t+1}^{(pre)}\right)\\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right)\end{pmatrix} = B\begin{pmatrix}y_t^{(pre)}\\ y_t^{(\neg pre)}\end{pmatrix} + Cz_t$$

have time index t in Sims notation. Sims notes on page 2, line 11 that "..., this paper uses a notation in which time arguments or subscripts relate consistently to the information structure: variables dated t are always known at t ." Second, the expectations at time t of the non-predetermined variables at time $t+1$ are denote by $\xi_t = \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right)$. Finally, we obtain

$$\begin{aligned}A_{11}y_t^{(pre)} + A_{12}\xi_t &= B_{11}y_{t-1}^{(pre)} + B_{12}\left(\xi_{t-1}+\eta_t\right) + C_1 z_t\\ A_{21}y_t^{(pre)} + A_{22}\xi_t &= B_{21}y_{t-1}^{(pre)} + B_{22}\left(\xi_{t-1}+\eta_t\right) + C_2 z_t\\ y_t^{(\neg pre)} &= \xi_{t-1}+\eta_t\end{aligned}$$

or in matrix notation

$$\begin{pmatrix}A & & 0\\ & & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix}y_t^{(pre)}\\ \xi_t\\ y_t^{(\neg pre)}\end{pmatrix} = \begin{pmatrix}B & & 0\\ & & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix}y_{t-1}^{(pre)}\\ \xi_{t-1}\\ y_{t-1}^{(\neg pre)}\end{pmatrix} + \begin{pmatrix}C_1\\ C_2\\ 0\end{pmatrix}z_t + \begin{pmatrix}B_{12}\\ B_{22}\\ I\end{pmatrix}\eta_t.$$

Approach in [11]. As already described in section 3.1.1, the model

$$A\begin{pmatrix}\mathbb{E}_t\left(y_{t+1}^{(pre)}\right)\\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right)\end{pmatrix} = B\begin{pmatrix}y_t^{(pre)}\\ y_t^{(\neg pre)}\end{pmatrix} + Cz_t$$

can be written, using $\begin{pmatrix}\mathbb{E}_t\left(y_{t+1}^{(pre)}\right)\\ \mathbb{E}_t\left(y_{t+1}^{(\neg pre)}\right)\end{pmatrix} = \begin{pmatrix}y_{t+1}^{(pre)}\\ y_{t+1}^{(\neg pre)}\end{pmatrix} - \begin{pmatrix}0\\ \eta_{t+1}\end{pmatrix}$, as

$$A\begin{pmatrix}y_{t+1}^{(pre)}\\ y_{t+1}^{(\neg pre)}\end{pmatrix} = B\begin{pmatrix}y_t^{(pre)}\\ y_t^{(\neg pre)}\end{pmatrix} + Cz_t + \begin{pmatrix}A_{12}\\ A_{22}\end{pmatrix}\eta_{t+1}$$

where the equation $\mathbb{E}_t\left(\eta_{t+1}\right) = 0$ is also part of the model specification.

Comparison. Note that the timing conventions in the two approaches do not coincide. The model

$$\begin{aligned}&y_t = A\mathbb{E}_t\left(y_{t+1}\right) + z_t = A\left(y_{t+1}-\eta_{t+1}\right) + z_t = Ay_{t+1} + z_t - A\eta_{t+1}\\ \iff\ &Ay_{t+1} = y_t - z_t + A\eta_{t+1}\\ \iff\ &Ay_t = y_{t-1} - z_{t-1} + A\eta_t.\end{aligned}$$

in [11] notation corresponds in Sims notation to

$$\underbrace{\Gamma_0}_{=A}y_t = \underbrace{\Gamma_1}_{=I}y_{t-1} + \underbrace{\Psi}_{=-I}z_t + \underbrace{\Pi}_{=A}\eta_t.$$

In this sense, the set of models considered by Sims is strictly larger than the one considered by [11] since firstly $z_t = u_{t-1}$ and including earlier time points is no problem and secondly the structure of Π is not influenced by the parameters pertaining to expectational variables. However, it should be noted that Sims does not specify how he obtains the canonical form (21) and it is not trivial to bring more complex models in the form required by Sims' method as we will see in section 4 starting on page 42.

3.4 Lubik and Schorfheide: Analysis of indeterminate equilibria

[34] elaborates on the analysis given in [37] and investigates the case of non-uniqueness in more, but as we will show not sufficient, detail. They demonstrate their findings on a simple New Keynesian DSGE model.

Model considered. By imposing stationary structure on the exogenous variables, they consider an extended vector of endogenous variables as described in Remark 28 on page 29 and thus consider the case $\mathbb{E}_t(z_{t+1}) = 0$ in Sims' notation. Lubik and Schorfheide consider the model

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t \tag{54}$$

where the endogenous variables y_t are n-dimensional, the inputs ε_t are q-dimensional and satisfy $\mathbb{E}_t(\varepsilon_{t+1}) = 0$, η_t is k-dimensional. They abstract from the constant term C and do not specify an index set which we take as $\mathbb{N}$ as before. Moreover, the non-explosiveness condition (22) applies to all components of the endogenous variables y_t, i.e.

$$\mathbb{E}_t\left(\xi^{-h} y_{t+h}\right) \xrightarrow{h\to\infty} 0,\ \xi > 1, \tag{55}$$

holds.

Conditioning set. The conditioning set at time t is not explicitly specified in [34]. We assume[42] it to be $H_{\varepsilon,\zeta}(t)$, where $(\zeta_t)_{t\in\mathbb{N}}$ is a p-dimensional stationary process of sunspots orthogonal to $(\varepsilon_t)_{t\in\mathbb{N}}$. We remind the reader of the consequences of a larger conditioning set described in Remark 7 on page 7. When Lubik and Schorfheide introduce "sunspot shocks" ([34], page 278, line 3) they do not assume that $(\zeta_t)_{t\in\mathbb{N}}$ is orthogonal to $(\varepsilon_t)_{t\in\mathbb{N}}$ but require only that $\mathbb{E}_t(\zeta_{t+1}) = 0$ hold; in particular, $(\zeta_t)_{t\in\mathbb{N}}$ could be a (linear) function of $(\varepsilon_t)_{t\in\mathbb{N}}$. The proof of their Proposition 1 ([34] page 278), however, would require $(\varepsilon_t)_{t\in\mathbb{N}}$ and $(\zeta_t)_{t\in\mathbb{N}}$ to be orthogonal and is thus in error; compare Proposition 31 (and remark 32) for a corrected version.

Analysis of existence condition. First, [34] analyzes for given ε_t the solutions of the system of equations $Q_{U\bullet}\Pi\eta_t = -Q_{U\bullet}\Psi\varepsilon_t$, where $Q_{U\bullet}\Pi$ and and $Q_{U\bullet}\Psi$ are of dimensions $(n(U)\times k)$ and $(n(U)\times q)$ respectively, pertaining to Sims' existence condition (34) on page 28. One obtains the set of all solutions η_t^* of the equation $Q_{U\bullet}\Pi\eta_t = Q_{U\bullet}\Psi\varepsilon_t$ as the sum of one particular solution η_t^p and the set of all homogenous solutions, i.e. the kernel of $Q_{U\bullet}\Pi$. Lubik and Schorfheide choose as particular solution the one with minimum Euclidean norm, i.e. they use the Moore-Penrose pseudo-inverse of $Q_{U\bullet}\Pi$.

Assuming that a solution exists and that $Q_{U\bullet}\Pi$ has rank $r \le \min\{n(U), k\}$, [34] define the dimension of the indeterminacy to be equal to the dimension $(k-r)$ of the right kernel of $Q_{U\bullet}\Pi$. This is unfortunate because a non-trivial kernel of $Q_{U\bullet}\Pi$ is necessary but not sufficient in order that there be multiple solutions of the rational expectations model. A necessary and sufficient condition is Sims' uniqueness condition (35) on page 29, i.e. there exists a matrix Φ of dimension $(n(S)\times n(U))$ such that

$$Q_{S\bullet}\Pi = \Phi Q_{U\bullet}\Pi$$

holds. A more accurate way for describing the dimension of the indeterminacy, and also the dimension of the solution set of the rational expectations model derived from the dimension of the indeterminacy, will be given in Theorem 36 on page 39 below.

However, let us first prove the following Proposition from [34] page 278, which constructs the set of all solutions η_t^* of the existence condition

$$Q_{U\bullet}\Psi\varepsilon_t + Q_{U\bullet}\Pi\eta_t = 0 \tag{56}$$

[42] Sunspots are defined similarly in [25], page 410, where they write the following below their equation (1): "We are interested in the random processes $y = \{y_t \mid t\in\mathbb{Z}\}$ satisfying the following equation

$$y_t = a\mathbb{E}_t(y_{t+1}) + z_t$$

where a is a given scalar ($a \neq 0$); $\mathbb{E}_t$ is the conditional expectation operator with respect to the current and past values: $\{w_t^1, w_{t-1}^1, \ldots; w_t^2, w_{t-1}^2, \ldots; w_t^k, w_{t-1}^k, \ldots\}$ of k given random processes $w^i = \{w_t^i \mid t\in\mathbb{Z}\}$, $i\in\{1,\ldots,k\}$ and $z = \{z_t \mid t\in\mathbb{Z}\}$ is a given random process such that, for any t, $\mathbb{E}_t(z_t) = z_t$. The latter condition $\mathbb{E}_t(z_t) = z_t$ means that z_t is a function of the current and past values of the processes $w^1, \ldots, w^k$. Usually, each variable w_t^i is either a perturbation or an exogenous variable. Some processes w^i may be independent of z; these w^i are sometimes called "sunspots"."

by using the minimum norm solution as the particular solution and subsequently parametrizing the right-kernel of $Q_{U\bullet}\Pi$ in terms of innovations ε_t and sunspot ζ_t.

The proof uses the singular value decomposition (SVD) of the $n(U) \times k$ dimensional matrix $(Q_{U\bullet}\Pi)$ is

$$\begin{aligned} Q_{U\bullet}\Pi &= \begin{bmatrix} U_{\bullet 1} & U_{\bullet 2} \end{bmatrix} \begin{pmatrix} D_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} V_{1\bullet} \\ V_{2\bullet} \end{bmatrix} \\ &= U_{\bullet 1} D_{11} V_{1\bullet} \end{aligned} \tag{57}$$

where D_{11} is a square $(r \times r)$-dimensional diagonal matrix of full rank. It is well known that $(V_{1\bullet})^T$ spans the orthogonal complement $(\ker(Q_{U\bullet}\Pi))^{\perp}$ of the kernel of $Q_{U\bullet}\Pi$, that $(V_{2\bullet})^T$ spans the kernel $\ker(Q_{U\bullet}\Pi)$ of $Q_{U\bullet}\Pi$, that $(U_{\bullet 1})$ spans the image $im(Q_{U\bullet}\Pi)$ of $Q_{U\bullet}\Pi$, and that $(U_{\bullet 2})$ spans the orthogonal complement $(im(Q_{U\bullet}\Pi))^{\perp}$ of the image of $Q_{U\bullet}\Pi$, see figure 1 below.

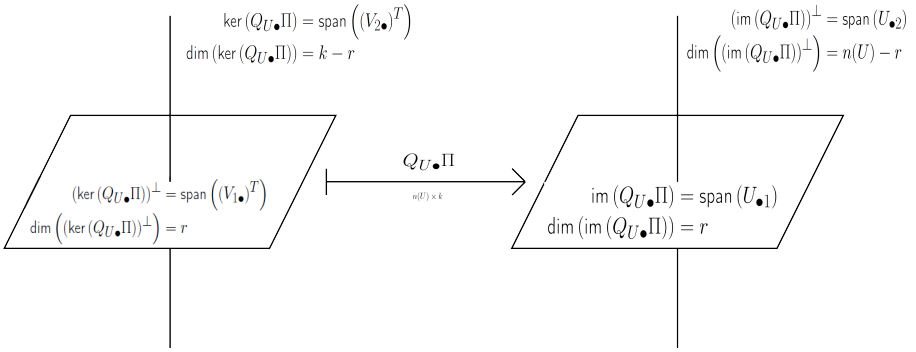

Figure 1:

Proposition 31. *Let $(\zeta_t)_{t\in\mathbb{N}}$ be a p-dimensional stochastic process which satisfies $\mathbb{E}_t(\zeta_{t+1}) = 0$ and is orthogonal to the inputs $(\varepsilon_t)_{t\in\mathbb{N}}$ in equation (54). Furthermore, assume that η_t is a linear function of ε_t and ζ_t, and that the existence condition (56) holds.*

Then, the set of all solutions η_t^* of the existence condition is

$$\left\{ -(V_{1\bullet})^T D_{11}^{-1} (U_{\bullet 1})^T Q_{U\bullet}\Psi\varepsilon_t + (V_{2\bullet})^T (M_1\varepsilon_t + M_2\zeta_t) \mid M_1 \in \mathbb{R}^{(k-r)\times q},\ M_2 \in \mathbb{R}^{(k-r)\times p} \right\}$$

where the first summand is the minimum norm solution, and $M_1 \in \mathbb{R}^{(k-r)\times q}$ and $M_2 \in \mathbb{R}^{(k-r)\times p}$ parametrize the kernel of $Q_{U\bullet}\Pi$. A sufficient (but not necessary[43]) condition for obtaining a unique solution is thus $k = r$.

Proof. The proof is divided into several steps.

Step 1: Transform the existence condition and apply SVD to $Q_{U\bullet}\Pi$ Note that since the existence condition $span(Q_{U\bullet}\Psi) \subseteq span(Q_{U\bullet}\Pi)$ holds, there exists a matrix λ of dimension $(k \times q)$ such that

$$Q_{U\bullet}\Psi = Q_{U\bullet}\Pi\lambda.$$

Thus, the existence condition is equivalent to

$$\begin{aligned} 0 = Q_{U\bullet}\Psi\varepsilon_t + Q_{U\bullet}\Pi\eta_t &= Q_{U\bullet}\Pi\lambda\varepsilon_t + Q_{U\bullet}\Pi\eta_t \\ &= Q_{U\bullet}\Pi(\lambda\varepsilon_t + \eta_t), \end{aligned}$$

which leads to (using the singular value decomposition of $Q_{U\bullet}\Pi$)

$$\begin{aligned} & 0 = U_{\bullet 1} D_{11} V_{1\bullet} (\lambda\varepsilon_t + \eta_t) \\ \iff\ & 0 = V_{1\bullet} (\lambda\varepsilon_t + \eta_t). \end{aligned}$$

[43] [34] states that "[i]f $k = r$ the second and third term drop out and the solution is unique.".

Step 2: Use the functional form $\eta_t = A_1\varepsilon_t + A_2\zeta_t$ and plug η_t into the existence condition. Assuming the functional form above for η_t, we obtain

$$\begin{aligned} & 0 = V_{1\bullet}\left(\lambda\varepsilon_t + [A_1\varepsilon_t + A_2\zeta_t]\right) \\ \iff & 0 = V_{1\bullet}\left(\lambda + A_1\right)\varepsilon_t + V_{1\bullet}A_2\zeta_t. \end{aligned} \tag{58}$$

Step 3: Conclude from the orthogonality of $(\varepsilon_t)_{t\in\mathbb{N}}$ and $(\zeta_t)_{t\in\mathbb{N}}$ on the structure of A_2. In order that the existence condition (58) be satisfied for all possible realizations of the sunspot shock ζ_t, the matrix A_2 has to be equal to $(V_{2\bullet})^T M_2$, where M_2 is an arbitrary matrix of dimension $(r \times p)$. Then, the existence condition is

$$\begin{aligned} & 0 = V_{1\bullet}\left(\lambda + A_1\right)\varepsilon_t + V_{1\bullet}A_2\zeta_t \\ \iff & 0 = V_{1\bullet}\left(\lambda + A_1\right)\varepsilon_t + \underbrace{V_{1\bullet}\left(V_{2\bullet}\right)^T}_{=0} M_2\zeta_t = V_{1\bullet}\left(\lambda + A_1\right)\varepsilon_t. \end{aligned}$$

Step 4: Given $A_2 = (V_{2\bullet})^T M_2$, get an expression A_1 by representing it as direct sum $A_1 = (V_{1\bullet})^T V_{1\bullet}A_1 + (V_{2\bullet})^T V_{2\bullet}A_1 = (V_{1\bullet})^T \tilde{A}_1 + (V_{2\bullet})^T M_1$. Substituting the expression above for A_1 in the existence condition gives

$$\begin{aligned} 0 &= V_{1\bullet}\left(\lambda + \left[(V_{1\bullet})^T \tilde{A}_1 + (V_{2\bullet})^T M_1\right]\right)\varepsilon_t \\ &= V_{1\bullet}\left(\lambda + \left[(V_{1\bullet})^T \tilde{A}_1 + (V_{2\bullet})^T M_1\right]\right) = V_{1\bullet}\lambda + \tilde{A}_1. \end{aligned}$$

It follows that

$$\tilde{A}_1 = -V_{1\bullet}\lambda.$$

Step 5: Express $\tilde{A}_1 = -V_{1\bullet}\lambda$ in terms of the matrices appearing in the SVD by using the existence condition $Q_{U\bullet}\Psi = Q_{U\bullet}\Pi\lambda$. Substituting the SVD of $(Q_{U\bullet}\Pi)$ in the existence condition, we obtain

$$U_{2\bullet}\Psi = U_{\bullet 1}D_{11}V_{1\bullet}\lambda \iff -V_{1\bullet}\lambda = -D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi.$$

Step 6: Obtain a parametrization of the solutions η_t of the existence condition. Finally, we obtain that

$$\begin{aligned} \eta_t &= A_1\varepsilon_t + A_2\zeta_t \\ &= \left[(V_{1\bullet})^T \underbrace{V_{1\bullet}A_1}_{=\tilde{A}_1=-V_{1\bullet}\lambda=-D_{11}^{-1}(U_{\bullet 1})^T Q_{2\bullet}\Psi} + (V_{2\bullet})^T \underbrace{V_{2\bullet}A_1}_{M_1}\right]\varepsilon_t + (V_{2\bullet})^T M_2\zeta_t \\ &= \underbrace{-\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\varepsilon_t}_{=\text{Minimum norm solution}} + \underbrace{\left(V_{2\bullet}\right)^T\left(M_1\varepsilon_t + M_2\zeta_t\right)}_{=\text{parametrization of kernel of } Q_{U\bullet}\Pi}. \end{aligned}$$

which proves the proposition. □

To summarize, we first obtain one particular solution of the existence condition $-Q_{U\bullet}\Pi\eta_t = Q_{U\bullet}\Psi\varepsilon_t$, i.e. the minimum norm solution $\eta_t^p = -\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\varepsilon_t$ obtained through the Moore-Penrose pseudo-inverse (compare [24] page 290) of $Q_{U\bullet}\Pi$. Subsequently, the kernel of $Q_{U\bullet}\Pi$ gets parametrized with M_1 and M_2 in order to describe how the endogenous forecast error η_t might depend on the innovations $(\varepsilon_t)_{t\in\mathbb{N}}$ of the exogenous process and on the sunspot shocks $(\zeta_t)_{t\in\mathbb{N}}$.

Remark 32 (Orthogonality assumption). Lubik and Schorfheide write on page 278 line 12 in [34] that "[i]n order to satisfy [the existence condition] for all ζ_t it is necessary that A_2 is orthogonal to $\left[(V_{1\bullet})^T\right]$". We assume that with "for all ζ_t" Lubik and Schorfheide mean "for all realizations of $(\zeta_t)_{t\in\mathbb{N}}$" because the process $(\zeta_t)_{t\in\mathbb{N}}$ is part of the model specification and therefore fixed. If (assuming A_2 to be square and non-singular) $\zeta_t = -A_2^{-1}\left(\lambda + A_1\right)\varepsilon_t$, the existence condition (58), i.e.

$$0 = V_{1\bullet}\left(\lambda + A_1\right)\varepsilon_t + V_{1\bullet}A_2\zeta_t,$$

holds for all realizations of $(\zeta_t)_{t\in\mathbb{N}}$. It is, thus, not necessary but sufficient that $A_2 = (V_{2\bullet})^T M_2$ for reducing the existence condition (58) to $0 = V_{1\bullet}(\lambda + A_1)\varepsilon_t$. Such an assumption, of course, would be ad hoc. Step 3 in the proof of Proposition 31, however, shows that requiring $(\varepsilon_t)_{t\in\mathbb{N}}$ and $(\zeta_t)_{t\in\mathbb{N}}$ to be orthogonal is sufficient to ensure that $A_2 = (V_{2\bullet})^T M_2$ holds.

Remark 33. Assuming that $M_2 = 0$, η_t is a function of the innovations ε_t of the exogenous variables only. Even though the minimum-norm solution suggests itself for solving an equation of this form, it is not necessarily a natural basis for the column space of $Q_{U\bullet}\Pi$. In [18], a similar ill-posed inverse problem is solved by choosing the first basis of the row space of a certain matrix, compare Section 4 starting on page 232 in [18]. This approach was chosen after realizing that the minimum norm solution in [14] may have some inconvenient properties.

Remark 34. Even though η_t might depend on sunspot shocks ζ_t, these shocks, of course, do not enter the second block of rows in

$$\begin{pmatrix}\Lambda_{SS} & \Lambda_{SU}\\ & \Lambda_{UU}\end{pmatrix}\begin{pmatrix}w_t^S\\ w_t^U\end{pmatrix} = \begin{pmatrix}\Omega_{SS} & \Omega_{SU}\\ & \Omega_{UU}\end{pmatrix}\begin{pmatrix}w_{t-1}^S\\ w_{t-1}^U\end{pmatrix} + \begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi z_t + \begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Pi\eta_t$$

because they are in the right-kernel of $Q_{U\bullet}\Pi$. They may, however, appear in the first block of rows if it is not possible to express the rows of $Q_{S\bullet}\Pi$ in terms of the rows of $Q_{U\bullet}\Pi$, i.e. if Sims' uniqueness condition (35) does not hold.

The uniqueness condition, the degree of indeterminacy, and the dimension of the solution set. The fact that Sims' uniqueness condition (35), i.e. there exists a matrix Φ of dimension $(n(S)\times n(U))$ such that $Q_{S\bullet}\Pi = \Phi Q_{U\bullet}\Pi$ holds, may be satisfied even though the kernel of $Q_{U\bullet}\Pi$ is not trivial was not further analyzed in [34]. In the same way as the singular value decomposition $Q_{U\bullet}\Pi = \begin{bmatrix}U_{\bullet 1} & U_{\bullet 2}\end{bmatrix}\begin{pmatrix}D_{11} & 0\\ 0 & 0\end{pmatrix}\begin{bmatrix}V_{1\bullet}\\ V_{2\bullet}\end{bmatrix}$ described in equation (57) on page 36 allows us to formulate the existence condition (56) conveniently[44], it provides insights into the non-uniqueness problem regarding solutions of rational expectations models. While the image of the map $Q_{U\bullet}\Pi$, and its relation to $Q_{U\bullet}\Psi$, is used to characterize the existence of a solution of the rational expectations model, the (right) kernel of the map $Q_{U\bullet}\Pi$ and its relation to the (right) kernel of $Q_{S\bullet}\Pi$ is used to describe the (non)-uniqueness problem of solutions of the rational expectations model. Indeed, introducing the singular value decomposition of $Q_{S\bullet}\Pi$ as

$$Q_{S\bullet}\Pi = \begin{bmatrix}\tilde{U}_{\bullet 1} & \tilde{U}_{\bullet 2}\end{bmatrix}\begin{pmatrix}\tilde{D}_{11} & 0\\ 0 & 0\end{pmatrix}\begin{bmatrix}\tilde{V}_{1\bullet}\\ \tilde{V}_{2\bullet}\end{bmatrix},$$

Sims' uniqueness condition (35) is equivalent to $rowspan\left(\tilde{V}_{1\bullet}\right) \subseteq rowspan\left(V_{1\bullet}\right)$, i.e. the orthogonal complement $(\ker(Q_{S\bullet}\Pi))^{\perp} = \left(\tilde{V}_{1\bullet}\right)^T$ of the kernel of $Q_{S\bullet}\Pi$ is contained in the orthogonal complement $(\ker(Q_{U\bullet}\Pi))^{\perp} = (V_{1\bullet})^T$ of the kernel of $Q_{U\bullet}\Pi$, and it is also equivalent to $rowspan\left(\tilde{V}_{2\bullet}\right) \supseteq rowspan\left(V_{2\bullet}\right)$, i.e. the kernel of $Q_{S\bullet}\Pi$ with basis $\left(\tilde{V}_{2\bullet}\right)^T$ is contained in the kernel of $Q_{U\bullet}\Pi$ with basis $(V_{2\bullet})^T$.

We define the dimension of the indeterminacy as the rank of the projection of the row space of $Q_{S\bullet}\Pi$ on the orthogonal complement of the row space of $Q_{U\bullet}\Pi$, i.e.

$$\begin{aligned} rank\left(Q_{S\bullet}\Pi - Proj\left(Q_{S\bullet}\Pi \,|\, Q_{U\bullet}\Pi\right)\right) &= rank\left(Q_{S\bullet}\Pi\left\{I_k - \left(\left(Q_{U\bullet}\Pi\right)^{\dagger}\left(Q_{U\bullet}\Pi\right)\right)\right\}\right)\\ &= rank\left(\tilde{U}_{\bullet 1}\tilde{D}_{11}\tilde{V}_{1\bullet}\left\{I_k - \left(\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T U_{\bullet 1}D_{11}V_{1\bullet}\right)\right\}\right),\\ &= rank\left(\tilde{V}_{1\bullet}\left\{\left(V_{2\bullet}\right)^T V_{2\bullet}\right\}\right)\end{aligned}$$

where $A^{\dagger}$ denotes the Moore-Penrose pseudo-inverse of a matrix A, compare [24] page 290, and $Proj\left(A|B\right)$ the projection of the row-space of A on the row-space of B. Intuitively, everything which is not contained in the kernel of $Q_{S\bullet}\Pi$ (the orthogonal complement $(\ker(Q_{S\bullet}\Pi))^{\perp} = \left(\tilde{V}_{1\bullet}\right)^T$ of the kernel of $Q_{S\bullet}\Pi$) is projected on the space through which indeterminacies appear in the model (the kernel of $Q_{U\bullet}\Pi$).

Remark 35. Equivalently, the dimension of the indeterminacy could be defined as the rank of $\left(V_{2\bullet} - V_{2\bullet}\left(\tilde{V}_{2\bullet}\right)^T\tilde{V}_{2\bullet}\right) = V_{2\bullet}\left(I_k - \left(\tilde{V}_{2\bullet}\right)^T\tilde{V}_{2\bullet}\right) = V_{2\bullet}\left(\left(\tilde{V}_{1\bullet}\right)^T\tilde{V}_{1\bullet}\right)$, i.e. the rank of the projection of the kernel of $Q_{U\bullet}\Pi$ (through which

[44] $Q_{U\bullet}\Psi\varepsilon_t + Q_{U\bullet}\Pi\eta_t = 0$ has a solution η_t^* if and only if $span\left(Q_{U\bullet}\Pi\right) \subseteq span\left(U_{\bullet 1}\right)$.

indeterminacies appear in the model) on the orthogonal complement $\left(\tilde{V}_{1\bullet}\right)^T$ of the kernel of $Q_{S\bullet}\Pi$ (everything that actually affects the variables w_t^S).

For a given conditioning set, we define the dimension of the solution set of the rational expectations model as the number of free parameters in the parametrization of the indeterminacy $\tilde{\eta}_t = \left(Q_{S\bullet}\Pi - Proj\left(Q_{S\bullet}\Pi \,|\, Q_{U\bullet}\Pi\right)\right)\eta_t$ when it is expressed as linear function of the components of stochastic processes in the conditioning set. For example, when the dimension of the indeterminacy is, say, d, i.e. when there are d linearly independent components in $\tilde{\eta}_t$, and the conditioning set is $H_\varepsilon(t)$ where $(\varepsilon_t)_{t\in\mathbb{N}}$ is the q-dimensional white noise input process (the innovations of the exogenous process), then the dimension of the solution set is $d \cdot q$.

We will state this as

Theorem 36. *The degree of indeterminacy of the rational expectations model (54) is equal to* $rank\left(\tilde{V}_{1\bullet}\left\{\left(V_{2\bullet}\right)^T V_{2\bullet}\right\}\right)$, *where* $\left(\tilde{V}_{1\bullet}\right)^\perp$ *is an orthonormal basis of the orthogonal complement of the kernel of* $Q_{S\bullet}\Pi$ *and* $\left(V_{2\bullet}\right)^T$ *is an orthonormal basis of the kernel of* $Q_{U\bullet}\Pi$. *Furthermore, for given conditioning set, the dimension of the solution set of the rational expectations model (54) is equal to* $\left[rank\left(\tilde{V}_{1\bullet}\left\{\left(V_{2\bullet}\right)^T V_{2\bullet}\right\}\right)\right]\cdot q$, *where* q *is the rank of the innovation covariance matrix of the stochastic processes contained in the conditioning set.*

Analysis of transfer function. We define the reduced sunspot shocks[45] $\zeta_t^* = M_2\zeta_t$ and proceed to analyze the derivatives of the transfer function relating the innovations ε_t of the exogenous variables and the reduced sunspot shocks ζ_t^* to the endogenous variables. This derivation differs from the one in [34] in two ways. First, Lubik and Schorfheide do not consider derivatives of the endogenous variables with respect to the innovations and the reduced sunspot shocks but rather "derivatives of the system"[46], compare their equation (18) and (19) on page 279 in [34]. Second, Lubik and Schorfheide do not analyze the directions in which the endogenous variables y_t (or rather $\Gamma_0 y_t$) change.

We proceed analogously to section 3.3.4 on page 29. First, we obtain a solution for $\left(w_t^U\right)_{t\in\mathbb{N}}$ which satisfies the non-explosiveness condition (55) and is contained in $H_{\varepsilon,\zeta}(t)$ at time t. In the case treated in [34], i.e. $C = 0$ and $\mathbb{E}_t\left(\varepsilon_{t+1}\right) = 0$, we obtain $w_t^U = 0$. Second, in order to obtain a solution $\left(w_t^S\right)_{t\in\mathbb{N}}$ for a given solution $\left(w_t^U\right)_{t\in\mathbb{N}}$, we need $n(S)$ linear combinations of the whole system

$$\begin{pmatrix}\Lambda_{SS} & \Lambda_{SU}\\ & \Lambda_{UU}\end{pmatrix}\begin{pmatrix}w_t^S\\ w_t^U\end{pmatrix} = \begin{pmatrix}\Omega_{SS} & \Omega_{SU}\\ & \Omega_{UU}\end{pmatrix}\begin{pmatrix}w_{t-1}^S\\ w_{t-1}^U\end{pmatrix} + \begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi\varepsilon_t + \begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Pi\eta_t$$

such that the degree of indeterminacy is minimal. This is achieved by premultiplying the system above with

$$\left(I_{n(S)} \quad -Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^\dagger\right),$$

as described in Theorem 36. Thus, we obtain

$$\begin{aligned}\left(\Lambda_{SS} \quad \Lambda_{SU} - Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^\dagger\Lambda_{UU}\right)\begin{pmatrix}w_t^S\\ w_t^U\end{pmatrix} = \left(\Omega_{SS} \quad \Omega_{SU} - Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^\dagger\Omega_{UU}\right)\begin{pmatrix}w_{t-1}^S\\ w_{t-1}^U\end{pmatrix} + \cdots\\ \cdots + \left(I_{n(S)} \quad -Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^\dagger\right)\begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi\varepsilon_t + \cdots\\ \cdots + \left(Q_{S\bullet}\Pi\left\{I_k - \left(\left(Q_{U\bullet}\Pi\right)^\dagger\left(Q_{U\bullet}\Pi\right)\right)\right\}\right)\eta_t.\end{aligned}$$

Finally, and in analogy to equation (38) on page 30, we obtain by premultiplying $Z\begin{pmatrix}\Lambda_{SS} & \Lambda_{SU} - \Phi\Lambda_{UU}\\ & I_{n(U)}\end{pmatrix}^{-1} = \left(Z_{\bullet S}, Z_{\bullet U}\right)\begin{pmatrix}\Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}\left(\Lambda_{SU} - Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^\dagger\Lambda_{UU}\right)\\ & I_{n(U)}\end{pmatrix}$ the following:

[45] Remember that $M_2 \in \mathbb{R}^{k-r\times p}$.

[46] Note that, in general, the mapping from the structural form to the final form of a difference equation is not unique.

$$
\begin{aligned}
y_t = &\left[(Z_{\bullet S}, Z_{\bullet U})\begin{pmatrix}\Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}\left(\Lambda_{SU}-Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\Lambda_{UU}\right)\\ & I_{n(U)}\end{pmatrix}\begin{pmatrix}\Omega_{SS} & \left(\Omega_{SU}-Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\Omega_{UU}\right)\\ 0 & \end{pmatrix}Z^T\right]y_{t-1}+\cdots\\
&\cdots+Z\begin{pmatrix}\Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}\left(\Lambda_{SU}-Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\Lambda_{UU}\right)\\ & I_{n(U)}\end{pmatrix}\begin{pmatrix}I_{n(S)} & -Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\\ 0 & \end{pmatrix}\begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi\varepsilon_t+\cdots\\
&\cdots+Z\begin{pmatrix}\Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}\left(\Lambda_{SU}-Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\Lambda_{UU}\right)\\ & I_{n(U)}\end{pmatrix}\begin{pmatrix}Q_{S\bullet}\Pi\left\{I_k-\left(\left(Q_{U\bullet}\Pi\right)^{\dagger}\left(Q_{U\bullet}\Pi\right)\right)\right\}\\ 0\end{pmatrix}\eta_t\\
=&\underbrace{Z_{\bullet S}\left[\begin{pmatrix}\Lambda_{SS}^{-1}\Omega_{SS} & \Lambda_{SS}^{-1}\left(\Omega_{SU}-Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\Omega_{UU}\right)\end{pmatrix}Z^T\right]}_{=\Theta_y}y_{t-1}+\cdots\\
&\cdots+\underbrace{Z_{\bullet S}\begin{pmatrix}\Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\end{pmatrix}\begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi}_{=\Theta_\varepsilon}\varepsilon_t+\cdots\\
&\cdots+\underbrace{Z_{\bullet S}\left(\Lambda_{SS}^{-1}Q_{S\bullet}\Pi\left\{I_k-\left(\left(Q_{U\bullet}\Pi\right)^{\dagger}\left(Q_{U\bullet}\Pi\right)\right)\right\}\right)}_{=\Theta_\eta}\eta_t.
\end{aligned}
$$

Using Proposition 31 to substitute for $\eta_t = -\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\varepsilon_t + \left(V_{2\bullet}\right)^T\left(M_1\varepsilon_t+\zeta_t^*\right)$, we obtain for the effects of the white noise inputs ε_t and the reduced sunspots ζ_t^*

$$
\frac{\partial y_t}{\partial \zeta_t^*} = Z_{\bullet S}\left(\Lambda_{SS}^{-1}Q_{S\bullet}\Pi\left\{I_k-\left(\left(Q_{U\bullet}\Pi\right)^{\dagger}\left(Q_{U\bullet}\Pi\right)\right)\right\}\right)\left(V_{2\bullet}\right)^T
$$

and

$$
\begin{aligned}
\frac{\partial y_t}{\partial \varepsilon_t} = &Z_{\bullet S}\begin{pmatrix}\Lambda_{SS}^{-1} & -\Lambda_{SS}^{-1}Q_{S\bullet}\Pi\left(Q_{U\bullet}\Pi\right)^{\dagger}\end{pmatrix}\begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi-\cdots\\
&\cdots-Z_{\bullet S}\left(\Lambda_{SS}^{-1}Q_{S\bullet}\Pi\left\{I_k-\left(\left(Q_{U\bullet}\Pi\right)^{\dagger}\left(Q_{U\bullet}\Pi\right)\right)\right\}\right)\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi+\cdots\\
&\cdots+Z_{\bullet S}\left(\Lambda_{SS}^{-1}Q_{S\bullet}\Pi\left\{I_k-\left(\left(Q_{U\bullet}\Pi\right)^{\dagger}\left(Q_{U\bullet}\Pi\right)\right)\right\}\right)\left(V_{2\bullet}\right)^T M_1.
\end{aligned}
$$

Lubik and Schorfheide plug $\eta_t = -\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\varepsilon_t + \left(V_{2\bullet}\right)^T\left(M_1\varepsilon_t+\zeta_t^*\right)$ into the system

$$
\begin{aligned}
\Gamma_0 y_t &= \Gamma_1 y_{t-1} + \Psi\varepsilon_t + \Pi\eta_t\\
&= \Gamma_1 y_{t-1} + \Psi\varepsilon_t - \Pi\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\varepsilon_t + \Pi\left(V_{2\bullet}\right)^T\left(M_1\varepsilon_t+\zeta_t^*\right)\\
&= \Gamma_1 y_{t-1} + \left(I-\Pi\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\right)\Psi\varepsilon_t + \Pi\left(V_{2\bullet}\right)^T M_1\varepsilon_t + \Pi\left(V_{2\bullet}\right)^T\zeta_t^*
\end{aligned}
\tag{59}
$$

and obtain thus

$$
\frac{\partial\Gamma_0 y_t}{\partial\zeta_t^*} = \Pi\left(V_{2\bullet}\right)^T \text{ and } \frac{\partial\Gamma_0 y_t}{\partial\varepsilon_t} = \left(\Psi-\Pi\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\right)+\Pi\left(V_{2\bullet}\right)^T M_1
$$

without taking into account that theses equations could be further specialized to

$$
\frac{\partial\Gamma_0 y_t}{\partial\zeta_t^*} = \left(Q_{S\bullet}\right)^T Q_S\Pi\left(V_{2\bullet}\right)^T \tag{60}
$$

and

$$
\begin{aligned}
\frac{\partial\Gamma_0 y_t}{\partial\varepsilon_t} &= \left(\Psi-\Pi\left(V_{1\bullet}\right)^T D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\right)+\Pi\left(V_{2\bullet}\right)^T M_1\\
&= \left(\Psi-\Pi\begin{pmatrix}\left(V_{1\bullet}\right)^T & \left(V_{2\bullet}\right)^T\end{pmatrix}\begin{pmatrix}D_{11}^{-1}\left(U_{\bullet 1}\right)^T Q_{U\bullet}\Psi\\ -M_1\end{pmatrix}\right).
\end{aligned}
$$

Remark 37. If Sims' uniqueness condition $Q_{S\bullet}\Pi = \Phi Q_{U\bullet}\Pi$ holds, it follows in equation (60) that $\frac{\partial \Gamma_0 y_t}{\partial \zeta_t^*} = 0$ because

$$\frac{\partial \Gamma_0 y_t}{\partial \zeta_t^*} = (Q_{S\bullet})^T \Phi \underbrace{Q_{U\bullet}\Pi (V_{2\bullet})^T}_{=0} = 0.$$

Under this condition, thus, the sunspots do not appear in system (59).

4 A (constrained) system equivalent to an RE model (BGS)

Here, we start with deriving (in analogy to [11]) from the rational expectations model (1) a recursive equation in terms of the components of leads and lags of the endogenous process by writing the conditional expectation $\mathbb{E}_{t-k}(y_{t+h})$ as sum of the endogenous variable y_{t+h} and its $(h+k)$-step-ahead prediction error $v_{t+h,h+k} = y_{t+h} - \mathbb{E}_{t-k}(y_{t+h})$. Secondly, constraints implied by the rational expectations model on the revision processes $\varepsilon_{t-j}^{j} = \mathbb{E}_{t-j}(y_t) - \mathbb{E}_{t-j+1}(y_t)$ are derived in section 4.2. Subsequently, we show in section 4.3 that a process $(y_t)_{t\in\mathbb{Z}}$ for which the recursive equation holds and whose revision processes $\varepsilon_{t-j}^{j} = \mathbb{E}_{t-j}(y_t) - \mathbb{E}_{t-j+1}(y_t)$ satisfy the constraints implied by the rational expectations model (1) also solves the rational expectations model (in the sense that equation (1) holds for all points in time). Thus, the problem of finding processes $(y_t)_{t\in\mathbb{Z}}$ solving the rational expectations model (1) is reduced to the problem of finding processes $(y_t)_{t\in\mathbb{Z}}$ (which are restricted by the fact that its revision processes $\varepsilon_{t-j}^{j} = \mathbb{E}_{t-j}(y_t) - \mathbb{E}_{t-j+1}(y_t)$ have to satisfy certain constraints) solving a vector difference equation (for given exogenous process).

In section 4.4, we generalize[47] Property 5 on page 245 in [11] with respect to the number of "arbitrary martingale differences" to the case in which the exogenous process has a singular spectral density and correct their count of "auxiliary parameters", i.e. the dimension of the solution set, on page 247 below their formula (4.1) which is only correct if the exogenous process has a spectral density of full rank.

Up to this point, there is no assumption as to whether a process for which the rational expectations equation (1) holds for every $t \in \mathbb{Z}$ also has to be contained in $H_u(t)$ at time t or as to whether it has to satisfy a non-explosiveness condition. Imposing more general non-explosiveness conditions as in [34] (and as general as in [37]) and imposing that $y_t \in H_u(t)$, $t \in \mathbb{Z}$, for a process $(y_t)_{t\in\mathbb{Z}}$ for which the rational expectations equation (1) holds for every $t \in \mathbb{Z}$ is straightforward after having obtained the recursive equation together with the constraints on the revision processes. The latter fact and its suitability for identifiability analysis are the major advantages of model (1) relative to the other approaches described in this paper (which do not take different timing into account).

4.1 Recursive equation

We obtain from the rational expectations model (1) (by substituting for conditional expectations the variables themselves and the associated prediction errors) that

$$\underbrace{z^{J_1}\left(\sum_{i=J_0}^{J_1} A_i^* z^{-i}\right)}_{=\pi(z)} y_t = \pi(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \zeta_{t-J_1} - u_{t-J_1} \tag{61}$$

where

$$A_i^* = \sum_{k:(k,i)\in\mathcal{J}} A_{k,k+i},$$
$$\mathcal{J} = \{(k, h-k)\,|k \in \{0,\ldots,K\}, h \in \{0,\ldots,H\}\},$$
$$J_0 = argmin_i\{i \mid A_i^* \neq 0\},\ J_1 = argmax_i\{i \mid A_i^* \neq 0\},$$

and

$$\zeta_t = \sum_{k=0}^{K}\sum_{j=0}^{H-1}\sum_{h=0}^{j} A_{kh} z^{k+(j-h)} \varepsilon_t^j \text{ and } \varepsilon_t^j = \mathbb{E}_t(y_{t+j}) - \mathbb{E}_{t-1}(y_{t+j})$$

as follows.

[47] Note, however, that we impose (wide sense) stationarity on the exogenous process, whereas [11] do not impose any assumption on the exogenous process. As soon as they do impose "stationary (finite or infinite) moving average structure" (compare [11] page 246 line 27) on the exogenous process, however, they require that the inputs be independent while we only assume that they are uncorrelated (compare [19] page 92 for more detail on the relation between uncorrelated processes, martingale difference sequences, and independent processes).

First, note that

$$\begin{aligned} v_{t+h-k,h} &= y_{t+h-k} - \mathbb{E}_{t-k}\left(y_{t+h-k}\right) \\ &= \left(y_{t+h-k} - \mathbb{E}_{t+(h-k)-1}\left(y_{t+h-k}\right)\right) + \left(\mathbb{E}_{t+(h-k)-1}\left(y_{t+h-k}\right) - \mathbb{E}_{t+(h-k)-2}\left(y_{t+h-k}\right)\right) + \cdots \\ &\quad \cdots + \left(\mathbb{E}_{t-(h-k)-(h-1)}\left(y_{t+h-k}\right) - \mathbb{E}_{t+(h-k)-h}\left(y_{t+h-k}\right)\right) \\ &= \varepsilon^0_{t+h-k} + \varepsilon^1_{t+h-k-1} + \cdots + \varepsilon^{h-1}_{t-k+1} \end{aligned}$$

and thus equation (1) is transformed to

$$\begin{aligned} -A_{00}y_t &= \sum_{k=0}^{K}\sum_{h=1}^{H} A_{kh}\mathbb{E}_{t-k}\left(y_{t+h-k}\right) + \sum_{k=1}^{K} A_{k0}y_{t-k} + u_t \\ &= \sum_{k=0}^{K}\sum_{h=1}^{H} A_{kh}\left[y_{t+h-k} - \sum_{j=0}^{h-1}\varepsilon^j_{t+h-k-j}\right] + \sum_{k=1}^{K} A_{k0}y_{t-k} + u_t \end{aligned} \tag{62}$$

which is equivalent to

$$z^{J_1}\left(\sum_{i=-K}^{H} A^*_i z^{-i}\right) y_t = z^{J_1}\left(\sum_{k=0}^{K}\sum_{h=1}^{H} A_{kh}\sum_{j=0}^{h-1}\varepsilon^j_{t+h-k-j} - u_t\right). \tag{63}$$

where the parameter matrices A^*_i feature the forecasting horizon i more prominently, i.e. the matrices A^*_i, $i \in \{-K,\ldots,0,\ldots,H\}$, are obtained by summing over the diagonals of the big matrix in (1) containing the matrices A_{kh}, $k \in \{0,\ldots,K\}$, $h \in \{0,\ldots,H\}$, as elements. Reordering the sum $z^{J_1}\sum_{k=0}^{K}\sum_{h=1}^{H}\sum_{j=0}^{h-1} A_{kh}z^{k-(h-j)}$ appearing on the right hand side of equation (63) leads to

$$\begin{aligned} z^{J_1}\sum_{k=0}^{K}\sum_{h=1}^{H} A_{kh}\sum_{j=0}^{h-1}\varepsilon^j_{t+h-k-j} &= z^{J_1}\sum_{k=0}^{K}\sum_{j=0}^{H-1}\sum_{h=j+1}^{H} A_{kh}\varepsilon^j_{t+h-k-j} \\ &= z^{J_1}\sum_{k=0}^{K}\sum_{j=0}^{H-1}\left(\sum_{h=0}^{H} A_{kh}\varepsilon^j_{t+h-k-j} - \sum_{h=0}^{j} A_{kh}\varepsilon^j_{t+h-k-j}\right) \\ &= z^{J_1}\sum_{k=0}^{K}\sum_{j=0}^{H-1}\sum_{h=0}^{H} A_{kh}\varepsilon^j_{t+h-k-j} - z^{J_1}\underbrace{\sum_{k=0}^{K}\sum_{j=0}^{H-1}\sum_{h=0}^{j} A_{kh}\varepsilon^j_{t+h-k-j}}_{=\zeta_t} \\ &= \underbrace{\left(z^{J_1}\sum_{k=0}^{K}\sum_{h=0}^{H} A_{kh}z^{k-h}\right)}_{=\pi(z)}\sum_{j=0}^{H-1}\varepsilon^j_{t-j} - \zeta_{t-J_1} \end{aligned}$$

Remark 38 (Perfect foresight solution). For arbitrary processes $\left(\varepsilon^j_t\right)_{t\in\mathbb{Z}}$, the solutions $(y_t)_{t\in\mathbb{Z}}$ of (61) are not necessarily solutions in the wide sense of the rational expectations model (1). In particular, the perfect foresight solution for which $(\varepsilon_t)_{t\in\mathbb{Z}}$ is assumed to be identically zero, may not be a solution in the wide sense of the rational expectations model (1), compare [9] page 350.

Remark 39 (Zeros at infinity). Note that [11] does not use the notion of zeros at infinity because equation (61) is transformed in a way that no leads, i.e. negative powers of the backshift operator z, appear. King and Watson write this equation in terms of the forward shift F. Their zeros at infinity of the matrix pencil $AF - B$ correspond to the zeros at zero of $\pi(z)$.

Remark 40 (No redundant equations). The condition $\det(AF - B) \not\equiv 0$ in [31] corresponds to (modulo stacking the conditional expectations in (1)) $\det(\pi(z)) \not\equiv 0$. Note that the assumption $A_{00} = -I_s$, imposed in [11], does not necessarily imply that $\det(\pi(z)) \not\equiv 0$ holds[48] and thus does not exclude inconsistent equation systems.

[48] Consider, e.g., $A_{00} = -I_s = -A_{KH}$.

Example 41. As an easy example consider the univariate model with $K = H = 2$, i.e.

$$\begin{array}{rccc} y_t = & & a_{01}\mathbb{E}_t\left(y_{t+1}\right) & +a_{02}\mathbb{E}_t\left(y_{t+2}\right) \\ & +a_{10}y_{t-1} & +a_{11}\mathbb{E}_{t-1}\left(y_t\right) & +a_{12}\mathbb{E}_{t-1}\left(y_{t+1}\right) \\ & +a_{20}y_{t-2} & +a_{21}\mathbb{E}_{t-2}\left(y_{t-1}\right) & +a_{22}\mathbb{E}_{t-2}\left(y_t\right) + u_t. \end{array}$$

Replacing the conditional expectations by the variables themselves and the associated endogenous forecast errors leads to

$$\begin{array}{rccc} -u_t = & -y_t & +a_{01}\left(y_{t+1} - \varepsilon_{t+1}^0\right) & +a_{02}\left(y_{t+2} - \varepsilon_{t+2}^0 - \varepsilon_{t+1}^1\right) \\ & +a_{10}y_{t-1} & +a_{11}\left(y_t - \varepsilon_t^0\right) & +a_{12}\left(y_{t+1} - \varepsilon_{t+1}^0 - \varepsilon_t^1\right) \\ & +a_{20}y_{t-2} & +a_{21}\left(y_{t-1} - \varepsilon_{t-1}^0\right) & +a_{22}\left(y_t - \varepsilon_t^0 - \varepsilon_{t-1}^1\right), \end{array}$$

which eventually leads to the recursive equation

$$\begin{aligned} &\underbrace{a_2^*}_{=a_{02}} y_{t+2} + \underbrace{a_1^*}_{=a_{12}+a_{01}} y_{t+1} + \underbrace{a_0^*}_{=-1+a_{11}+a_{22}} y_t + \underbrace{a_{-1}^*}_{=a_{10}+a_{21}} y_{t-1} + \underbrace{a_{-2}^*}_{=a_{20}} y_{t-2} = \\ &\quad = a_{01}\varepsilon_{t+1}^0 + a_{02}\left(\varepsilon_{t+2}^0 + \varepsilon_{t+1}^1\right) + a_{11}\varepsilon_t^0 + a_{12}\left(\varepsilon_{t+2}^0 + \varepsilon_{t+1}^1\right) + a_{21}\varepsilon_{t-1}^0 + a_{22}\left(\varepsilon_t^0 + \varepsilon_{t-1}^1\right) + u_t. \end{aligned}$$

4.2 Constraints on the revision process

In this subsection, the constraints for the revision processes $\varepsilon_t^j = \mathbb{E}_t\left(y_{t+j}\right) - \mathbb{E}_{t-1}\left(y_{t+j}\right)$ are derived by taking conditional expectations of the recursive equation (61) with respect to different information sets, and taking subsequently differences.

We follow [11], page 244ff. We start from equation (61), i.e.

$$\underbrace{z^{J_1}\left(\sum_{i=J_0}^{J_1} A_i^* z^{-i}\right)}_{=\pi(z)} y_t = \pi(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \zeta_{t-J_1} - u_{t-J_1}$$

where $J_0 = argmin_i\left\{i \,|\, A_i^* \neq 0\right\}$, $J_i = argmax_i\left\{i \,|\, A_i^* \neq 0\right\}$, $\zeta_t = \sum_{k=0}^{K}\sum_{j=0}^{H-1}\sum_{h=0}^{j} A_{kh} z^{k+(j-h)}\varepsilon_t^j$, and $\varepsilon_t^j = \mathbb{E}_t\left(y_{t+j}\right) - \mathbb{E}_{t-1}\left(y_{t+j}\right)$ and write the Smith canonical form of $\pi(z)$ as

$$\pi(z) = P(z)\alpha(z)\Phi(z)Q(z),$$

where $P(z)$ and $Q(z)$ are unimodular[49] matrices of dimension $(s \times s)$, and $\alpha(z) = \begin{pmatrix} \alpha_1(z) & & \\ & \ddots & \\ & & \alpha_s(z) \end{pmatrix}$ and $\Phi(z) = \begin{pmatrix} \phi_1(z) & & \\ & \ddots & \\ & & \phi_s(z) \end{pmatrix}$ are diagonal polynomial matrices whose i-th diagonal element divides the $(i+1)$-th diagonal element[50]. Moreover, the entries of $\alpha(z)$ have only zeros at zero. Thus, we will work with the equation

$$P(z)\alpha(z)\Phi(z)Q(z)y_t = P(z)\alpha(z)\Phi(z)Q(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \zeta_{t-J_1} - u_{t-J_1} \tag{64}$$

Theorem 42. *Assume that $(y_t)_{t\in\mathbb{Z}}$ is a solution in the wide sense of the rational expectations model (1). Then, H revision processes of dimension s satisfy the conditions*

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \ldots, H-1\}$$

or equivalently

$$\begin{aligned} &\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\zeta_{t-J_1}\right) - \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\zeta_{t-J_1}\right) = \\ &\quad = -\left[\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}u_{t-J_1}\right) - \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}u_{t-J_1}\right)\right], \quad i \in \{0, \ldots, H-1\}. \end{aligned} \tag{65}$$

Proof. The proof is divided into several steps.

[49] A unimodular matrix is a matrix whose elements are polynomials but its determinant is a non-zero constant. For further background on polynomial and rational matrices see [21] Chapter VI, [30] Chapter 6, [22, 23], and [28] Chapter 2.

[50] Let $\phi_i(z)$ and $\phi_{i+1}(z)$ denote the i-th and $(i+1)$-th diagonal element. If $\phi_i(z)$ divides $\phi_{i+1}(z)$, there exists a polynomial $p(z)$ such that $\phi_{i+1}(z) = p(z)\phi_i(z)$.

Step 1: Apply $(P(z)\alpha(z))^{-1}$ to the recursive equation (64) (which was derived from (1)). The equation we will work with is

$$\underbrace{\Phi(z)Q(z)}_{=\Omega(z)} y_t = \underbrace{\Phi(z)Q(z)}_{=\Omega(z)} \left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \alpha(z)^{-1}P(z)^{-1}\left(\zeta_{t-J_1} - u_{t-J_1}\right). \tag{66}$$

Step 2: Take conditional expectations of $\Omega(z)y_t$ with respect to the information at time $(t-i)\,,\ i \in \{0,\ldots,H\}\,,$ and subtract equation $(i+1)$ from equation i for $i \in \{0,\ldots,H-1\}\,.$ Note that lags of y_t appearing in $\Omega(z)y_t = \omega_0 y_t + \omega_1 y_{t-1} + \cdots + \omega_i y_{t-i} + \omega_{i+1} y_{t-(i+1)} + \cdots + \omega_{\deg(\Omega(z))} y_{t-\deg(\Omega(z))}$ that are larger than i, have the same conditional expectation with respect to information sets up to time $(t-i)$ and up to time $(t-(i+1))$. Thus, we obtain for the left hand side of equation (66)

$$\begin{aligned}
&\mathbb{E}_{t-i}\left(\Omega(z)y_t\right) - \mathbb{E}_{t-(i+1)}\left(\Omega(z)y_t\right) = \\
&\quad = \mathbb{E}_{t-i}\left(\omega_0 y_t + \omega_1 y_{t-1} + \cdots + \omega_i y_{t-i}\right) - \mathbb{E}_{t-(i+1)}\left(\omega_0 y_t + \omega_1 y_{t-1} + \cdots + \omega_i y_{t-i}\right) \qquad (67)\\
&\quad = \omega_0\left[\mathbb{E}_{t-i}\left(y_t\right) - \mathbb{E}_{t-(i+1)}\left(y_t\right)\right] + \cdots + \omega_i\left[\mathbb{E}_{t-i}\left(y_{t-i}\right) - \mathbb{E}_{t-(i+1)}\left(y_{t-i}\right)\right] \\
&\quad = \omega_0\varepsilon_{t-i}^i + \omega_1\varepsilon_{t-i}^{i-1} + \cdots + \omega_i\varepsilon_{t-i}^0. \qquad (68)
\end{aligned}$$

Step 3: Take the conditional expectation of $\Omega(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right)$ with respect to the information at time $(t-i)\,,\ i \in \{0,\ldots,H\}\,,$ and subtract equation $(i+1)$ from equation i for $i \in \{0,\ldots,H-1\}\,.$ Considering the term $\Omega(z)\varepsilon_{t-j}^j$, we note that lags larger than $(i-j)$ are contained in both information sets[51] which contain information up to time $(t-i)$ and up to time $(t-(i+1))$. Thus, we obtain for $i \leq j$

$$\begin{aligned}
&\mathbb{E}_{t-i}\left(\Omega(z)\varepsilon_{t-j}^j\right) - \mathbb{E}_{t-(i+1)}\left(\Omega(z)\varepsilon_{t-j}^j\right) = \\
&\quad = \mathbb{E}_{t-i}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-j-(i-j-1)}^j + \omega_{i-j}\varepsilon_{t-j-(i-j)}^j\right) - \cdots \\
&\qquad \cdots - \mathbb{E}_{t-(i+1)}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-j-(i-j-1)}^j + \omega_{i-j}\varepsilon_{t-j-(i-j)}^j\right) \\
&\quad = \mathbb{E}_{t-i}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-i-1)}^j + \omega_{i-j}\varepsilon_{t-i}^j\right) - \cdots \\
&\qquad \cdots - \mathbb{E}_{t-(i+1)}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-i-1)}^j + \omega_{i-j}\varepsilon_{t-i}^j\right) \\
&\quad = \omega_{i-j}\varepsilon_{t-i}^j
\end{aligned}$$

such that

$$\begin{aligned}
&\mathbb{E}_{t-i}\left(\Omega(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right)\right) - \mathbb{E}_{t-(i+1)}\left(\Omega(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right)\right) = \\
&\quad = \omega_i\varepsilon_{t-i}^0 + \omega_{i-1}\varepsilon_{t-i}^1 + \cdots + \omega_1\varepsilon_{t-i}^{i-1} + \omega_0\varepsilon_{t-i}^i
\end{aligned}$$

which is equal to (68), i.e. $\mathbb{E}_{t-i}\left(\Omega(z)y_t\right) - \mathbb{E}_{t-(i+1)}\left(\Omega(z)y_t\right)$, from above. On the right hand side of equation (66) remains thus

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)$$

from which the theorem follows. □

4.3 Constrained solutions of the recursive equation

In this subsection, we characterize the solutions in the wide sense of rational expectations model (1). They comprise all solutions of the recursive equation (61) where the uncorrelated processes $\left(\varepsilon_t^j\right)_{t\in\mathbb{Z}}$ satisfy the constraints (65). We follow [11] page 244ff. and prove

[51] $(t-j)-(i-j) = t-i$

Theorem 43. *Assume that the process $(y_t)_{t\in\mathbb{Z}}$ satisfies the equation*

$$\underbrace{\pi(z)}_{=P(z)\alpha(z)\Phi(z)Q(z)} y_t = \pi(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \zeta_{t-J_1} - u_{t-J_1},$$

where H (arbitrary) martingale difference processes $\left(\varepsilon_t^j\right)_{t\in\mathbb{Z}}$, $j \in \{0,\ldots,H-1\}$, of dimension s satisfy the conditions

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0,\ldots,H-1\}$$

or equivalently

$$\begin{aligned}&\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\zeta_{t-J_1}\right) - \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\zeta_{t-J_1}\right) = \\ &\quad = -\left[\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}u_{t-J_1}\right) - \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}u_{t-J_1}\right)\right], \quad i \in \{0,\ldots,H-1\}.\end{aligned}$$

It follows that the process $(y_t)_{t\in\mathbb{Z}}$ is also a solution in the wide sense of the rational expectations model (1), i.e.

$$\begin{pmatrix}1 & z & \cdots & z^k & \cdots & z^K\end{pmatrix}\begin{pmatrix}A_{00} & A_{01} & \cdots & A_{0h} & \cdots & A_{0H}\\ A_{10} & \ddots & & & & \\ \vdots & & & & & \\ A_{k0} & & & A_{kh} & & A_{kH}\\ \vdots & & & & & \vdots\\ A_{K0} & A_{K1} & \cdots & & & A_{KH}\end{pmatrix}\begin{pmatrix}\mathbb{E}_t\left(y_t\right)\\ \mathbb{E}_t\left(y_{t+1}\right)\\ \vdots\\ \mathbb{E}_t\left(y_{t+h}\right)\\ \vdots\\ \mathbb{E}_t\left(y_{t+H}\right)\end{pmatrix} = -u_t.$$

Proof. We start by premultiplying $\alpha^{-1}(z)P^{-1}(z)$ on $\pi(z)y_t = \pi(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \zeta_{t-J_1} - u_{t-J_1}$ in order to obtain

$$\underbrace{\Phi(z)Q(z)}_{=\Omega(z)} y_t = \Omega(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \alpha^{-1}(z)P^{-1}(z)\zeta_{t-J_1} - \alpha^{-1}(z)P^{-1}(z)u_{t-J_1}.$$

Step 1: Take conditional expectations of $\Omega(z)y_t = \Omega(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right) + \alpha^{-1}(z)P^{-1}(z)\zeta_{t-J_1} - \alpha^{-1}(z)P^{-1}(z)u_{t-J_1}$ with respect to $H_u(t-i),\ i \in \{0,\ldots,H\}$, and subtract each projection from the preceding. The left hand side of the equation is evidently $\mathbb{E}_{t-i}\left(\Omega(z)y_t\right) - \mathbb{E}_{t-(i+1)}\left(\Omega(z)y_t\right)$, where only lags of y_t up to time $(t-i)$ have to be considered because $y_{t-j},\ j > i$ is contained in both $H_u(t-i)$ and $H_u\left(t-(i+1)\right)$ and thus cancels out.

For the right hand side, we consider the term $\Omega(z)\varepsilon_{t-j}^j$ and note that lags larger than $(i-j)$ are contained in both information sets [52] which contain information up to time $(t-i)$ and up to time $(t-(i+1))$. Thus, we obtain for $i \leq j$

$$\begin{aligned}
\mathbb{E}_{t-i}\left(\Omega(z)\varepsilon_{t-j}^j\right) - \mathbb{E}_{t-(i+1)}\left(\Omega(z)\varepsilon_{t-j}^j\right) &= \mathbb{E}_{t-i}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-j-(i-j-1)}^j + \omega_{i-j}\varepsilon_{t-j-(i-j)}^j\right) - \cdots \\
&\quad \cdots - \mathbb{E}_{t-(i+1)}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-j-(i-j-1)}^j + \omega_{i-j}\varepsilon_{t-j-(i-j)}^j\right) \\
&= \mathbb{E}_{t-i}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-i-1)}^j + \omega_{i-j}\varepsilon_{t-i}^j\right) - \cdots \\
&\quad \cdots - \mathbb{E}_{t-(i+1)}\left(\omega_0\varepsilon_{t-j}^j + \omega_1\varepsilon_{t-j-1}^j + \cdots + \omega_{i-j-1}\varepsilon_{t-i-1)}^j + \omega_{i-j}\varepsilon_{t-i}^j\right) \\
&= \omega_{i-j}\varepsilon_{t-i}^j
\end{aligned}$$

such that

$$\mathbb{E}_{t-i}\left(\Omega(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right)\right) - \mathbb{E}_{t-(i+1)}\left(\Omega(z)\left(\varepsilon_t^0 + \varepsilon_{t-1}^1 + \cdots + \varepsilon_{t-H+1}^{H-1}\right)\right) = \omega_i\varepsilon_{t-i}^0 + \omega_{i-1}\varepsilon_{t-i}^1 + \cdots + \omega_1\varepsilon_{t-i}^{i-1} + \omega_0\varepsilon_{t-i}^i.$$

Likewise, applying conditional expectations with respect to information up to time $(t-i)$ and up to time $(t-(i+1))$ on $\alpha^{-1}(z)P^{-1}(z)\zeta_{t-J_1} - \alpha^{-1}(z)P^{-1}(z)u_{t-J_1}$ gives $\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right),\quad i \in \{0,\ldots,H-1\}$.

Step 2: Use the constraints and obtain a system of equations relating the revision processes to some conditional expectations. Using the constraints and denoting the matrix-valued coefficients of $\Omega(z)$ by ω_i, we thus obtain

$$\mathbb{E}_{t-i}\left(\left(\omega_0 + \omega_1 z + \cdots + \omega_i z^i\right)y_t\right) - \mathbb{E}_{t-(i+1)}\left(\left(\omega_0 + \omega_1 z + \cdots + \omega_i z^i\right)y_t\right) = \omega_i\varepsilon_{t-i}^0 + \omega_{i-1}\varepsilon_{t-i}^1 + \cdots + \omega_1\varepsilon_{t-i}^{i-1} + \omega_0\varepsilon_{t-i}^i,\quad i \in \{0,\ldots,H-1\},$$

[52] $(t-j)-(i-j) = t-i$

and equivalently the following system of equations

$$\begin{aligned}
\omega_0 y_t - \mathbb{E}_{t-1}(\omega_0 y_t) &= \omega_0 \varepsilon_t^0 \\
\mathbb{E}_{t-1}(\omega_0 y_t + \omega_1 y_{t-1}) - \mathbb{E}_{t-2}(\omega_0 y_t + \omega_1 y_{t-1}) &= \omega_0 \varepsilon_t^1 + \omega_1 \varepsilon_{t-1}^0 \\
\mathbb{E}_{t-2}(\omega_0 y_t + \omega_1 y_{t-1} + \omega_2 y_{t-2}) - \mathbb{E}_{t-3}(\omega_0 y_t + \omega_1 y_{t-1} + \omega_2 y_{t-2}) &= \omega_0 \varepsilon_t^2 + \omega_1 \varepsilon_{t-1}^1 + \omega_2 \varepsilon_{t-2}^0 \\
&\vdots \\
\mathbb{E}_{t-i}(\omega_0 y_t + \cdots + \omega_i y_{t-i}) - \mathbb{E}_{t-(i+1)}(\omega_0 y_t + \cdots + \omega_i y_{t-i}) &= \omega_0 \varepsilon_t^i + \cdots + \omega_i \varepsilon_{t-i}^0 \\
&\vdots \\
\mathbb{E}_{t-(H-1)}\left(\omega_0 y_t + \cdots + \omega_{H-1} y_{t-(H-1)}\right) - \mathbb{E}_{t-H}\left(\omega_0 y_t + \cdots + \omega_{H-1} y_{t-(H-1)}\right) &= \omega_0 \varepsilon_t^{H-1} + \cdots + \omega_{H-1} \varepsilon_{t-(H-1)}^0
\end{aligned}$$

Step 3: Reorder the system of equations in order to conclude. If the i-th equation is shifted $H-i$ periods backwards, this system can be written as

$$\begin{aligned}
z^{H-1}\left[\omega_0\left(y_t - \mathbb{E}_{t-1}(y_t)\right)\right] &= \omega_0 \varepsilon_{t-H+1}^0 \\
z^{H-2}\left[\omega_0\left(\mathbb{E}_{t-1}(y_t) - \mathbb{E}_{t-2}(y_t)\right) + \omega_1\left(\mathbb{E}_{t-1}(y_{t-1}) - \mathbb{E}_{t-2}(y_{t-1})\right)\right] &= \omega_0 \varepsilon_{t-H+2}^1 + \omega_1 \varepsilon_{t-H+1}^0 \\
z^{H-3}\left[\omega_0\left(\mathbb{E}_{t-2}(y_t) - \mathbb{E}_{t-3}(y_t)\right) + \omega_1\left(\mathbb{E}_{t-2}(y_{t-1}) - \mathbb{E}_{t-3}(y_{t-1})\right) + \omega_2\left(\mathbb{E}_{t-2}(y_{t-2}) - \mathbb{E}_{t-3}(y_{t-2})\right)\right] &= \omega_0 \varepsilon_{t-H+3}^2 + \omega_1 \varepsilon_{t-H+2}^1 + \omega_2 \varepsilon_{t-H+1}^0 \\
&\vdots \\
z^{H-1-(i+1)}\left[\omega_0\left(\mathbb{E}_{t-i}(y_t) - \mathbb{E}_{t-(i+1)}(y_t)\right) + \cdots + \omega_i\left(\mathbb{E}_{t-i}(y_{t-i}) - \mathbb{E}_{t-(i+1)}(y_{t-i})\right)\right] &= \omega_0 \varepsilon_{t-H+i+1}^i + \cdots + \omega_i \varepsilon_{t-H+1}^0 \\
&\vdots \\
\omega_0\left(\mathbb{E}_{t-(H-1)}(y_t) - \mathbb{E}_{t-H}(y_t)\right) + \cdots + \omega_{H-1}\left(\mathbb{E}_{t-(H-1)}\left(y_{t-(H-1)}\right) - \mathbb{E}_{t-H}\left(y_{t-(H-1)}\right)\right) &= \omega_0 \varepsilon_t^{H-1} + \cdots + \omega_{H-1} \varepsilon_{t-(H-1)}^0
\end{aligned}$$

or equivalently

$$\begin{aligned}
\omega_0\left(y_{t-H+1} - \mathbb{E}_{t-H}(y_{t-H+1})\right) &= \omega_0 \varepsilon_{t-H+1}^0 \\
\omega_0\left(\mathbb{E}_{t-H+1}(y_{t-H+2}) - \mathbb{E}_{t-H}(y_{t-H+2})\right) + \omega_1\left(\mathbb{E}_{t-H+1}(y_{t-H+1}) - \mathbb{E}_{t-H}(y_{t-H+1})\right) &= \omega_0 \varepsilon_{t-H+2}^1 + \omega_1 \varepsilon_{t-H+1}^0 \\
\omega_0\left(\mathbb{E}_{t-H+1}(y_{t-H+3}) - \mathbb{E}_{t-H}(y_{t-H+3})\right) + \omega_1\left(\mathbb{E}_{t-H+1}(y_{t-H+2}) - \mathbb{E}_{t-H}(y_{t-H+2})\right) + \omega_2\left(\mathbb{E}_{t-H+1}(y_{t-H+1}) - \mathbb{E}_{t-H}(y_{t-H+1})\right) &= \omega_0 \varepsilon_{t-H+3}^2 + \omega_1 \varepsilon_{t-H+2}^1 + \omega_2 \varepsilon_{t-H+1}^0 \\
&\vdots \\
\omega_0\left(\mathbb{E}_{t-H+1}(y_{t-H-i+2}) - \mathbb{E}_{t-H}(y_{t-H-i+2})\right) + \cdots + \omega_i\left(\mathbb{E}_{t-H+1}(y_{t-H+1}) - \mathbb{E}_{t-H}(y_{t-H+1})\right) &= \omega_0 \varepsilon_{t-H+i+1}^i + \cdots + \omega_i \varepsilon_{t-H+1}^0 \\
&\vdots \\
\omega_0\left(\mathbb{E}_{t-(H-1)}(y_t) - \mathbb{E}_{t-H}(y_t)\right) + \cdots + \omega_{H-1}\left(\mathbb{E}_{t-(H-1)}\left(y_{t-(H-1)}\right) - \mathbb{E}_{t-H}\left(y_{t-(H-1)}\right)\right) &= \omega_0 \varepsilon_t^{H-1} + \cdots + \omega_{H-1} \varepsilon_{t-(H-1)}^0
\end{aligned}$$

or

$$\begin{pmatrix} \omega_0 & 0 & & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \omega_i & & \omega_0 & & \\ \vdots & & & \ddots & 0 \\ \omega_{H-1} & \omega_{H-2} & \cdots & \omega_1 & \omega_0 \end{pmatrix} \left\{ \mathbb{E}_{t-H+1} \begin{pmatrix} y_{t-H+1} \\ \vdots \\ y_{t-H-i+2} \\ \vdots \\ y_t \end{pmatrix} - \mathbb{E}_{t-H} \begin{pmatrix} y_{t-H+1} \\ \vdots \\ y_{t-H-i+2} \\ \vdots \\ y_t \end{pmatrix} \right\} = \begin{pmatrix} \omega_0 & 0 & & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \omega_i & & \omega_0 & & \\ \vdots & & & \ddots & 0 \\ \omega_{H-1} & \omega_{H-2} & \cdots & \omega_1 & \omega_0 \end{pmatrix} \begin{pmatrix} \varepsilon^0_{t-H+1} \\ \vdots \\ \varepsilon^i_{t-H+i+1} \\ \vdots \\ \varepsilon^{H-1}_t \end{pmatrix}.$$

Since ω_0 is a non-singular matrix, we obtain that

$$\mathbb{E}_{t-i}\left(y_t\right) - \mathbb{E}_{t-(i+1)}\left(y_t\right) = \varepsilon^i_{t-i}.$$

□

Remark 44. Note the similar structure of the proof of Theorem 42. While we assumed in Theorem 42 that the martingale difference processes are derived from the solutions in the wide sense of the rational expectations model, we prove here that for martingale difference sequences satisfying the constraints, the solutions of the recursive equations are also solutions in the wide sense of the rational expectations model.

4.4 Dimension of the solution set with general restrictions

In this section, we generalize Property 5 on page 245 in [11] with respect to the number of "arbitrary martingale differences" to the case in which the exogenous process has a singular spectral density.

Moving average structure of exogenous process. We consider first the case where the exogenous process has an infinite moving average representation, i.e.

$$u_t = \sum_{i=0}^{\infty} w_i \varepsilon_{t-i} = w(z)\varepsilon_t$$

where $w_0 = I_s$, $\sum_{i=0}^{\infty} w_i w_i' < \infty$ (component wise), and $\mathbb{E}(\varepsilon_t \varepsilon_s) = \delta_{ts}\Sigma > 0$ and search for solutions $y_t = \sum_{j=-\infty}^{\infty} k_j \varepsilon_{t-j}$ such that $\sum_{j=-\infty}^{\infty} k_j k_j' < \infty$ (component wise)[53].

Note that the revision process ε_t^j satisfies

$$\varepsilon_{t-j}^j = \mathbb{E}_{t-j}(y_t) - \mathbb{E}_{t-(j+1)}(y_t) = k_j \varepsilon_{t-j}, \quad j \geq 0.,$$

where $k_j \in \mathbb{R}^{s \times q}$.

Assumptions on the parameter space. We assume that there exists an $h \in \{0, \ldots, H\}$ such that $A_{Kh} \neq 0$ and a $k \in \{0, \ldots, K\}$ such that $A_{kH} \neq 0$, and that

$$\pi(z) = z^{J_1}\left(\sum_{i=J_0}^{J_1} A_i^* z^{-i}\right) = A_{J_0}^* z^{J_1 - J_0} + A_{J_0+1}^* z^{(J_1-J_0)-1} + \cdots + A_0^* z^{J_1} + \cdots + A_{J_1-1}^* z + A_{J_1}^*$$

has determinant not identically zero[54]. We will consider two different kinds of parameter restrictions, namely zero restrictions, i.e. the entries of the matrices A_{kh}, $k \in \{0, \ldots, K\}$ and $h \in \{0, \ldots, H\}$, may only be constrained to be zero, and rational restrictions, i.e. we require that their entries are of the form $A_{kh}^{ij} = \frac{p_{kh}^{ij}(\theta_1, \ldots, \theta_p)}{q_{kh}^{ij}(\theta_1, \ldots, \theta_p)}$ where p_{kh}^{ij} and q_{kh}^{ij} are polynomials in $(\theta_1, \ldots, \theta_p)$ and q_{kh}^{ij} is not identically zero. Both of these restrictions guarantee that J_1 and G_1 (the number of zeros at zero of $\pi(z)$) are well defined on the parameter space in the sense that both are constant on the complement of a subset (of the parameter space) of lower dimension. Furthermore, we assume that there is a point in the parameter space such that the matrix (72) on page 58 has full row rank[55].

Remember that

$$\pi(z) = P(z)\alpha(z)\Phi(z)Q(z)$$

where $P(z)$ and $Q(z)$ are unimodular matrices of dimension $(s \times s)$, and $\alpha(z) = \begin{pmatrix} \alpha_1(z) & & \\ & \ddots & \\ & & \alpha_s(z) \end{pmatrix}$ and $\Phi(z) = \begin{pmatrix} \phi_1(z) & & \\ & \ddots & \\ & & \phi_s(z) \end{pmatrix}$ are diagonal polynomial matrices whose i-th diagonal element divides the $(i+1)$-th diagonal element. Moreover, the entries of $\alpha(z)$ have only zeros at zero.

[53] As soon as "stationary (finite or infinite) moving average structure" (compare [11] page 246 line 27) is imposed on the exogenous process in [11], however, it is required that the inputs be independent while we only assume that they are uncorrelated (compare [19] page 92 for more detail on the relation between uncorrelated processes, martingale difference sequences, and independent processes). Moreover, while [9] imposes a summability condition on the the coefficients in $u_t = \sum_{i=0}^{\infty} w_i \varepsilon_{t-i}$ ($\sum_{j=0}^{\infty} |w_j| < \infty$, [9] page 351, which is stronger than our $\sum_{j=0}^{\infty} w_j^2 < \infty$) [11] do not make such an assumption.

[54] Note that the non-singularity of A_{00} does not imply that $\det(\pi(z)) \not\equiv 0$, compare remark 40 on page 43.

[55] In [11] it is assumed that $A_{00} = -I_s$ and (implicitly by only allowing for zero restrictions) that the point for which all (unrestricted) matrices are zero is contained in the parameter space. If we allow for rational restrictions this has to be assumed explicitly.

Theorem 45. *We consider the rational expectations model (1), i.e.*

$$\begin{pmatrix} I_s & I_s z & \cdots & I_s z^k & \cdots & I_s z^K \end{pmatrix} \begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0h} & \cdots & A_{0H} \\ A_{10} & \ddots & & & & \\ \vdots & & & & & \\ A_{k0} & & & A_{kh} & & A_{kH} \\ \vdots & & & & & \vdots \\ A_{K0} & A_{K1} & \cdots & & & A_{KH} \end{pmatrix} \begin{pmatrix} y_t \\ \mathbb{E}_t\left(y_{t+1}\right) \\ \vdots \\ \mathbb{E}_t\left(y_{t+h}\right) \\ \vdots \\ \mathbb{E}_t\left(y_{t+H}\right) \end{pmatrix} = -u_t$$

and assume that (together with the assumptions on the parameter space above)

- *the entries of the parameter matrices above are of the form* $A_{kh}^{ij} = \frac{p_{kh}^{ij}(\theta_1,\ldots,\theta_p)}{q_{kh}^{ij}(\theta_1,\ldots,\theta_p)}$ *where* p_{kh}^{ij} *and* q_{kh}^{ij} *are polynomials in* $(\theta_1,\ldots,\theta_p)$ *and* q_{kh}^{ij} *is not identically zero, and that*
- $rk\left(f_u(\lambda)\right) = q \leq s$ *holds.*

The rational expectations model has a reduced form involving generically $(J_1 s - G_1)\, q$ *free parameters, where*

- s *is the number of equations of the model,*
- J_1 *is such that* $t + J_1$ *is the largest time index of expected endogenous variables appearing in the model, and*
- G_1 *is the number of zero roots of* $\det\left(\pi(z)\right)$.

Proof. We consider the constraints derived from the rational expectations model in Theorem 42, i.e.

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)=\mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right),\quad i\in\{0,\dots,H-1\}$$

or equivalently

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\zeta_{t-J_1}\right)-\mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\zeta_{t-J_1}\right)=-\left[\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}u_{t-J_1}\right)-\mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}u_{t-J_1}\right)\right],\quad i\in\{0,\dots,H-1\},$$

and derive explicit constraints on the martingale difference sequences $\left(\varepsilon_t^i\right)_{t\in\mathbb{Z}}$, $i\in\{0,\dots,H-1\}$, by writing $\zeta_t=-\sum_{k=0}^{K}\sum_{j=0}^{H-1}\sum_{h=0}^{j}A_{kh}z^{k+(j-h)}\varepsilon_t^j$ in an intelligent way. We start by proving the case $\alpha(z)=I_s$ (or $G_1=0$, case A), then $G_1=1$ (case B), and finally $G_1=2$ (case C) from which the more general cases are obvious.

Step A.1: Note that some constraints are trivially satisfied. It is easy to see that the first J_1 equations in

$$\left.\begin{aligned}\mathbb{E}_t\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-1}\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\\&\vdots\\\mathbb{E}_{t-J_1+1}\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-J_1}\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\end{aligned}\right\}J_1\text{ equation systems}$$

$$\left.\begin{aligned}\mathbb{E}_{t-J_1}\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-J_1-1}\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\\&\vdots\\\mathbb{E}_{t-H+1}\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-H}\left(P(z)^{-1}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\end{aligned}\right\}(H-J_1)\text{ equation systems}$$

are trivially satisfied (such that $(H-J_1)$ equations remain), because ζ_{t-J_1} and u_{t-J_1} are contained in the linear span of $\{u_{t-J_1},u_{t-J_1-1},\dots\}$ and supersets thereof.

Step A.2: First simplification of the last $(H-J_1)$ equation: Discard unnecessary lags. Write $P(z)^{-1}=P_0+P_1z+\cdots$ and rewrite the last $(H-J_1)$ equations from above as

$$\begin{aligned}\mathbb{E}_{t-J_1}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-J_1-1}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\\\mathbb{E}_{t-J_1-1}\left(\left\{P_0+P_1z\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-J_1-2}\left(\left\{P_0+P_1z\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\\&\vdots\\\mathbb{E}_{t-J_1-i+1}\left(\left\{P_0+P_1z+\cdots+P_{i-1}z^{i-1}\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-J_1-i}\left(\left\{P_0+P_1z+\cdots+P_{i-1}z^{i-1}\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\\&\vdots\\\mathbb{E}_{t-H+1}\left(\left\{P_0+P_1z+\cdots+P_{H-J_1-1}z^{H-J_1-1}\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)&=\mathbb{E}_{t-H}\left(\left\{P_0+P_1z+\cdots+P_{H-J_1-1}z^{H-J_1-1}\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)\end{aligned}$$

by noting that the i-th lags of ζ_{t-J_1} and u_{t-J_1} are contained in the linear span of $\{u_{t-J_1-i},u_{t-J_1-i-1},\dots\}$ and supersets thereof. Thus these lags cancel out in the i-th equation.

Step A.3: Second simplification of the last $(H-J_1)$ equations: Since $P(z)$ is (as unimodular matrix) non-singular for all[56] $z \in \mathbb{C}$ the same is true for $P(z)^{-1}$ and in particular for $P(0)^{-1} = P_0$. Moreover, note that, e.g., in the second equation the term

$$\begin{aligned} & \mathbb{E}_{t-J_1-1}\left(P_1 z\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) = \mathbb{E}_{t-J_1-2}\left(P_1 z\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) \\ \iff & P_1\mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1-1}-u_{t-J_1-1}\right) = P_1\mathbb{E}_{t-J_1-2}\left(\zeta_{t-J_1-1}-u_{t-J_1-1}\right) \\ \iff & P_1\mathbb{E}_{t-J_1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) = P_1\mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) \end{aligned}$$

is already satisfied because of the first equation, i.e.

$$\begin{aligned} & \mathbb{E}_{t-J_1}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) = \mathbb{E}_{t-J_1-1}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) \\ \iff & \mathbb{E}_{t-J_1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) = \mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}-u_{t-J_1}\right), \end{aligned}$$

such that only

$$\begin{aligned} & \mathbb{E}_{t-J_1-1}\left(\left\{P_0+P_1 z\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) = \mathbb{E}_{t-J_1-2}\left(\left\{P_0+P_1 z\right\}\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) \\ \iff & \mathbb{E}_{t-J_1-1}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) = \mathbb{E}_{t-J_1-2}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) \\ \iff & \mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) = \mathbb{E}_{t-J_1-2}\left(\zeta_{t-J_1}-u_{t-J_1}\right) \end{aligned} \tag{69}$$

remains. The same procedure is performed on equations $i \in \{3,\dots,H-J_1\}$, i.e. we use equation one to get rid of the lag $(i-1)$ in $\left\{P_0+P_1 z+\cdots+P_{i-1}z^{i-1}\right\}$ in equation i, then use equation two to get rid of lag $(i-2)$ in $\left\{P_0+P_1 z+\cdots+P_{i-1}z^{i-1}\right\}$ in equation i, and more generally use equation $j<i$ to get rid of lag $(i-j)$ in $\left\{P_0+P_1 z+\cdots+P_{i-1}z^{i-1}\right\}$ such that only

$$\mathbb{E}_{t-J_1-i+1}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) = \mathbb{E}_{t-J_1-i}\left(P_0\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right)$$

remains.

Finally, we obtain

$$\begin{aligned} \mathbb{E}_{t-J_1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) &= \mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) \\ \mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) &= \mathbb{E}_{t-J_1-2}\left(\zeta_{t-J_1}-u_{t-J_1}\right) \\ &\vdots \\ \mathbb{E}_{t-J_1-i+1}\left(\zeta_{t-J_1}-u_{t-J_1}\right) &= \mathbb{E}_{t-J_1-i}\left(\zeta_{t-J_1}-u_{t-J_1}\right) \\ &\vdots \\ \mathbb{E}_{t-H+1}\left(\left[\zeta_{t-J_1}-u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H}\left(\zeta_{t-J_1}-u_{t-J_1}\right) \end{aligned}$$

[56] This is not a "genericity argument" as argued in [11] page 255.

which is equivalent to

$$
\begin{aligned}
\mathbb{E}_{t-J_1}\left(\zeta_{t-J_1}\right)-\mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1}\left(u_{t-J_1}\right)-\mathbb{E}_{t-J_1-1}\left(u_{t-J_1}\right)\right] \\
\mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}\right)-\mathbb{E}_{t-J_1-2}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1-1}\left(u_{t-J_1}\right)-\mathbb{E}_{t-J_1-2}\left(u_{t-J_1}\right)\right] \\
&\vdots \\
\mathbb{E}_{t-J_1-i+1}\left(\zeta_{t-J_1}\right)-\mathbb{E}_{t-J_1-i}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1-i+1}\left(u_{t-J_1}\right)-\mathbb{E}_{t-J_1-i}\left(u_{t-J_1}\right)\right] \\
&\vdots \\
\mathbb{E}_{t-J_1-H+1}\left(\zeta_{t-J_1}\right)-\mathbb{E}_{t-J_1-H}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1-H+1}\left(u_{t-J_1}\right)-\mathbb{E}_{t-J_1-H}\left(u_{t-J_1}\right)\right]
\end{aligned}
\tag{70}
$$

Step A.4: Using the definition of ζ_t to obtain a system of equation which gives restrictions on the martingale difference sequences $\left(\varepsilon_t^j\right)_{t\in\mathbb{Z}}$. Let us start by writing

$$
\begin{aligned}
\zeta_t &= -\sum_{k=0}^{K}\sum_{j=0}^{H-1}\sum_{h=0}^{j} A_{kh} z^{k+(j-h)}\varepsilon_t^j. \\
&= \left(A_{00}, A_{01}, \ldots, A_{0,H-1}\right)\begin{pmatrix}\varepsilon_t^0\\ \varepsilon_t^1\\ \vdots\\ \varepsilon_t^{H-1}\end{pmatrix} && k=0 \\
&\quad + \begin{pmatrix} I_s & I_s\end{pmatrix}\begin{pmatrix} 0 & A_{00} & A_{01} & \cdots & A_{0,H-2}\\ A_{10} & A_{11} & \cdots & A_{1,H-2} & A_{1,H-1}\end{pmatrix}\begin{pmatrix}\varepsilon_{t-1}^0\\ \varepsilon_{t-1}^1\\ \vdots\\ \varepsilon_{t-1}^{H-2}\\ \varepsilon_{t-1}^{H-1}\end{pmatrix} && k=\begin{pmatrix}0\\1\end{pmatrix} \\
&\quad + \begin{pmatrix} I_s & I_s & I_s\end{pmatrix}\begin{pmatrix} 0 & 0 & A_{00} & A_{01} & \cdots & A_{0,H-3}\\ 0 & A_{10} & A_{11} & \cdots & A_{1,H-3} & A_{1,H-2}\\ A_{20} & A_{21} & \cdots & & A_{2,H-2} & A_{2,H-1}\end{pmatrix}\begin{pmatrix}\varepsilon_{t-2}^0\\ \varepsilon_{t-2}^1\\ \varepsilon_{t-2}^2\\ \vdots\\ \varepsilon_{t-2}^{H-2}\\ \varepsilon_{t-2}^{H-1}\end{pmatrix} && k=\begin{pmatrix}0\\1\\2\end{pmatrix} \\
&\quad \vdots \\
&\quad + \underbrace{\begin{pmatrix} I_s & \cdots & I_s\end{pmatrix}}_{(r+1)\text{ elements}}\left(\begin{array}{ccccc|ccc} 0 & & & & & 0 & A_{00} & A_{01} & \cdots & A_{0,H-J_1-1-r} \\ & & & & & A_{10} & A_{11} & & & \\ & & & & & & & & & \\ & & & \ddots & & & & & & \\ 0 & A_{i-1,0} & A_{i-1,1} & & & & & & & \\ A_{r0} & A_{r1} & A_{r,2} & & & A_{rr} & A_{r,r+1} & \cdots & A_{r,H-J_1-1} \end{array}\right.\\ &\qquad\qquad \left.\begin{array}{ccc} A_{0,H-J_1-r} & \cdots & A_{0,H-1-r} \\ & & \\ & & \\ & & \\ & & \\ A_{r,H-J_1} & \cdots & A_{r,H-1}\end{array}\right)\begin{pmatrix}\varepsilon_{t-r}^0\\ \vdots\\ \varepsilon_{t-r}^{H-J_1-1}\\ \hline \varepsilon_{t-r}^{H-J_1}\\ \vdots\\ \varepsilon_{t-r}^{H-1}\end{pmatrix} && k=\begin{pmatrix}0\\1\\ \vdots\\ r\end{pmatrix} \\
&\quad \vdots \\
&\quad + \underbrace{\begin{pmatrix} I_s & \cdots & I_s\end{pmatrix}}_{(H-J_1)\text{ elements}}\left(\begin{array}{ccccc|ccc} 0 & & & & & 0 & A_{00} & A_{01} & \cdots & A_{0,H-(H-J_1)} \\ & & & & & A_{10} & A_{11} & & & \\ & & & & & & & & & \\ & & & \ddots & & & & & & \\ 0 & A_{H-J_1-2,0} & A_{H-J_1-2,1} & & & & & & & \\ A_{H-J_1-1,0} & A_{H-J_1-1,1} & A_{H-J_1-1,2} & & & A_{H-J_1-1,H-J_1-1} & A_{H-J_1-1,H-J_1} & \cdots & A_{H-J_1-1,H-1} \end{array}\right)\\ &\qquad\qquad \begin{pmatrix}\varepsilon_{t-(H-J_1-1)}^0\\ \vdots\\ \varepsilon_{t-(H-J_1-1)}^{H-J_1-1}\\ \hline \varepsilon_{t-(H-J_1-1)}^{H-J_1}\\ \vdots\\ \varepsilon_{t-(H-J_1-1)}^{H-1}\end{pmatrix} && , \ k=\begin{pmatrix}0\\1\\ \vdots\\ H-J_1-1\end{pmatrix}
\end{aligned}
$$

Now, it is easy to see that the $(H-J_1)$ equations

$$
\begin{aligned}
\mathbb{E}_{t-J_1}\left(\zeta_{t-J_1}\right) - \mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1}\left(u_{t-J_1}\right) - \mathbb{E}_{t-J_1-1}\left(u_{t-J_1}\right)\right] \\
\mathbb{E}_{t-J_1-1}\left(\zeta_{t-J_1}\right) - \mathbb{E}_{t-J_1-2}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1-1}\left(u_{t-J_1}\right) - \mathbb{E}_{t-J_1-2}\left(u_{t-J_1}\right)\right] \\
&\vdots \\
\mathbb{E}_{t-J_1-i+1}\left(\zeta_{t-J_1}\right) - \mathbb{E}_{t-J_1-i}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1-i+1}\left(u_{t-J_1}\right) - \mathbb{E}_{t-J_1-i}\left(u_{t-J_1}\right)\right] \\
&\vdots \\
\mathbb{E}_{t-J_1-H+1}\left(\zeta_{t-J_1}\right) - \mathbb{E}_{t-J_1-H}\left(\zeta_{t-J_1}\right) &= -\left[\mathbb{E}_{t-J_1-H+1}\left(u_{t-J_1}\right) - \mathbb{E}_{t-J_1-H}\left(u_{t-J_1}\right)\right]
\end{aligned}
$$

are equivalent to

$$\left(A_{00}, A_{01}, \ldots, A_{0,H-1}\right)\begin{pmatrix}\varepsilon_t^0\\ \varepsilon_t^1\\ \vdots\\ \varepsilon_t^{H-1}\end{pmatrix} = -\left[\mathbb{E}_t\left(u_t\right) - \mathbb{E}_{t-1}\left(u_t\right)\right]$$

$$z^{-1}\left\{(I_s \quad I_s)\begin{pmatrix}0 & A_{00} & A_{01} & \cdots & A_{0,H-2}\\ A_{10} & A_{11} & \cdots & A_{1,H-2} & A_{1,H-1}\end{pmatrix}\begin{pmatrix}\varepsilon_{t-1}^0\\ \varepsilon_{t-1}^1\\ \vdots\\ \varepsilon_{t-1}^{H-2}\\ \varepsilon_{t-1}^{H-1}\end{pmatrix}\right\} = -z^{-1}\left[\mathbb{E}_{t-1}\left(u_t\right) - \mathbb{E}_{t-2}\left(u_t\right)\right]$$

$$\vdots$$

$$z^{-(i-1)}\left\{\underbrace{(I_s \quad \cdots \quad I_s)}_{i \text{ elements}}\left(\begin{array}{ccc|c|ccc|ccc}0 & & & 0 & A_{00} & A_{01} & \cdots & A_{0,H-J_1-1-(i-1)} & A_{0,H-J_1-(i-1)} & \cdots & A_{0,H-1-(i-1)}\\ & & & A_{10} & A_{11} & & & & & & \\ & & \iddots & & & & & & & & \\ 0 & A_{i-2,0} & A_{i-2,1} & & & & & & & & \\ A_{i-1,0} & A_{i-1,1} & A_{i-1,2} & & A_{i-1,i-1} & A_{i-1,i} & \cdots & A_{i-1,H-J_1-1} & A_{i-1,H-J_1} & \cdots & A_{i-1,H-1}\end{array}\right)\left(\begin{array}{c}\varepsilon_{t-(i-1)}^0\\ \vdots\\ \varepsilon_{t-(i-1)}^{i-2}\\ \hline \varepsilon_{t-(i-1)}^{i-1}\\ \hline \varepsilon_{t-(i-1)}^{i}\\ \vdots\\ \varepsilon_{t-(i-1)}^{H-J_1-1}\\ \hline \varepsilon_{t-(i-1)}^{H-J_1}\\ \vdots\\ \varepsilon_{t-(i-1)}^{H-1}\end{array}\right)\right\} = -z^{-(i-1)}\left[\mathbb{E}_{t-(i-1)}\left(u_t\right) - \mathbb{E}_{t-i}\left(u_t\right)\right]$$

$$\vdots$$

$$z^{-(H-J_1-1)}\underbrace{(I_s \quad \cdots \quad I_s)}_{(H-J_1) \text{ elements}}\left(\begin{array}{ccc|cc|ccc}0 & & & 0 & A_{00} & A_{01} & \cdots & A_{0,H-(H-J_1)}\\ & & & A_{10} & A_{11} & & & \\ & & \iddots & & & & & \\ 0 & A_{H-J_1-2,0} & A_{H-J_1-2,1} & & & & & \\ A_{H-J_1-1,0} & A_{H-J_1-1,1} & A_{H-J_1-1,2} & & A_{H-J_1-1,H-J_1-1} & A_{H-J_1-1,H-J_1} & \cdots & A_{H-J_1-1,H-1}\end{array}\right)\left(\begin{array}{c}\varepsilon_{t-(H-J_1-1)}^0\\ \vdots\\ \varepsilon_{t-(H-J_1-1)}^{H-J_1-1}\\ \hline \varepsilon_{t-(H-J_1-1)}^{H-J_1}\\ \vdots\\ \varepsilon_{t-(H-J_1-1)}^{H-1}\end{array}\right) = -z^{-(H-J_1-1)}\left[\mathbb{E}_{t-H+1}\left(u_t\right) - \mathbb{E}_{t-H}\left(u_t\right)\right].$$

or

$$\left(A_{00}, A_{01}, \ldots, A_{0,H-1}\right)\begin{pmatrix}\varepsilon_t^0\\ \varepsilon_t^1\\ \vdots\\ \varepsilon_t^{H-1}\end{pmatrix} = -\left[\mathbb{E}_t\left(u_t\right) - \mathbb{E}_{t-1}\left(u_t\right)\right]$$

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & A_{00} & A_{01} & \cdots & A_{0,H-2}\\ A_{10} & A_{11} & \cdots & A_{1,H-2} & A_{1,H-1}\end{pmatrix}\begin{pmatrix}\varepsilon_t^0\\ \varepsilon_t^1\\ \vdots\\ \varepsilon_t^{H-2}\\ \varepsilon_t^{H-1}\end{pmatrix} = -\left[\mathbb{E}_t\left(u_{t+1}\right) - \mathbb{E}_{t-1}\left(u_{t+1}\right)\right]$$

$$\vdots$$

$$\underbrace{\begin{pmatrix}I_s & \cdots & I_s\end{pmatrix}}_{i \text{ elements}}\left(\begin{array}{ccc|c|ccc|ccc}
0 & & & 0 & A_{00} & A_{01} & \cdots & A_{0,H-J_1-1-(i-1)} & A_{0,H-J_1-(i-1)} & \cdots & A_{0,H-1-(i-1)}\\
 & & & A_{10} & A_{11} & & & & & & \\
 & & \iddots & & & & & & & & \\
0 & A_{i-2,0} & A_{i-2,1} & & & & & & & & \\
A_{i-1,0} & A_{i-1,1} & A_{i-1,2} & & A_{i-1,i-1} & A_{i-1,i} & \cdots & A_{i-1,H-J_1-1} & A_{i-1,H-J_1} & \cdots & A_{i-1,H-1}
\end{array}\right)\left(\begin{array}{c}\varepsilon_t^0\\ \vdots\\ \varepsilon_t^{i-2}\\ \hline \varepsilon_t^{i-1}\\ \hline \varepsilon_t^{i}\\ \vdots\\ \varepsilon_t^{H-J_1-1}\\ \hline \varepsilon_t^{H-J_1}\\ \vdots\\ \varepsilon_t^{H-1}\end{array}\right) = -\left[\mathbb{E}_t\left(u_{t+(i-1)}\right) - \mathbb{E}_{t-1}\left(u_{t+(i-1)}\right)\right] \tag{71}$$

$$\vdots$$

$$\underbrace{\begin{pmatrix}I_s & \cdots & I_s\end{pmatrix}}_{(H-J_1) \text{ elements}}\left(\begin{array}{ccc|cc|ccc}
0 & & & 0 & A_{00} & A_{01} & \cdots & A_{0,H-(H-J_1)}\\
 & & & A_{10} & A_{11} & & & \\
 & & \iddots & & & & & \\
0 & A_{H-J_1-2,0} & A_{H-J_1-2,1} & & & & & \\
A_{H-J_1-1,0} & A_{H-J_1-1,1} & A_{H-J_1-1,2} & & A_{H-J_1-1,H-J_1-1} & A_{H-J_1-1,H-J_1} & \cdots & A_{H-J_1-1,H-1}
\end{array}\right)\left(\begin{array}{c}\varepsilon_t^0\\ \vdots\\ \varepsilon_t^{H-J_1-1}\\ \hline \varepsilon_t^{H-J_1}\\ \vdots\\ \varepsilon_t^{H-1}\end{array}\right) = -\left[\mathbb{E}_t\left(u_{t+(H-J_1-1)}\right) - \mathbb{E}_{t-1}\left(u_{t+(H-J_1-1)}\right)\right].$$

Step A.5: Use the stationary structure of the exogenous process to conclude on the number of restrictions. We write the system above as

$$\left(\begin{array}{c|c} M & N \end{array}\right)\left(\begin{array}{c}\varepsilon_t^0\\ \vdots\\ \varepsilon_t^{H-J_1-1}\\ \hline \varepsilon_t^{H-J_1}\\ \vdots\\ \varepsilon_t^{H-1}\end{array}\right) = \left(\begin{array}{ccc|ccc}
m_{11} & & m_{1,H-J_1} & n_{1,1} & & n_{1,J_1}\\
 & m_{ij} & & & n_{ij} & \\
m_{H-J_1,0} & & m_{H-J_1,H-J_1} & n_{H-J_1,1} & & n_{H-J_1,J_1}
\end{array}\right)\left(\begin{array}{c}\varepsilon_t^0\\ \vdots\\ \varepsilon_t^{H-J_1-1}\\ \hline \varepsilon_t^{H-J_1}\\ \vdots\\ \varepsilon_t^{H-1}\end{array}\right) = \begin{pmatrix}-\left[\mathbb{E}_t\left(u_t\right) - \mathbb{E}_{t-1}\left(u_t\right)\right]\\ \vdots\\ -\left[\mathbb{E}_t\left(u_{t+(H-J_1-1)}\right) - \mathbb{E}_{t-1}\left(u_{t+(H-J_1-1)}\right)\right]\end{pmatrix} \tag{72}$$

where M and N have $(H-J_1)$ blocks of rows, and $(H-J_1)$ and J_1 blocks of columns respectively. The entries $m_{ij},\ n_{ij}$ are $(s \times s)$-dimensional matrices.

The diagonal elements of M are of the form

$$m_{ii} = \sum_{k=0}^{i-1} A_{kk},$$

the lower triangular elements, i.e. $i > j$, are of the form[57]

$$m_{ij} = \sum_{l=0}^{j-1} A_{i-1-l,j-1-l}$$

and the upper triangular elements, i.e. $j > i$, are of the form[58]

$$m_{ij} = \sum_{l=0}^{i-1} A_{i-1-l,j-1-l}.$$

The elements of N are of the form

$$n_{ij} = \sum_{l=0}^{i-1} A_{i-1-l,H-J_1+(j-1)-l}.$$

Using the assumption on the parameter space that there exists a point in the parameter space for which $(M \,|\, N)$ in equation (72) has full row rank (w.l.o.g. we assume that M is invertible) and the assumptions on the structure of u_t and y_t, we obtain

$$\begin{pmatrix} k_0 \\ \vdots \\ k_{H-J_1-1} \end{pmatrix} = M^{-1}N \begin{pmatrix} k_{H-J_1} \\ \vdots \\ k_{H-1} \end{pmatrix} - M^{-1} \begin{pmatrix} w_0 \\ \vdots \\ w_{H-J_1-1} \end{pmatrix}.$$

Thus, for given $\begin{pmatrix} w_0 \\ \vdots \\ w_{H-J_1-1} \end{pmatrix}$ and freely chosen $\begin{pmatrix} k_{H-J_1} \\ \vdots \\ k_{H-1} \end{pmatrix}$, containing $J_1 sq$ parameters, we obtain $\begin{pmatrix} k_0 \\ \vdots \\ k_{H-J_1-1} \end{pmatrix}$, i.e. $(H - J_1)\, sq$ restricted parameters.

Step B: The Smith-form in the case where $G_1 = 1$. If $\det(\pi(z))$ has one zero at zero, the diagonal matrix $\alpha(z)$ has as elements only ones except for the last entry, which is equal to z. Thus, the last equation for every $i \in \{0, \dots, H-1\}$ in

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \dots, H-1\}$$

is of the form

$$\begin{aligned} &\mathbb{E}_{t-i}\left(z^{-1}\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(z^{-1}\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \dots, H-1\} \\ \iff &\mathbb{E}_{t-i}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-(J_1-1)} - u_{t-(J_1-1)}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-(J_1-1)} - u_{t-(J_1-1)}\right]\right), \quad i \in \{0, \dots, H-1\} \end{aligned}$$

Whereas these constraints were trivially satisfied for $i \in \{0, \dots, J_1 - 1\}$, i.e. the first J_1 equations in the case where $G_1 = 0$, here they are only satisfied for

[57] Start summing from the bottom of the matrix in equation (71)
[58] Again, start summing from the bottom of the matrix in equation (71)

$i \in \{0, \ldots, J_1 - 2\}$. Indeed, it is easy to see that now only the first $(J_1 - 1)$ equations in

$$\left.\begin{aligned}
\mathbb{E}_t\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \\
&\vdots \\
\mathbb{E}_{t-J_1+2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-(J_1-1)}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right)
\end{aligned}\right\} (J_1 - 1) \text{ equation systems}$$

$$\left.\begin{aligned}
\mathbb{E}_{t-J_1+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \\
\mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \\
&\vdots \\
\mathbb{E}_{t-H+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right)
\end{aligned}\right\} (H - J_1 + 1) \text{ equation systems}$$

are trivially satisfied (such that $(H - J_1 + 1)$ equations remain), because ζ_{t-J_1+1} and u_{t-J_1+1} are contained in the linear span of $\{u_{t-J_1+1}, u_{t-J_1}, \ldots\}$ and supersets thereof. Thus, there is one additional constraint if $G_1 = 1$.

Shifting the last $(H - J_1 + 1)$equations, i.e. for $i \in \{J_1 - 1, \ldots, H - 1\}$, by one period, we obtain

$$\left.\begin{aligned}
\mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
\mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-J_1-2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
&\vdots \\
\mathbb{E}_{t-H+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)
\end{aligned}\right\} H \text{ equations}$$

$$\mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \text{ One equation}$$

Thus, the system (70) obtained in the case $G_1 = 0$ is left unchanged in the case $G_1 = 1$; however, there is an additional constraint, i.e.

$$\mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)$$

which does not appear in the case $G_1 = 0$.

Note that we may use, in analogy to the case $G_1 = 0$, see equation (69), the j-th, $j < i$, equation to get rid of lag $(i - j)$ in $\left\{P_0 + P_1 z + \cdots + P_{i-1} z^{i-1}\right\}$ such that only

$$\begin{aligned}
&\mathbb{E}_{t-J_1-i+1}\left(P_0\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-J_1-i}\left(P_0\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
\iff &\mathbb{E}_{t-J_1-i+1}\left(\zeta_{t-J_1} - u_{t-J_1}\right) = \mathbb{E}_{t-J_1-i}\left(\zeta_{t-J_1} - u_{t-J_1}\right) \quad i \in \{1, \ldots, H - J_1\}
\end{aligned}$$

remains; however, we have an additional equation, i.e.

$$\mathbb{E}_{t-H}\left(\left(P_0\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-H-1}\left(\left(P_0\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right).$$

Step C.1: $G_1 = 2$, $\alpha_s(z) = z^2$**:** In analogy to the case $G_1 = 1,\ \alpha_s(z) = z$, we see that the last equation for every $i \in \{0, \dots, H-1\}$ in

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1}P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \dots, H-1\}$$

is of the form

$$\begin{aligned}
&\mathbb{E}_{t-i}\left(z^{-2}\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(z^{-2}\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \dots, H-1\} \\
\iff\ &\mathbb{E}_{t-i}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-(J_1-2)} - u_{t-(J_1-2)}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right), \quad i \in \{0, \dots, H-1\}
\end{aligned}$$

Again, these constraints (for the last equation) were trivially satisfied for $i \in \{0, \dots, J_1 - 1\}$. Here, only the first $(J_1 - 2)$ constraints, i.e. for $i \in \{0, \dots, J_1 - 3\}$, are trivially satisfied. Indeed, it is easy to see that now only the first $(J_1 - 2)$ equations in

$$\left.\begin{aligned}
\mathbb{E}_t\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) &= \mathbb{E}_{t-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) \\
&\vdots \\
\mathbb{E}_{t-J_1+3}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) &= \mathbb{E}_{t-(J_1-2)}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right)
\end{aligned}\right\} (J_1 - 2) \text{ equations}$$

$$\left.\begin{aligned}
\mathbb{E}_{t-J_1+2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) &= \mathbb{E}_{t-J_1+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) \\
\mathbb{E}_{t-J_1+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) &= \mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) \\
&\vdots \\
\mathbb{E}_{t-H+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right) &= \mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+2} - u_{t-J_1+2}\right]\right)
\end{aligned}\right\} (H - J_1 + 2) \text{ equations}$$

are trivially satisfied (such that $(H - J_1 + 2)$ equations remain), because ζ_{t-J_1+2} and u_{t-J_1+2} are contained in the linear span of $\{u_{t-J_1+2}, u_{t-J_1+1}, \dots\}$ and supersets thereof. Thus, there are two additional constraints if $G_1 = 2$ and $\alpha_s(z) = z^2$.

Shifting the last $(H - J_1 + 2)$equations, i.e. for $i \in \{J_1 - 2, \dots, H-1\}$, two periods backwards, we obtain

$$\left.\begin{aligned}
\mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
\mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-J_1-2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
&\vdots \\
\mathbb{E}_{t-H+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)
\end{aligned}\right\} H \text{ equations}$$

$$\left.\begin{aligned}
\mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
\mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)
\end{aligned}\right\} 2 \text{ equations.}$$

Thus, the system (70) obtained in the case $G_1 = 0$ is left unchanged in the case $G_1 = 2,\ \alpha_s(z) = z^2$; however, there are two additional constraints, i.e.

$$\begin{aligned}
\mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
\mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)
\end{aligned}$$

which do not appear in the case $G_1 = 0$.

Again, we may use, in analogy to the case $G_1 = 0$, see equation(69), the j-th, $j < i$, equation to get rid of lag $(i-j)$ in $\left\{P_0 + P_1 z + \cdots + P_{i-1} z^{i-1}\right\}$ such that only

$$\begin{aligned} &\mathbb{E}_{t-J_1-i+1}\left(P_0\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-J_1-i}\left(P_0\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\ \iff &\mathbb{E}_{t-J_1-i+1}\left(\zeta_{t-J_1} - u_{t-J_1}\right) = \mathbb{E}_{t-J_1-i}\left(\zeta_{t-J_1} - u_{t-J_1}\right) \quad i \in \{1, \ldots, H-J_1\} \end{aligned}$$

remains. For $i = H - J_1 + 1$ and $i = H - J_1 + 2$, we obtain

$$\begin{aligned} \mathbb{E}_{t-H}\left(\left(P_0\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-1}\left(\left(P_0\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\ \mathbb{E}_{t-H-1}\left(\left(P_0\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-2}\left(\left(P_0\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \end{aligned}$$

as additional constraints.

Step C.2: $G_1 = 2$**,** $\alpha_s(z) = \alpha_{s-1}(z) = z$**:** Similarly to the cases B and C.1, the last two equations for every $i \in \{0, \ldots, H-1\}$ in

$$\mathbb{E}_{t-i}\left(\alpha(z)^{-1} P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\alpha(z)^{-1} P(z)^{-1}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \ldots, H-1\}$$

are of the form

$$\begin{aligned} &\mathbb{E}_{t-i}\left(z^{-1}\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(z^{-1}\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \ldots, H-1\} \\ \iff &\mathbb{E}_{t-i}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-(J_1-1)} - u_{t-(J_1-1)}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right), \quad i \in \{0, \ldots, H-1\} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}_{t-i}\left(z^{-1}\left(P(z)^{-1}\right)_{[s-1,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) = \mathbb{E}_{t-(i+1)}\left(z^{-1}\left(P(z)^{-1}\right)_{[s-1,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right), \quad i \in \{0, \ldots, H-1\} \\ \iff &\mathbb{E}_{t-i}\left(\left(P(z)^{-1}\right)_{[s-1,\bullet]}\left[\zeta_{t-(J_1-1)} - u_{t-(J_1-1)}\right]\right) = \mathbb{E}_{t-(i+1)}\left(\left(P(z)^{-1}\right)_{[s-1,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right), \quad i \in \{0, \ldots, H-1\} \end{aligned}$$

where again the subscripts $[s,\bullet]$ and $[s-1,\bullet]$ in the equations above denote the last and second to last row of $P(z)^{-1}$.

These constraints (for the last and second to last equation) were trivially satisfied for $i \in \{0, \ldots, J_1 - 1\}$. Here, only the first $(J_1 - 1)$ constraints for the last and second to last component, i.e. for $i \in \{0, \ldots, J_1 - 2\}$, are trivially satisfied. Indeed, it is easy to see that now only the first $(J_1 - 1)$ equations in

$$\left.\begin{aligned} \mathbb{E}_t\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \\ &\vdots \\ \mathbb{E}_{t-J_1+2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-(J_1-1)}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \end{aligned}\right\} (J_1-1) \text{ equations}$$

$$\left.\begin{aligned} \mathbb{E}_{t-J_1+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \\ \mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \\ &\vdots \\ \mathbb{E}_{t-H+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) &= \mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1+1} - u_{t-J_1+1}\right]\right) \end{aligned}\right\} (H-J_1+1) \text{ equations}$$

are trivially satisfied (such that $(H - J_1 + 1)$ equations remain). In this case, the same holds true if we replace $\left(P(z)^{-1}\right)_{[s,\bullet]}$ with $\left(P(z)^{-1}\right)_{[s-1,\bullet]}$, i.e. there are also two additional constraints if $G_1 = 2$ and $\alpha_s(z) = \alpha_{s-1}(z) = z$.

Shifting the last $(H - J_1 + 1)$equations, i.e. for $i \in \{J_1 - 1, \dots, H - 1\}$, one periods backwards, we obtain

$$\left.\begin{array}{rl}
\mathbb{E}_{t-J_1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) & = \mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
\mathbb{E}_{t-J_1-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) & = \mathbb{E}_{t-J_1-2}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
& \vdots \\
\mathbb{E}_{t-H+1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) & = \mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)
\end{array}\right\} H \text{ equations}$$

$$\left.\begin{array}{rl}
\mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) & = \mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right)
\end{array}\right\} 2 \text{ equations.}$$

and again the same holds if we replace $\left(P(z)^{-1}\right)_{[s,\bullet]}$ with $\left(P(z)^{-1}\right)_{[s-1,\bullet]}$.

Thus, the system (70) obtained in the case $G_1 = 0$ is left unchanged in the case $G_1 = 2,\ \alpha_s(z) = \alpha_{s-1}(z) = z$; however, there are two additional constraints, i.e.

$$\begin{aligned}
\mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s-1,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,-1\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) \\
\mathbb{E}_{t-H}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right) &= \mathbb{E}_{t-H-1}\left(\left(P(z)^{-1}\right)_{[s,\bullet]}\left[\zeta_{t-J_1} - u_{t-J_1}\right]\right).
\end{aligned}$$

which do not appear in the case $G_1 = 0$.

Other cases for $G_1 > 2$ are derived in the same way. □

Corollary 46. *Under the assumptions in Theorem 45, except that there are only zero restrictions on the entries of the parameter matrices and that $A_{00} = -I_s$ holds, it follows that the rational expectations model has a reduced form involving generically $(J_1 s - G_1)\, q$ free parameters, where*

- *s is the number of equations of the model,*
- *J_1 is such that $t + J_1$ is the largest time index of expected endogenous variables appearing in the model, and*
- *G_1 is the number of zero roots of $\det(\pi(z))$.*

Corollary 47. *Assume that in the rational expectations model (1) only zero restrictions are imposed and that additionally $A_{00} = -I_s$ and $rk(f_u(\lambda)) = s$ holds, i.e.*

$$\begin{pmatrix} I_s & I_s z & \cdots & I_s z^k & \cdots & I_s z^K \end{pmatrix} \begin{pmatrix} -I_s & A_{01} & \cdots & A_{0h} & \cdots & A_{0H} \\ A_{10} & \ddots & & & & \\ \vdots & & & & & \\ A_{k0} & & & A_{kh} & & A_{kH} \\ \vdots & & & & & \vdots \\ A_{K0} & A_{K1} & \cdots & & & A_{KH} \end{pmatrix} \begin{pmatrix} \mathbb{E}_t(y_t) \\ \mathbb{E}_t(y_{t+1}) \\ \vdots \\ \mathbb{E}_t(y_{t+h}) \\ \vdots \\ \mathbb{E}_t(y_{t+H}) \end{pmatrix} = -u_t.$$

It has a reduced form involving generically $(J_1 s - G_1)$ arbitrary martingale differences, where

- *s is the number of equations of the model,*
- *J_1 is such that $t + J_1$ is the largest time index of expected endogenous variables appearing in the model, and*
- *G_1 is the number of zero roots of $\det(\pi(z))$.*

Thus, the solution set has dimension $(J_1 s - G_1)\, s$.

4.5 Causal and non-explosive solutions

Up to this point, there is no assumption as to whether a process for which the rational expectations equation (1) holds for every $t \in \mathbb{Z}$ also has to be contained in $H_u(t)$ at time t or as to whether it has to satisfy a non-explosiveness condition. Imposing more general non-explosiveness conditions as in [34] (and as general as in [37]) and imposing that $y_t \in H_u(t)$, $t \in \mathbb{Z}$, for a process $(y_t)_{t\in\mathbb{Z}}$ for which the rational expectations equation (1) holds for every $t \in \mathbb{Z}$ is straightforward in this framework. In the same way, it is obvious how causality can be imposed.

First, we impose causality on the solutions of the recursive equation (61) (in which redundant martingale difference sequences have been replaced) by only considering solutions for which the determinant of $\pi(z)$ is developed in terms of non-negative powers of the backward shift, i.e.

$$y_t = \det\left(\pi(z)\right)^{-1} adj(\pi(z))g(\varepsilon),$$

where $g(\varepsilon)$ denotes a polynomial matrix depending on present and past values of the innovations of the exogenous process. Second, the non-explosiveness conditions, which are given in the form of an $(r \times s)$-dimensional, $r \leq s$, matrix G of full (row) rank, are taken into account by requiring that Gy_t does not explode faster than a given rate of growth $\xi > 1$. If it is possible to cancel roots λ of $\det\left(\pi(z)\right)$ (by adjusting free parameters) for which $|\lambda|^{-1} > \xi$ a causal, non-explosive solution exists. This solution is unique if there are no remaining free parameters.

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