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Resilience of the Interbank Network to Shocks and Optimal Bail-Out Strategy: Advantages of "Tiered" Banking Systems

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Abstract

Systemic risk and the scale of systemic breakdown in the banking system are the key concern for central banks charged with safeguarding overall financial stability. This paper focuses on the risk and potential impact of system-wide defaults in the frequently observed "tiered" banking system, where relatively few first-tier head institutions are connected with second-tier "peripheral" banks and are also connected with each other, while the peripheral banks are almost exclusively connected with the head banks. The banking network is constructed from a number of banks which are linked by interbank exposures with a certain predefined probability. In this framework, the tiered structure is represented either by a network with negative correlation in connectivity of neighboring banks, or alternatively, by a network with a scale-free distribution of connectivity across banks. The main finding of the paper highlights the advantages of tiering within the banking system in terms of both the resilience of the banking network to systemic shocks and the extent of necessary government intervention should a crisis evolve. Specifically, the tiered network structure, showing negative correlations in bank connectivity, is found to be less prone to systemic breakdown than other structures, showing either positive or zero correlations. Moreover, in the scale-free tiered system, the resilience of the system to shocks increases as the *level of tiering* grows. Also, the *targeted* bail-out policy of the government aimed at rescuing the most connected failing banks in the first place, is expected to be more effective and induce lower costs in a tiered system with high level of tiering.

JEL Classification: C63, D85, G01, G21

Keywords: Banking crisis, contagion, default, bail-out, random network, assortative mixing, scale-free distribution

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1 Introduction

One of the major concerns in recent policy debates over financial stability is how ”tiering” in the banking system may impact on so-called systemic risk, the large-scale breakdown of financial intermediation due to the domino effect of insolvency.[1] The tiered banking system is commonly defined as an organization of lending-borrowing relations/linkages between banks, where relatively few first-tier or ”head” institutions have a large number of interbank linkages, whereas many second-tier or ”peripheral” banks have only few links. First-tier banks are connected to second-tier banks and are also connected with each other, whereas second-tier banks are almost exclusively connected to first-tier banks. Interbank linkages may act as a device for co-insurance against uncertain liquidity shocks (Bhattachrya and Gale (1987)) and/or improve market discipline by providing incentives for peer-monitoring (Flannery (1996) and Rochet and Tirole (1996)) but can also serve as a channel through which problems in one bank spread to another.

Tiered banking systems are found in a range of countries but the empirical evidence of contagion risk in these systems is mixed. In their 2003 Financial System Stability assessment of the United Kingdom, the International Monetary Fund (IMF) highlights the potential contagion risk arising from the highly tiered structure of the U.K. large-value payment systems. However, several subsequent studies including Wells (2004), Harrison et al. (2005), Lasaosa and Tudela (2008) report relatively limited scope for contagion among U.K. banks. Boss et al. (2004b) and Degryse and Nguyen (2005) find that tiered banking systems in Austria and Belgium are stable and systemic crises are unlikely. By contrast, Upper and Worms (2004) suggests that in the structurally similar banking system in Germany, ”the effects of the breakdown of a single bank could potentially be very strong” (p. 847) and system-wide bank failures are possible. In addition, somewhat differently from other studies, Elsinger et al. (2006) and Mistrulli (2007) find that while contagious failures in tiered Austrian and Italian banking systems are relatively rare, large parts of the system are affected in the worst-case scenarios.

The controversy in the empirical literature leaves the question of benefits and risks of tiering open for further investigation. In this paper, I aim to shed more light on this issue by proposing an analytical and simulations-based approach. I develop a simple theoretical model to study how structural characteristics of a tiered banking system may affect its susceptibility to systemic breakdown and the scope of the breakdown, particularly in comparison with other types of systems.[2] This enables

[1]The domino effect of insolvency occurs when the non-repayment of interbank obligations by the failing bank jeopardizes the ability of its creditor banks to meet their obligations to interbank creditors.

[2]In fact, the susceptibility to systemic breakdown of every system is only evaluated in *relative* terms. This allows *comparative* assessment of risk across various types of banking systems.

an analysis of the scale of bank bail-outs that might be needed to sustain the stability of tiered banking system and allows comparing the costs of bail-outs across systems.

The banking system is modelled with a *random* network where nodes represent banks and links are interbank exposures that connect any two banks with a certain predefined probability. Links in the network are directed, reflecting the fact that interbank exposures comprise assets as well as liabilities. An important assumption about the structure of links is that with probability 1 the number of incoming and outgoing links of each bank is the same. Furthermore, interbank assets of a bank are assumed to be evenly distributed over its incoming links and the total amount of these assets is set equal across banks.[3] Admittedly, this is a restrictive framework. However, it allows deriving some results of the paper analytically and brings attention to effects of structural features of banking systems on contagious defaults in otherwise symmetric setting.

I examine the impact of tiering in the banking system on the stability of the system by studying the consequences of negative correlation in connectivity (known as *degree*) between neighboring banks/nodes and of the highly right-skewed distribution of connectivity across banks/nodes, such as for example, in the case of a power-law, or scale-free, distribution.

First, I model a tiered banking system with a network displaying negative degree correlations (a *disassortative* network),[4] and compare this network with other types of structures, showing either positive degree correlations (an *assortative* network) or no correlations (a *neutral* network). For these three types of banking networks, I employ a numerical analysis to investigate (i) the resilience of the networks to systemic failure, (ii) the scale of the failure if systemic crisis occurs, and (iii) the optimal bail-out strategy of the government or of the central bank to guarantee the financial stability of the system at minimal costs.[5] In studying the optimal bail-out strategy of the government, I restrict attention to a common practice of "targeted" bail-outs, that is, the policy to first rescue the most connected failing banks. While the cost of each bail-out is not modelled explicitly, the optimal bail-out strategy minimizes the *aggregate* cost of bail-outs since by construction, it minimizes the total number of targeted bank rescues.

I find that both, the risk of systemic crisis and the scope of the crisis in the banking system are minimized when the network is disassortative/tiered. Intuitively, in a disassortative banking network,

[3]The assumptions of even distribution of interbank assets over incoming links and identical interbank asset positions of all banks in the system are the same as in Gai and Kapadia (2010).

[4]In the next section, it is argued that even with negative degree correlation between neighboring nodes, first-tier highly connected banks are still likely to be connected with each other, while the probability of connectedness between second-tier low-degree banks is very small.

[5]The recent events triggered by the sub-prime crisis of August 2007 highlighted the criticality of these questions. For example, recent rescue of some institutions, such as American International Group (AIG), remains a highly disputable issue. As a rule, the main argument of policymakers in favour of these rescues is that many yet unaffected banks (across the national or international financial system) might be exposed to the defaulting institutions. But in fact, no rigorous assessment is suggested of how far contagion could have spread had AIG been allowed to fail.

high-degree banks are broadly distributed over the network and therefore, presumably form links on many paths between other banks. As a result, with high probability an initial shock hitting a random bank reaches a high-degree bank in a small number of "steps", gets absorbed by that bank and does not spread any further. This implies that disassortative networks are relatively resilient to shocks. For the same reason, targeted bail-outs are less costly in a disassortative network: while many bank rescues may be required in assortative and neutral banking networks, in a disassortative network, bail-outs may not be needed at all.

Further, I consider a tiered banking system as represented by the *scale-free network* and study both analytically and numerically the impact of tiering in such a system on its susceptibility to shocks. I focus on the effects of variation in the *level of tiering* in scale-free networks, where the level of tiering is represented by the inverse of a parameter of the scale-free distribution.[6] Some empirical studies find close parallels to scale-free networks in real-world tiered banking systems. For example, Boss et al. (2004) confirms that the Austrian interbank market has a scale-free structure. An advantage of using the scale-free distribution to study tiered network structures is its analytical tractability. In the simplified framework of absent degree correlations, the scale-free distribution allows obtaining the exact closed-form solution for the threshold at which systemic failure can occur and government intervention is needed.

I find that the resilience of scale-free banking networks to systemic breakdown is *increasing* and the maximal expected number of required bail-outs is *decreasing* in the level of tiering. When the system is more tiered, links emanating from and arriving at highly-connected banks span a larger part of the banking system. This implies that initial shocks at any part of the system reach highly-connected banks "quickly" and with large probability subside at these banks. Therefore, the resilience of such a system to global breakdown is higher and the expected number of necessary targeted bail-outs is lower than in a system with lower tiering.

Strong resilience to shocks of a highly tiered system is confirmed by a simulation exercise performed for the Austrian banking network where the exponent parameter is equal to 2.01 (Boss et al. (2004)). In accordance with Boss et al. (2004, 2004b), I find that the large-scale breakdown of Austrian banking system is very unlikely.

Thus, the findings of this paper demonstrate the advantages of tiering in the banking network. This highlights some specific regulatory issues. For example, with regard to stability-improvement of the interbank market, the results suggest that regulation should promote the tiered structure of

[6]The parameter of the scale-free degree distribution governs the rate at which probability decays with connectivity. Therefore, for smaller values of this parameter, the fraction of highly-connected banks in the network is larger and so is the probability that poorly-connected banks are linked with highly-connected banks rather than with each other.

interbank relations since this can reduce systemic risk. Furthermore, when designing the optimal bail-out policy, regulators should consider specific features of the degree distribution underlying the pattern of connections in the banking network. This may help to determine the limits of government regulation when confronted with systemic shock, so as to guarantee the global stability of the system at minimal cost.

To solve the model, I apply techniques from the literature on complex networks (Strogatz (2001), Newman et al. (2001), Vega-Redondo (2007)) to a financial system setting. I use these techniques to model contagion stemming from unexpected shock to a single institution in complex banking networks. The banking system and transmission of shocks are modelled as in Gai and Kapadia (2010). However, I focus on the impact of tiering in the banking system on the risk and potential scope of defaults, while Gai and Kapadia (2010) examine the susceptibility to shocks of a *uniform* (Poisson) banking network, where each possible directed link is present with *independent* and *identical* probability. I show that when the degree distribution is not highly skewed, disregarding degree correlations between banks may substantially change the predictions for the risk and scale of systemic defaults in the tiered banking network.

Within the complex network literature, I exploit results from the standard epidemic/ information diffusion and percolation literature. Specifically, the framework builds on the generating-function techniques used in the SI(R) models[7] of Watts (2002), Callaway et al. (2000), Cohen et al. (2001), and Newman (2002b), and in the structured network model of Newman (2002a). This literature describes the behavior of connected groups of nodes in a random network, with or without internode degree correlations, and characterizes *phase transition*, the point at which extensive contagion outbreaks occur, as well as the size of a susceptible cluster beyond that point.

Unlike the generic, undirected network models of Watts (2002) and Newman (2002a), the model in this paper provides an explicit characterization of balance sheets, which specifies the direction of links connecting banks in the financial system. The distinction between incoming links (claims) and outgoing links (obligations) implies that in contrast to epidemiological and percolation models, greater connectivity does not only create more channels through which contagion can spread but also improves counteracting risk-sharing benefits, since exposures are diversified across a larger set of banks. Moreover, whereas in most epidemiological models the susceptibility of a node to contagion is determined solely by the total number of its infected neighbors, in the present setup the *share* of neighbors that default determines the contagion risk.

[7]SI(R) (*susceptible-infected (-recovered)*) models are canonical epidemiological models, where the life history of each node passes from being susceptible (S), to becoming infected (I) and, in the SIR setting, to finally being recovered (the definition based on Vega-Redondo (2007, p.75)). The primary theoretical approach used in the SI(R) context is the generating-function analysis.

An alternative approach to studying contagion in financial networks is represented by the seminal contribution of Allen and Gale (2000) and by more recent literature on endogenous network formation (Leitner (2005), Castiglionesi and Navarro (2007)). These studies are based on small networks with rigid structures and provide key insights into the mechanism through which the pattern of interconnectedness between banks affects the spread of defaults. However, being restricted to the analysis of simple networks, this literature is limited in addressing the case of real-world contagion in large and complex banking systems. For the same reason the representation of tiered systems in this approach is confined to a very limited set of cases.[8] In contrast, the complex network approach allows for a wide range of network structures with arbitrarily large number of banks. Furthermore, given the scarce information that policymakers have about the true interlinkages involved, the connections between banks are, perhaps, best captured by a random network.

At the same time a natural critique of the approach in this paper is that it assumes that interbank connections are formed randomly and exogenously and are static in nature. This leads to modelling contagion in a relatively mechanical manner, keeping balance sheets and the structure of interbank linkages fixed as default propagates through the system. Yet, as suggested by Gai and Kapadia (2008), this framework still yields a useful and realistic benchmark for the analysis. Arguably, in crises contagion spreads rapidly through the system and banks have little time to change their behavior before they are affected. Moreover, banks have no choice over whether to default, which precludes strategic behavior on networks of the type assumed in Morris (2000), Jackson and Yariv (2007) and Galeotti and Goyal (2007).

The remainder of the paper is organized as follows. Section 2 introduces the model and describes the structure of the banking network, the transmission process for contagion, and the generating-function approach to measure the extent of contagion and phase transition. Section 3 uses the introduced techniques to study the effects of a failure of an individual institution on the risk and extent of system-wide contagion in tiered and other banking systems. Section 4 summarizes the findings and suggests potential policy implications.

[8]For example, one representation of a tiered structure in the deterministic network setting is the "money center bank" structure proposed by Freixas et al. (2000). They consider a model with three banks, where the banks on the periphery are linked to the bank at the center but not to each other. In this framework, they find that for some parameter values the failure of a bank on the periphery will not trigger the breakdown of other institutions while the failure of the money center bank would.

2 The model

2.1 Interbank network

Consider a banking system represented by a network where each node is a bank and each link represents a directional lending relationship between two banks. Two crucial properties of this network are that it is (i) *directed* and (ii) *random.* The first property implies that links in the network are directed, so that every node has two degrees: an *in-degree*, the number of links that point into the node, and an *out-degree*, the number of links that point out. Incoming links to a bank reflect the interbank assets of the bank, that is, funds owed to the bank by a counterparty. In contrast, outgoing links from a bank indicate its interbank liabilities. Randomness of the network means that any two banks are linked with a certain predefined probability, so that the connectivity, or degree, of each bank is not deterministic but random. In case of a directed random network, the connectivity of nodes is described by the *joint probability distribution of in- and out-degrees.* I assume a specific form of the joint distribution such that with probability 1 in- and out-degrees of each node are the same:

Assumption 1 $\quad p_{jk} = p(k)\delta_{jk}$

where j denotes in-degree of a bank, k denotes out-degree, and δ_{jk} is the Kronecker delta, i.e. $\delta_{jk} = 1$ if $j = k$ but 0 otherwise. By definition, $p(k)$ is the probability that a randomly chosen node has in- and out-degrees each equal to k. The assumption of almost surely equal number of lenders and borrowers of each bank in the interbank market is imposed for two reasons. First, it allows for the notion of highly-connected and "peripheral" banks in an intuitive manner: both in- and out-degree of highly-connected banks are large and both in- and out-degree of peripheral banks are low.[9] Secondly, it simplifies the discussion and the derivation of the results thereafter.[10]

Also in this framework, the network of interbank linkages is assumed to be formed exogenously: issues related to endogenous network formation, optimal network structures and network efficiency are left to one side.

2.2 Tiered network structure

A tiered banking network is defined as a network in which relatively few high-degree (first-tier) banks\nodes are connected with low-degree (second-tier) banks\nodes and also connected with each other, whereas low-degree banks are almost exclusively connected with high-degree banks. This property of tiering can be modelled by assuming that the banking network shows *disassortative mixing* on

[9] It also seems likely (although this question could be for future research) that in the real-world interbank market, banks with a large number of outgoing links also have a large number of incoming ones and peripheral banks have a low number of both incoming and outgoing links.

[10] For example, it makes the definition of *internode* degree correlation and joint probability distribution of the neighboring nodes more intuitive.

its degrees, that is, degrees of neighboring banks in the network are negatively correlated. It should be noticed that even with negative degree correlations, first-tier highly connected banks are still likely to be linked with each other. This is explained by the fact that in *any* random network, the probability of a given source node being connected to a target node is an increasing function of the degree of the target node. Disassortativity implies that the probability of connectedness between second-tier low-degree banks is decreased and that the probability of connectedness between low-degree and high-degree banks is increased relative to the situation when degree correlations between neighboring nodes are either absent or positive. In fact, if the network displays zero or positive internode degree correlations, low-degree banks are likely to be connected with each other, especially if the proportion of these low-degree banks in the network is sufficiently large.

Alternatively, as suggested by Boss et al. (2004) and Nier et al. (2007), the same property of tiering in banking networks can be captured by the assumption that nodes' degree distribution is highly right-skewed. One example of such distribution is a power-law, or scale-free, distribution. In *scale-free networks*, the vast majority of nodes have small degree and nodes with high degree, although relatively few, display connectivity that greatly exceeds the average. As a result, high-degree nodes attach mainly to low-degree ones and those low-degree nodes rarely have links to other nodes, including the nodes of low-degree.

2.3 Bank's solvency condition

An individual bank's assets consist of external assets (investors' borrowing), denoted by A_i^E, and interbank assets (other banks' borrowing), denoted by A_i^{IB}. As in Gai and Kapadia (2009), the total interbank asset position of each bank is assumed to be evenly distributed over its incoming links and independent of the number of links the bank has[11]:

Assumption 2 $\quad A_i^{IB} = j_i w_i, \quad$ and $\quad A_i^{IB} = A_k^{IB} = A^{IB} \quad \forall i \neq k$

where j_i is the number of incoming links of bank i and w_i is the interbank assets held by i against any of its debtor banks. These assumptions provide a useful benchmark for studying the effects of tiering in the banking network on the potential for contagion spread. By setting the size of interbank assets equal for all banks, they isolate the effects of risk sharing and risk spreading as predefined by the structure of interbank links. In particular, they allow me to show that widespread contagious defaults may still occur even if risk sharing between banks in the system is maximized.

A bank's liabilities are composed of interbank liabilities, denoted by L_i^{IB}, and customer deposits, denoted by D_i. Since every interbank liability of a bank is another bank's asset, interbank liabilities

[11]Without loss of generality, each bank is assumed to have at least one incoming link, so that interbank assets of any bank are strictly positive.

are determined endogenously. Customer deposits are instead exogenous.

For any bank i the condition to be solvent can be written as[12]:

$$(1-\varphi)A^{IB} + A_i^E - L_i^{IB} - D_i \geq 0 \tag{2.1}$$

where φ is the fraction of banks with obligations to bank i that have defaulted. Here I assume that when a linked bank defaults, bank i loses *all* of its interbank assets held against that bank.[13] Alternatively, the solvency condition can be stated as:

$$\varphi \leq \frac{K_i}{A^{IB}} \tag{2.2}$$

where $K_i = A^{IB} + A_i^E - L_i^{IB} - D_i$ is the capital buffer of bank i, or the net worth of a bank, equal to the difference between the book value of bank's assets and liabilities.

2.4 The transmission of shocks and bank's vulnerability

I examine the consequences of an unexpected idiosyncratic shock hitting one of the banks in the system. This shock can be thought of as resulting from operational risk (fraud) or credit risk.[14] Although for credit risk in particular, aggregate or correlated shocks affecting all banks at the same time may be more relevant in practice, idiosyncratic shocks are a clearer starting point for studying contagious defaults due to interbank exposures. Moreover, a single bank failure may actually result from an aggregate shock which has particularly adverse consequences for one institution.[15]

The failure of one bank reduces the interbank asset positions of all creditor banks and may give rise to a wave of contagious defaults in the network. In fact, the larger the number of *outgoing* links of a defaulting institution, the larger the potential for the spread of shocks. On the other hand, the impact of a bank default on the solvency of its creditors is determined by the degree of diversification of creditors' asset portfolio. The larger the number of *incoming* links of a creditor bank, the smaller the losses it incurs due to a default of a borrowing neighbor and the lower the risk of contagious failures in the system.

To demonstrate the latter inference, let us denote by j_i the number of incoming links of bank i. Since creditor banks each lose a fraction $1/j_i$ of their interbank assets when a single counterparty

[12]The price of external assets is fixed at 1 so that liquidity effects associated with the knock-on defaults are ruled out. In principle, Acharya and Yorulmazer (2007) show that the asset price may be depressed (become less than 1) when failed banks' assets need to be sold but financial markets have limited overall liquidity to absorb the assets.

[13]The same "zero recovery" assumption is made in Gai and Kapadia (2009). The authors argue that this assumption is realistic in the midst of a crisis: in the immediate aftermath of a default, the rate and timing of recovery are highly uncertain and banks' funders are likely to assume the worst-case scenario.

[14]See for example, the case of a failure of Barings in the U.K., Drexel Burnham Lambert in the U.S. or Société Générale in France.

[15]In this model, the aggregate shocks can be captured through a simultaneous reduction in the capital stocks of all banks, with a major loss for one particular bank.

defaults, inequality (2.2) suggests that a default can only spread if there is at least one creditor bank for which

$$\frac{1}{j_i} > \frac{K_i}{A^{IB}}, \qquad \text{or} \qquad j_i < \frac{A^{IB}}{K_i} \tag{2.3}$$

Following Gai and Kapadia (2009), bank i is called *vulnerable* if its in-degree fulfils (2.3), that is, if i is exposed to the default of a single neighbor. The other banks are called *safe*. According to the figures for developed countries reported by Upper (2007) and to the calibration results in Gai and Kapadia (2009) based on data for published accounts of large, international financial institutions, the ratio $\frac{A^{IB}}{K_i}$ lies in the range between 5 and 6.[16] In section 3 I will use this finding to demonstrate some of the model's results.

For simplicity, in the following I assume that capital buffers of all banks are identical:

Assumption 3 $\quad K_i = K_j = K \quad \forall i \neq j$

This allows denoting the right-hand side of (2.3) with a constant identical for all banks. Let $\overline{K} = \frac{A^{IB}}{K}$. Then from condition (2.3) it is clear that given $\overline{K}$, the vulnerability of a bank is fully determined by its in-degree j_i.

Thus, incoming and outgoing links of each bank play opposite roles in determining the extent of contagion. Their joint effect is predetermined by the specific features of the network structure as specified by the degree distribution. In the next section, I introduce various types of network structures and later on study the impact of structural characteristics on the spread of defaults in more detail.

2.5 The reach of contagious defaults

To compare the extent of contagious defaults and the risk of system-wide contagion across tiered and other types of banking structures, I introduce the notion of internode degree correlation and define three types of networks according to the pattern of this correlation.

Assortative, disassortative and neutral networks

Recall that the joint distribution of in- and out-degrees in the banking network is such that with probability 1 each node has identical in- and out-degree. Let $\xi(k, k')$ be the probability that a randomly selected link goes from a node with in- and out-degree k (node (k, k)) to a node with in- and out-degree

[16] Further details are available from the author.

k' (node (k', k')). This quantity obeys the sum rules:

$$\sum_{k,k'=1}^{\infty} \xi(k,k') = 1, \tag{2.4}$$

$$\sum_{k'=1}^{\infty} \xi(k,k') = \zeta_d(k), \quad k = 1, 2, \ldots \tag{2.5}$$

$$\sum_{k=1}^{\infty} \xi(k,k') = \zeta_c(k'), \quad k' = 1, 2, \ldots \tag{2.6}$$

where $\zeta_d(k)$ represents the marginal frequency of neighboring *debtor* banks with in- and out-degree k and $\zeta_c(k')$ is the marginal frequency of neighboring *creditor* banks with in- and out-degree k'. Notice however that every debtor with in- and out-degree k is simultaneously a creditor with the same degree. This implies that the frequency of debtor and creditor banks with degree k is the same: $\zeta_d(k) = \zeta_c(k)$. I denote this quantity $\zeta(k)$ and call it the marginal frequency of neighboring nodes with in- and out-degree k. Notice that $\zeta(k)$ is different from $p(k)$, the probability that a *randomly* chosen node on a network has in- and out-degree k. Instead it is biased in favor of nodes of high degree since more edges end (and start) at a high-degree node than at a low-degree one. Therefore, the degree distribution of a neighboring node is proportional to $kp(k)$ and the correctly normalized distribution is given by:

$$\zeta(k) = \frac{kp(k)}{\sum_{k=1}^{\infty} kp(k)} \tag{2.7}$$

If internode degree correlations are absent, the network is called *neutral* and $\xi(k,k')$ takes the value $\zeta(k) \cdot \zeta(k')$. Otherwise, $\xi(k,k')$ is different from this value and the network is either *assortative* or *disassortative*. The network is called assortative if high-degree nodes show tendency to be connected to other high-degree nodes. Conversely, the disassortative network is one where high-degree nodes tend to attach to low-degree ones. The amount of assortative/disassortative mixing can be quantified by the function which represents the degree correlation between neighboring nodes:

$$\langle kk' \rangle - \langle k \rangle \langle k' \rangle = \sum_{k,k'=1}^{\infty} kk'(\xi(k,k') - \zeta(k)\zeta(k')) \tag{2.8}$$

where $\langle \ldots \rangle$ indicates an average over links. This correlation is equal to zero for neutral networks and positive or negative for assortative or disassortative networks, respectively. For convenience of comparing different networks, the function in (2.8) can be normalized by dividing by its maximal value; the value it achieves on a perfectly assortative network. In the perfectly assortative network, $\xi(k,k') = \zeta(k)\delta_{kk'}$ and the degree correlation between neighboring nodes is equal to the variance of the distribution $\zeta(k)$:

$$\sigma_\zeta^2 = \sum_{k=1}^{\infty} k^2 \zeta(k) - \left[\sum_{k=1}^{\infty} k\zeta(k) \right]^2 \tag{2.9}$$

So, the normalized correlation function is

$$r = \frac{1}{\sigma_\zeta^2} \sum_{k,k'=1}^{\infty} kk'(\xi(k,k') - \zeta(k)\zeta(k')) \tag{2.10}$$

This is the standard Pearson correlation coefficient of the degrees at either end of a link and $-1 \leq r \leq 1$.

For the purpose of further analysis, I also use the joint degree distribution of the linked nodes, $\xi(k, k')$, to obtain the collection of conditional probability distributions $p(k'|k)$, $k = 1, 2, ..$, where $p(k'|k)$ denotes the conditional probability that there is a link going from a node with in- and out-degree k to a node with in- and out-degree k'. This probability is given by:

$$p(k'|k) = \frac{\xi(k,k')}{\sum_{k''=1}^{\infty} \xi(k,k'')} = \frac{\xi(k,k')}{\zeta(k)}, \quad k = 1, 2, \ldots \tag{2.11}$$

Notice that if internode degree correlations are absent,

$$p(k'|k) = \zeta(k') \tag{2.12}$$

Throughout the remainder of section 2.5, I study the extent of contagion and thresholds for the system-wide contagion, conditional on a general form of the joint probability distribution for the degrees of the neighboring nodes, $\xi(k, k')$. Then in section 3.1, I return to the question of variations in the pattern of degree correlations and study the effect of these variations on the spread of contagious defaults.

Generating functions

To evaluate the extent of contagion in banking networks, I use the approach based on probability generating functions.[17] First, assume that the components outgoing from the defaulting node are tree-like in structure, that is, contain no cycles, or closed loops.[18] This assumption is likely to hold if the number of nodes/banks in the financial network, n, is sufficiently large since the chances of a component containing a closed loop decrease as n^{-1}. The tree-like structure of the components outgoing from the defaulting node guarantees that any bank in the system can be exposed to a default of no more than one borrowing neighbor and no second round of contagion can occur. This implies that safe banks, resistant to a default of a single debtor, never default. As a result, contagion in the banking system propagates only through vulnerable banks: from one vulnerable bank to another.

Given this feature of the model, characterizing the spread of contagious defaults in the banking network comes to deriving the distribution of the sizes of components, or clusters, of vulnerable banks

[17] A detailed description of the key properties of the probability generating functions can be found in Newman et al. (2001) and Vega-Redondo (2007).

[18] The assumption of absent closed loops in the financial network is supported by empirical research which implies that clustering coefficients in real-world banking networks are relatively low (for example, Boss et al. (2004)).

that can be reached after an initial default. First, consider the clusters of vulnerable banks that can be reached by following a *randomly chosen directed link*, after an initial default. Let $H_1(y|k)$ be the function generating this probability distribution conditional on the fact that a randomly chosen link via which the vulnerable components are accessed is outgoing from the bank with in- and out-degree k. The pattern of shock transmission can take many different forms. A random link emanating from the node/bank (k,k) may lead to a component consisting of a single bank, either safe or vulnerable, or it may lead to a vulnerable bank with one, two, or more other tree-like vulnerable clusters emanating from it via single outgoing edges. Clusters are reached *independently* for each of the outgoing edges. Then, assuming that all clusters are almost surely of finite size, $H_1(y|k)$ must satisfy a self-consistency condition of the form:

$$H_1(y|k) = Pr[\text{reach safe bank}|(k,k)] + y\sum_{j'=1}^{\overline{K}}\sum_{k'=1}^{\infty} p[(j',k')|(k,k)]\big(H_1(y|k')\big)^{k'}, \quad k = 1,2,\ldots \tag{2.13}$$

where $p[(j',k')|(k,k)]$ is the conditional probability that a node (j',k') is reached via the outgoing link from a node (k,k). By definition, $p[(k',k')|(k,k)] = p(k'|k)$, where $p(k'|k)$ is given by (2.11). The leading factor y in (2.13) accounts for the first node encountered along the initial edge and I have used the fact that a generating function of the sum of m i.i.d. random variables is equal to the m^{th} power of the generating function of a single random variable. Notice that the probability of reaching a safe bank from the initial defaulting bank (k,k), $Pr[\text{reach safe bank}|(k,k)]$, is equal to 1 minus the probability of the complementary event that a vulnerable bank is reached:

$$Pr[\text{reach safe bank}|(k,k)] = 1 - \sum_{j'=1}^{\overline{K}}\sum_{k'=1}^{\infty} p[(j',k')|(k,k)] \tag{2.14}$$

Using (2.14) and recalling that with probability 1 all nodes in the network have identical in- and out-degree, we can write (2.13) as:

$$H_1(y|k) = \left(1 - \sum_{k'=1}^{\overline{K}} \frac{\xi(k,k')}{\zeta(k)}\right) + y\sum_{k'=1}^{\overline{K}} \frac{\xi(k,k')}{\zeta(k)}\big(H_1(y|k')\big)^{k'}, \quad k = 1,2,\ldots \tag{2.15}$$

Based on functions $H_1(y|k)$, $k = 1,2,\ldots$, it is now possible to characterize the distribution of sizes of the outgoing vulnerable components reachable from a *randomly selected bank*, that is, the distribution of sizes of the vulnerable components to which a randomly selected bank belongs. This distribution is generated by:

$$
\begin{aligned}
H_0(y) &= Pr[\text{bank is safe}] + y\sum_{j=1}^{\overline{K}}\sum_{k=1}^{\infty} p_{j,k}\big(H_1(y|k)\big)^k \qquad (2.16)\\
&= \left(1-\sum_{k=1}^{\overline{K}} p(k)\right) + y\sum_{k=1}^{\overline{K}} p(k)\big(H_1(y|k)\big)^k \quad k=1,2,\dots
\end{aligned}
$$

Two possibilities are contemplated in (2.16): either a randomly chosen bank is safe or it is vulnerable with in- and out-degree k, $k = 1, 2, \dots$, and each of its k outgoing links leads to a vulnerable cluster with size distribution generated by $H_1(y|k)$.

In principle, one can now solve equations (2.15) for the collection of $H_1(y|k)$, $k = 1, 2, \dots$, and substitute into (2.16) to get $H_0(y)$. In practice, this is usually impossible: (2.15) is a complicated and often transcendental equation, which rarely has a known solution.[19] One way to tackle this problem is to find a numerical approximation of $H_1(y|k)$ for any $k = 1, 2, \dots$, using iterations of (2.15) with initial set of values $H_1 = 1$. However, for the purposes of this paper, it is enough to obtain the first moment of the distribution of the vulnerable cluster sizes, the expression for the *average* size of the vulnerable cluster.

Average size of the vulnerable component and phase transition

I now derive the expression for the average size of the vulnerable cluster in the banking network and study the conditions for the emergence of the *giant vulnerable component*, when the size of the vulnerable cluster diverges. Formally, a giant component is a unique component whose relative size remains bounded *above* zero as the number of nodes in the network increases indefinitely. In the framework of this model, the formation of the giant vulnerable cluster can be interpreted as a threshold condition for the possibility of system-wide contagion, or "global" banking crisis: with positive probability, a random initial default of one bank can cause failure of a substantial fraction of vulnerable institutions in the banking system. Then the relative size of the giant vulnerable component can be regarded as a scope of the crisis should systemic failure occur.

Due to a basic property of generating functions, the average size of the vulnerable cluster, S, can be computed as:

$$S = H_0'(1) \qquad (2.17)$$

Taking a derivative of $H_0'(y)$ in (2.16) and evaluating it at $y = 1$, we obtain that

$$S = \sum_{k=1}^{\overline{K}} p(k)\big(H_1(1|k)\big)^k + \sum_{k=1}^{\overline{K}} kp(k)\big(H_1(1|k)\big)^{k-1}H_1'(1|k) \qquad (2.18)$$

[19] For reference, see for example, Newman et al. (2001).

$H_1(y|k)$ is a standard generating function so that $H_1(1|k) = 1$ for all k. Also, $kp(k) = \langle k\rangle\zeta(k)$ as follows from the definition of $\zeta(k)$ in (2.7). Therefore, (2.18) becomes:

$$S = \sum_{k=1}^{\overline{K}} p(k) + \langle k\rangle \sum_{k=1}^{\overline{K}} \zeta(k) H_1'(1|k) \tag{2.19}$$

Differentiating (2.15) and solving for the set of values $H_1'(1|k)$, $k = 1, 2, \ldots$, the average size of the vulnerable cluster can be written in a form of the following matrix expression:

$$S = \mathbf{1}'\mathbf{p} + \langle k\rangle \zeta' \mathbf{M}^{-1} \zeta \tag{2.20}$$

where $\mathbf{1}$ is the vector of 1's, $\mathbf{p} = \{p(k)\}_{k=1}^{\overline{K}}$ and $\zeta = \{\zeta(k)\}_{k=1}^{\overline{K}}$ are the vectors of degree frequencies for a randomly chosen and neighboring node, respectively, and $\mathbf{M}$ is the matrix of elements $\{m_{k,k'}\}_{k,k'=1}^{\overline{K}}$ such that

$$m_{k,k'} = k\xi(k,k') - \zeta(k')\delta_{k,k'}, \quad k, k' = 1, 2, \ldots \tag{2.21}$$

Expression (2.20) diverges when $\det(\mathbf{M}) = 0$. This condition marks the *phase transition* at which a giant vulnerable component first appears in the network. By studying the behavior of (2.20) close to the transition, where S must be large and positive in the absence of a giant component, one can show that a giant vulnerable component exists in the network when $\det(\mathbf{M}) > 0$.[20]

So, in the framework of a given model, when $\det(\mathbf{M}) < 0$, the vulnerable clusters in the banking system are small and contagion stops quickly. However, if $\det(\mathbf{M}) \geq 0$, a giant vulnerable cluster arises and occupies a finite fraction of the network. The next section presents the analytical base for computing the relative, or *fractional* size of the giant vulnerable component. I employ this analysis later, in section 3.1, to compare the fractional size of the giant vulnerable component across tiered and other banking networks, so as to compare the susceptibility to risk and the scope of global crises in various types of banking structures.

Relative size of the giant vulnerable component

When a giant vulnerable component exists in the financial network, it is straightforward to assess its relative size, ω, – the fraction of banks in the network that belong to the giant vulnerable component. To find ω I first define $1 - \widehat{\omega}_k$ – the probability that a randomly selected link outgoing from a node (k, k) does *not* lead to the giant vulnerable component. This means that the endnode of this randomly selected link is either a safe bank or a vulnerable bank that does not belong to the giant vulnerable component. By consistency, in the latter case none of the outgoing links of the endnode bank leads

[20]See Newman (2002a) and Molloy and Reed (1995) for more details.

to the giant vulnerable component. This results in the following set of equations:[21]

$$1-\widehat{\omega}_k = \left(1-\sum_{k'=1}^{\overline{K}} p(k'|k)\right) + \sum_{k'=1}^{\overline{K}} p(k'|k)\big(1-\widehat{\omega}_{k'}\big)^{k'}, \quad k=1,2,\ldots \tag{2.22}$$

Using the definition of $p(k'|k)$ in (2.11), (2.22) can be written as

$$1-\widehat{\omega}_k = \left(1-\frac{1}{\zeta(k)}\sum_{k'=1}^{\overline{K}} \xi(k,k')\right) + \frac{1}{\zeta(k)}\sum_{k'=1}^{\overline{K}} \xi(k,k')\big(1-\widehat{\omega}_{k'}\big)^{k'}, \quad k=1,2,\ldots \tag{2.23}$$

Having obtained $\{\widehat{\omega}_k\}_{k=1}^{\overline{K}}$ from (2.23), let us now determine the probability ω that a randomly selected bank belongs to the giant vulnerable component, that is, the fractional size of the component. The probability $(1-\omega)$ that a randomly selected node does *not* belong to the giant vulnerable component reflects the possibility of two events. First, a randomly selected bank may be safe and hence does not span even a vulnerable singleton. Alternatively, the bank may be vulnerable but such that none of its outgoing links leads to the giant vulnerable component. The two possibilities sum up to:

$$1-\omega = \left(1-\sum_{k=1}^{\overline{K}} p(k)\right) + \sum_{k=1}^{\overline{K}} p(k)\big(1-\widehat{\omega}_k\big)^{k}, \quad k=1,2,\ldots \tag{2.24}$$

As with equations (2.15), (2.16), it is usually not possible to solve for ω in closed form, but the approximate solution can be found by numerical iteration from a suitable set of starting values for $\{1-\widehat{\omega}_k\}_{k=1}^{\overline{K}}$. Section 3.1 presents the results of the numerical analysis to compare the phase transition thresholds and the sizes of the giant vulnerable components in assortative, disassortative and neutral networks.

3 Results

3.1 The giant vulnerable component and optimal bail-out strategy in assortative, disassortative and neutral banking networks

The numerical analysis is conducted following Newman (2002a). Let us consider the symmetric binomial form of the joint probability distribution for the degrees of the neighboring nodes:

$$\xi(k,k') = Ne^{-(k+k')/\nu}\left[\binom{k+k'}{k}\alpha^k\beta^{k'} + \binom{k+k'}{k'}\alpha^{k'}\beta^{k}\right] \tag{3.1}$$

where $\alpha+\beta=1$, $\nu>0$, and $N=\frac{1}{2}(1-e^{-1/\nu})$ is a normalizing constant. The choice of this distribution function is explained by analytical tractability. Moreover, as argued by Newman (2002a), the behavior of this distribution is also quite natural: the distribution of the sum of the degrees, $k+k'$, decreases as a simple exponential, while that sum is distributed between the two neighboring nodes binomially.

[21]The first term in brackets on the right-hand side of (2.22) is the probability that a randomly chosen edge emanating from a node (k,k) leads to a safe bank.

The parameter α controls the assortative, disassortative or neutral mixing in the network. From equation (2.10), the correlation coefficient of the degrees of the neighboring nodes, r, is equal to

$$r = \frac{8\alpha\beta - 1}{2e^{1/\nu} - 1 + 2(\alpha - \beta)^2} \tag{3.2}$$

The value of r can be positive or negative, passing through zero when $\alpha = \frac{1}{2} \mp \frac{1}{4}\sqrt{2} \approx 0.1464$ or 0.8536.

Using the specification (3.1) of the joint degree distribution of the neighboring nodes, and equations (2.23), (2.24), I calculate numerically the fractional size of the giant vulnerable component, ω, for $\alpha = 0.05$, where the network is disassortative, $\alpha = 0.1464$, where it is neutral, and $\alpha = 0.5$, where it is assortative. Figure 1 demonstrates the results, showing the size of the giant vulnerable component as a function of the degree cutoff parameter $\overline{K}$. The three panels of the figure represent cases of different values of the degree scale parameter ν, where $1/\nu$ measures the rate at which the probability of the aggregate degree of neighboring nodes, $k + k'$, declines as $k + k'$ becomes larger.

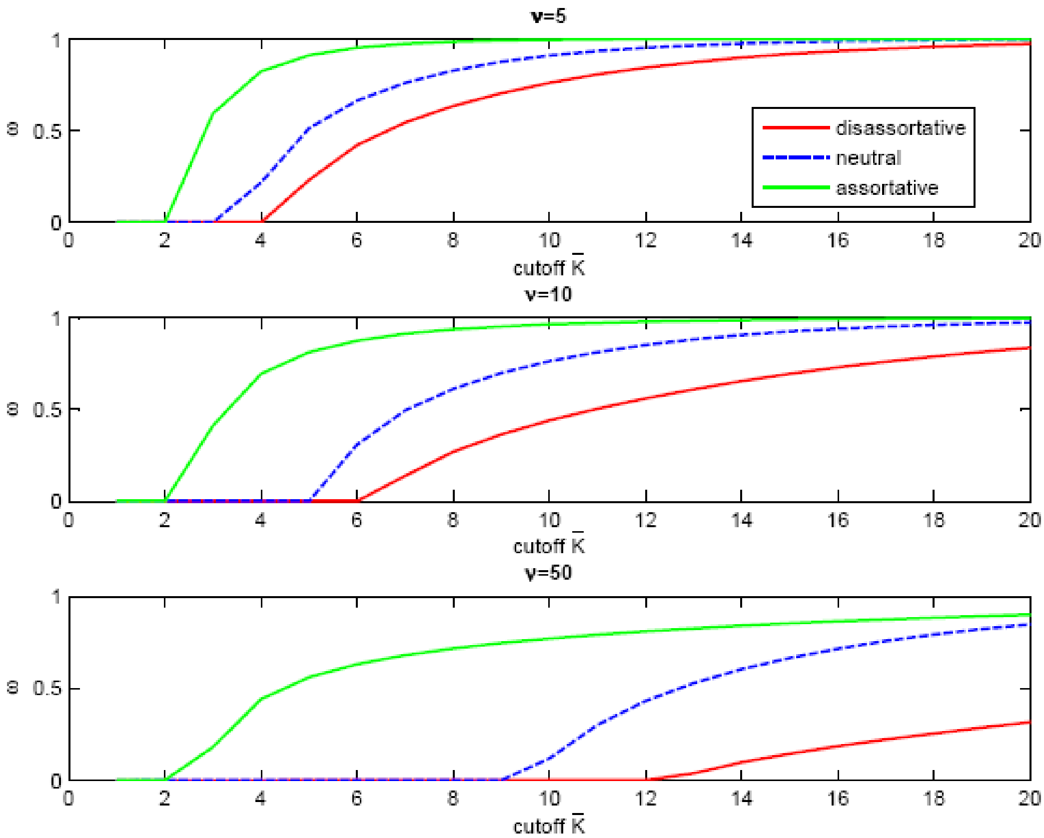

Figure 1: Fractional size of the giant vulnerable component as a function of $\overline{K}$ for the joint degree distribution of the neighboring nodes given in (3.1).

When $\overline{K}$ increases, more and more banks in the network are characterized as vulnerable and at some point, the giant vulnerable component is formed and a risk of global financial breakdown is realized. Two important outcomes are suggested by Figure 1. First, the phase transition, the instance at which a giant vulnerable component arises, shifts to the left as the network becomes more assortative. That is, the giant vulnerable component arises more easily and the risk of "global" crisis in the banking system is higher if poorly-connected banks preferentially associate with other poorly-connected banks. Conversely, the disassortative/tiered banking systems show more resilience to global defaults. For example, when $\nu = 10$ or $\nu = 15$ and $5 \leq \overline{K} \leq 6$, as it is suggested by the empirical evidence and by the calibration results in Gai and Kapadia (2009), no risk of system-wide contagion exists in the disassortative/tiered banking network but in the assortative network, the risk of global default is positive. Second, for the same value of degree cutoff $\overline{K}$, at least in the plausible range of values, the size of the giant vulnerable component as a fraction of the whole banking network, is larger in the assortatively mixed network. This means that as soon as contagious defaults become system-wide (the giant vulnerable component emerges), the fraction of defaulting institutions in the banking network, or the scope of the crisis, is larger if the network is assortative. As a result, in the disassortative/tiered banking structure, both the risk of systemic crisis and the scope of the crisis are lower than in the other types of structures.

These results are intuitively reasonable. In a disassortative banking network, high-degree banks are broadly distributed over the network and therefore, presumably form links on many paths between other banks. This implies that with high probability an initial shock hitting a random bank reaches a high-degree bank in a small number of "steps". Since high-degree banks are relatively resilient to neighbors' defaults, the shock gets absorbed at that bank and does not spread any further. Therefore, both the risk of systemic defaults and the scale of these defaults in the disassortative network are lower than in the other structures. In fact, this simple intuition suggests that the finding of relative resilience to shocks of disassortative networks, derived here for the specific distribution in (3.1), can be generalized to a wider range of degree distributions, that is, to a larger class of assortative, disassortative and neutral networks. However, the formal investigation of this question is beyond the scope of this paper.

Furthermore, the comparison of the susceptibility to crises of assortative, disassortative and neutral banking networks allows for the evaluation of the relative effectiveness and costs of the optimal government bail-out strategy in these networks. Here I focus on *targeted* bail-outs, when the government first rescues the defaulting institutions that have a particularly large number of interbank connections and therefore, represent the highest risk in terms of the spread of shocks through the network.[22] Let $K_{\max}$ be the threshold degree which satisfies the condition that bailing-out all defaulting banks with in- and

[22]Notice that in the framework of this model, bailing-out defaulting banks simply means "transforming" these initially vulnerable institutions into safe institutions. After having been rescued by the government, they become "immune" from defaults and do not transmit shocks to their neighbors in the banking network.

out-degree higher than this threshold guarantees that no global crisis will emerge (with probability 1 the giant vulnerable component will not arise). Then the banks with in- and out-degree lower or equal to $K_{\max}$ either do not default or default but do not need to be rescued for the global stability of the system to be preserved. In this sense, $K_{\max}$ represents the highest in- and out-degree among defaulting institutions whose failure does not endanger the stability of the banking system.

The threshold $K_{\max}$ determines *a priori optimal bail-out strategy* of the government in the following sense. Given only the degree distribution in the banking network, it minimizes the number of targeted bail-outs (with in- and out-degree higher than $K_{\max}$) subject to the constraint that the system stability is preserved. Any higher threshold degree would not guarantee the stability of the system, whereas a lower threshold would induce unnecessary bail-out costs.

The definition of $K_{\max}$ implies that it is nothing but the point on the scale of possible values of $\overline{K}$ at which the giant vulnerable component first forms. It can, therefore, be found from the conditions which determine the phase transition. Furthermore, for any degree cutoff $\overline{K}$ in the banking system, the value $K_{\max} \geq \overline{K}$ implies that no bail-outs are required since by definition, all banks with in- and out-degree larger or equal to $\overline{K}$ are safe. Conversely, if $1 \leq K_{\max} < \overline{K}$, banks with degrees in the range between $K_{\max}$ and $\overline{K}$ are vulnerable and need to be rescued upon default since their failure endangers the stability of the system.

In view of the aforesaid, Figure 1 suggests that in the assortative and neutral banking networks targeted bail-outs are needed for lower values of $K_{\max}$ than in the disassortative network. For example, when $\nu = 5$, the threshold degree $K_{\max} = 2$ if the network is assortative, 3 if it is neutral and 4 if it is disassortative. Moreover, when $\nu = 10$ or $\nu = 50$ and $5 \leq \overline{K} \leq 6$, no bail-outs are needed to preserve the stability of the disassortative banking network, while the assortative network requires rescues of all failing banks with in- and out-degree exceeding 2.

3.2 Resilience to systemic breakdown and targeted bail-outs in tiered banking system: scale-free networks

Let us now consider the alternative way of representing the tiered banking network. Suppose that there are *no degree correlations* between neighboring nodes but the degree distribution is specified to capture (i) the tendency of highly-connected nodes to be connected to poorly-connected ones and vice versa, and (ii) the numerical superiority of the low-degree nodes over the high-degree ones. Both characteristics are inherent in the power-law, or scale-free, degree distribution defined by:

$$p(k) = (\lambda - 1)k^{-\lambda}, \quad k = 1, 2, \ldots \quad \text{and} \quad 2 < \lambda < 3 \tag{3.3}$$

where, as before, $p(k)$ is the probability that a randomly chosen node has in- and out-degrees each equal to k, λ governs the rate at which the probability decays with connectivity and the assumption $2 < \lambda < 3$ means that the degree distribution is so broad that it displays unbounded second moments but still possesses a well-defined average degree. Arguably, it is more reasonable to think of a banking network as being generated by the scale-free degree distribution with such low values of λ, since this implies that $p(k)$ does not decline "too fast" with k and the substantial share of banks in the network have more than one in- and out-going connection. This assumption is supported by the finding of Boss et al. (2004) which suggests that the power-law degree distribution of the Austrian interbank market has exponent parameter $\lambda = 2.01$.

In the framework of this model, λ has an important interpretation as the inverse of a tiering level in the scale-free network. Indeed, as the value of λ declines, the fraction of highly-connected banks in the network increases and so does the probability that poorly-connected banks link with highly-connected ones rather than with each other. This implies that by varying the value of λ, one can study and compare the risks of systemic crisis and optimal bail-out strategy of the government in scale-free banking networks with different levels of tiering. Below, I consider these questions in detail. Specifically, I find the closed-form expression for the threshold degree $K_{\max}$ and study the functional dependence of $K_{\max}$ on parameter λ. Hereafter, for simplicity, I regard the degree of each node, k, as a continuous variable and the probability $p(k)$ in (3.3) as a continuous density function.

Generating functions

Given the assumption of absent degree correlations between neighboring nodes and the continuous framework of the degree distribution, the generating functions H_1 and H_0 in (2.15) and (2.16) can be written as follows:

$$\begin{aligned} H_1(y) &= \left(1 - \int_1^{\overline{K}} \zeta(k) dk\right) + y \int_1^{\overline{K}} \zeta(k) \big(H_1(y)\big)^k dk \\ &= \left(1 - \frac{1}{\langle k \rangle} \int_1^{\overline{K}} p(k) k dk\right) + y \frac{1}{\langle k \rangle} \int_1^{\overline{K}} p(k) k \big(H_1(y)\big)^k dk, \qquad (3.4) \\ H_0(y) &= \left(1 - \int_1^{\overline{K}} p(k) dk\right) + y \int_1^{\overline{K}} p(k) \big(H_1(y)\big)^k dk \qquad (3.5) \end{aligned}$$

where the second equality in (3.4) uses the definition of $\zeta(k)$ in (2.7). Both expressions can be formulated more compactly by means of the other two generating functions: $G_0(y)$, for the number of links leaving a randomly chosen vulnerable bank, and $G_1(y)$, for the number of links leaving a vulnerable bank reached by following a randomly chosen incoming link. Since the degree distribution of a randomly selected bank is given by $p(k)$, and the degree distribution of a *neighboring* bank reached by following a randomly chosen link is $\zeta(k) = kp(k)/\langle k\rangle$, $G_0(y)$ and $G_1(y)$ can be defined as:

$$G_0(y) = \int_1^{\overline{K}} p(k)y^k dk, \tag{3.6}$$

$$G_1(y) = \frac{1}{\langle k\rangle}\int_1^{\overline{K}} kp(k)y^k dk \tag{3.7}$$

Functions $G_0(y)$ and $G_1(y)$ allow $H_0(y)$ and $H_1(y)$ to be rewritten in the following form:

$$H_1(y) = (1 - G_1(1)) + yG_1(H_1(y)), \tag{3.8}$$

$$H_0(y) = (1 - G_0(1)) + yG_0(H_1(y)), \tag{3.9}$$

Now, as in (2.17), the average size of the vulnerable cluster, S, can be computed as:

$$S = H_0'(1) = G_0(H_1(1)) + G_0'(H_1(1))H_1'(1) = G_0(1) + G_0'(1)H_1'(1) \tag{3.10}$$

where $H_1(1) = 1$ due to the basic property of generating functions. From (3.8) we know that

$$H_1'(1) = \frac{G_1(1)}{1 - G_1'(1)} \tag{3.11}$$

Hence, the average size of the vulnerable cluster becomes:

$$S = G_0(1) + \frac{G_0'(1)G_1(1)}{1 - G_1'(1)} \tag{3.12}$$

Phase transition and threshold degree $K_{\max}$

The term $G_1'(1)$ on the right-hand side of (3.12) is the average out-degree of a vulnerable first neighbor. If this quantity is less than one, all vulnerable clusters in the network are small and contagion dies out quickly. But as $G_1'(1) \nearrow 1$, the average size of the vulnerable cluster, S, increases unboundedly. It diverges when $G_1'(1) = 1$. It is at this point that the giant vulnerable component, whose size scales linearly with the size of the whole network, first forms. The threshold degree $K_{\max}$ for vulnerable banks, above which systemic risk is realized and massive contagion is likely, can be determined from this equation. Using the definition of $G_1(y)$ in (3.7) and the definition of $p(k)$ in (3.3), we obtain:

$$G_1'(1) = \frac{1}{\langle k\rangle}\int_1^{K_{\max}} k^2 p(k)dk = \frac{\lambda-1}{\langle k\rangle}\int_1^{K_{\max}} k^{2-\lambda}dk = \frac{\lambda-1}{\langle k\rangle}\left(K_{\max}^{3-\lambda} - 1\right) \tag{3.13}$$

Since $\langle k\rangle = (\lambda-1)/(\lambda-2)$, condition $G_1'(1) = 1$ leads to the closed-form expression for $K_{\max}$.

Proposition 3.2.1. *In the scale-free network, the threshold degree $K_{\max}$ at which the giant vulnerable component first forms is given by*

$$K_{\max} = \left[\frac{1}{\lambda-2}\right]^{\frac{1}{3-\lambda}} \tag{3.14}$$

$K_{\max}$ is monotonically decreasing in λ for all $2 < \lambda < 3$

So, the critical level for the formation of the giant vulnerable component and the targeted bail-out strategy of the government are determined by the value of λ, the level of tiering in the banking system. Furthermore, $K_{\max}$ is decreasing in λ, that is, it is increasing in the level of tiering.[23] This functional dependence of $K_{\max}$ on λ is illustrated with Figure 2.

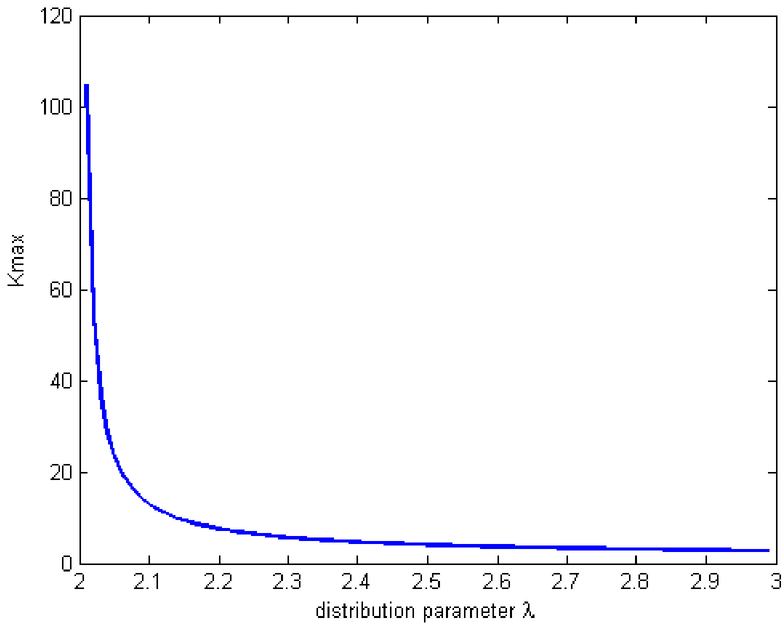

Figure 2: Threshold degree $K_{\max}$ as a function of λ.

For λ sufficiently close to 2, the threshold degree $K_{\max}$ is large but it declines sharply with λ. For example, when λ is 2.1, $K_{\max}$ is approximately 13 but it is only around 4 for λ equal to 2.4 and around 3 for λ equal to 2.7. This finding suggests that the resilience to global crisis of the scale-free banking system with high level of tiering (λ is close to 2) is very high but declines strongly when the level of tiering decreases (λ becomes large). Specifically, since all banks in the network with in- and out-degree greater or equal to $\overline{K}$ are safe and $\overline{K}$ is estimated to be in the range between 5 and 6 for real-world banking systems, no system-wide contagion occurs and hence, no bail-outs are needed in the highly-tiered scale-free network with λ close to 2. This result agrees with the conclusion of the empirical studies by Boss et al. (2004, 2004b) for the scale-free Austrian banking system where $\lambda = 2.01$. They report that "the banking system is very stable and default events that could be classified as a "systemic crisis" are unlikely" (Boss et al. (2004b)). In contrast, for the scale-free network with lower level of tiering, when λ is around 2.4 or larger, the risk of global contagion exists unless the government rescues all defaulting banks with in- and out-degree exceeding $K_{\max}$. This means that the required bail-outs comprise all defaulting institutions with in- and out-degree higher

[23]This simple comparative statics result is derived in the Appendix.

than 4 if λ is 2.4, higher than 3 if λ is 2.7 and higher than 2 if λ is arbitrarily close to 3.[24]

These conclusions are confirmed by Figure 3, which shows the threshold $K_{\max}$ at which the giant vulnerable component first forms, and the size of the giant vulnerable component beyond this threshold, ω, for three values of λ.[25]

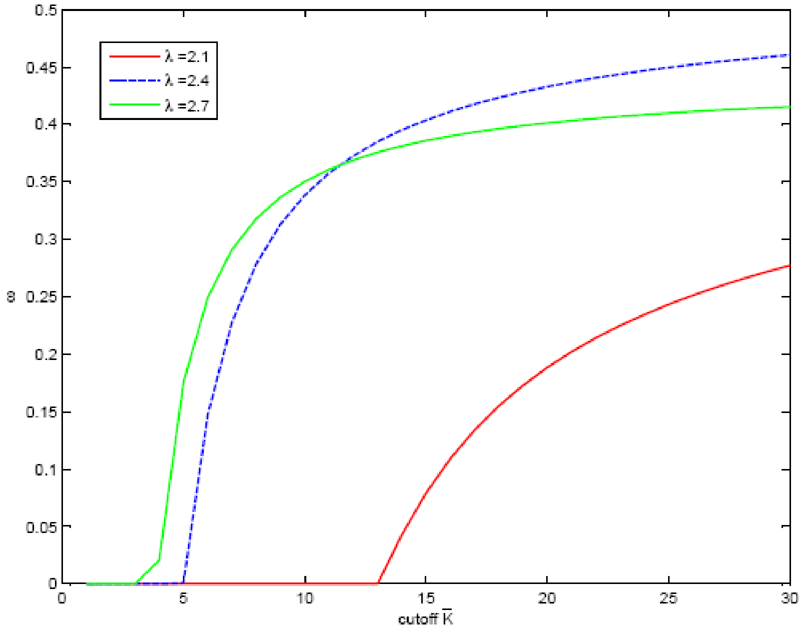

Figure 3: Fractional size of the giant vulnerable component as a function of $\overline{K}$, for $\lambda = 2.1$, $\lambda = 2.4$, and $\lambda = 2.7$.

Intuitively, a decline in resilience of the scale-free network to shocks as the level of tiering lowers (parameter λ grows) can be explained as follows. When the level of tiering is high (λ is small), the fraction of highly-connected banks in the network is relatively large and so is the number of paths between other banks in the network which "pass" through the highly-connected banks. As a result, in a scale-free network with higher level of tiering, initial shocks in any part of the network reach highly-connected banks and are absorbed by these banks "faster" than in a less tiered network. Therefore, more tiered systems are more resilient to massive defaults.

Figures 2 and 3 shed some light on the empirical evidence about the susceptibility of tiered banking

[24] Notice, that all these values of $K_{\max}$ in the scale-free network with λ equal to 2.4, 2.7 and λ arbitrarily close to 3 exceed the mean degree, $\langle k \rangle$, of the scale-free distribution with the corresponding parameter λ.

[25] The size of the giant vulnerable component is computed numerically from the equations analogous to (2.23) and (2.24). Under the assumption of absent degree correlations and continuity of the degree distribution, (2.23) and (2.24) become:

$$1 - \widehat{\omega} = \left(1 - \frac{1}{\langle k \rangle} \int_1^{\overline{K}} p(k) k dk\right) + \frac{1}{\langle k \rangle} \int_1^{\overline{K}} p(k) k \big(1 - \widehat{\omega}\big)^k dk, \quad (3.15)$$

$$1 - \omega = \left(1 - \int_1^{\overline{K}} p(k) dk\right) + \int_1^{\overline{K}} p(k) \big(p(k)(1 - \widehat{\omega})\big)^k dk \quad (3.16)$$

As before, $\widehat{\omega}$ represents the fraction of links leading to the giant vulnerable component and ω is the fraction of nodes in the giant vulnerable component.

networks to systemic shocks. They suggest that at least in the tiered banking systems represented by the scale-free networks, the extent of resilience to shocks is determined by the level of tiering. While a highly-tiered scale-free network is extremely resilient to systemic defaults, less tiered structures are fragile and financial contagion propagates through such structures easily.

Finally, given the gap between $K_{\max}$ and $\overline{K}$, I calculate the *maximal expected share* of banks, ρ, which *may need to be rescued* to preserve the stability of the scale-free banking network. This maximal expected share of rescued banks, or the upper bound for the percentage of bail-outs in the system, corresponds to the situation when *all* vulnerable banks with in- and out-degree exceeding $K_{\max}$ actually default.

Proposition 3.2.2. *In the scale-free network, the maximal expected share of banks which may need to be rescued to preserve the stability of the system is equal to*

$$\rho = \int_{K_{\max}}^{\overline{K}} p(k)dk = \left[\frac{1}{K_{\max}}\right]^{\lambda-1} - \left[\frac{1}{\overline{K}}\right]^{\lambda-1} \tag{3.17}$$

ρ is monotonically increasing in λ for all $2 < \lambda < 3$

So, the upper bound for the percentage of bail-outs in the scale-free networks is increasing in λ for any $2 < \lambda < 3$, or equivalently, it is decreasing in the level of tiering.[26]

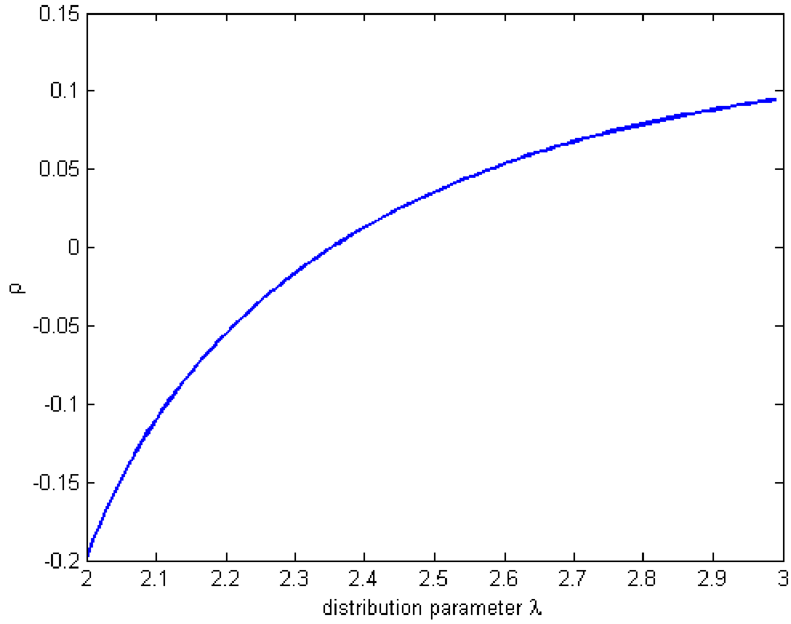

Figure 4: The upper bound for the percentage of bail-outs in scale-free networks as a function of λ, for $\overline{K} = 5$. Negative values of ρ correspond to $K_{max} < \overline{K}$.

For example, when $\overline{K}$ is 5, ρ is equal to 1.3% of all banks if λ is 2.4, and 6.8% of all banks if λ is 2.7. Thus, the total number of bail-outs needed can be as high as 6.8% of the system for λ equal

[26]The short proof of this result is provided in the Appendix.

to 2.7 and at the other extreme, bail-outs may not be needed at all if λ is sufficiently close to 2. As before, when the system is more tiered (λ is smaller), highly connected banks form a larger fraction of the system and the number of targeted bail-outs sufficient to stop the massive contagion is smaller.

4 Policy implications and conclusions

This paper develops a model of contagion in a banking system and studies the effects of tiering in the system on the risk and potential impact of system-wide defaults. High policy relevance of this issue and controversial empirical findings provide strong motivation for this research.

The banking system is constructed by a random directed network, where nodes represent banks and links represent claims and obligations of banks to each other. In this framework, the tiered banking system is modelled in two ways: first, by a disassortative network, displaying negative degree correlations between neighboring nodes, and then by a scale-free network. Using the first modelling approach, I compare the resilience to systemic defaults and optimal bail-out strategy of the government in the tiered disassortative network with those in other types of networks: the assortative network, displaying positive degree correlations, and the neutral network, displaying no correlations. Subsequently, using the second approach, I concentrate on the tiered banking systems displaying various levels of tiering. I argue that the level of tiering in the scale-free network can be approximated by the inverse of the exponent parameter. By changing this parameter, I evaluate the effects of the level of tiering on the susceptibility of the banking system to shocks and on the maximal expected number of targeted bail-outs.

The key feature of the model, which highlights the importance of network structure in determining the spread of contagious defaults, is the counteracting effects of bank connectivity. While greater connectivity increases the spread of contagion in the banking network, it also improves risk sharing among neighboring banks and thereby reduces the susceptibility of banks to defaults. These opposing effects of risk sharing and risk spreading interact differently in varying structures. As a result, the resilience of a banking system to shocks and the optimal number of bank rescues depend on the features of the underlying degree distribution.

Specifically, for the degree distribution which generates assortative, disassortative, and neutral networks, I find that the disassortative/tiered banking network is more resilient to shocks and in the event of a crisis, shows a lower frequency of failures than other types of structures. Consequently, the threshold degree above which the defaulting banks endanger the stability of the system and require government assistance is higher in the disassortative network, so that the overall number of bail-outs and the associated bail-out costs are expected to be lower.

Further, for the scale-free banking system, I find that the higher level of tiering implies higher resilience to shocks and, in the worst-case scenario, lower number of bail-outs. In particular, when the level of tiering is relatively low, the required number of bail-outs may be as high as 6.8% of the system, whereas if the level of tiering is higher, as in the case of the Austrian banking network, bail-outs may not be needed at all.

These findings suggest not only the advantages of tiering in the banking network over other types of network organization, but also the advantages of relatively high levels of tiering regardless.

These insights provide the basis for specific policy recommendations. First, the relatively high resilience of the tiered system to contagion and the minor extent of government intervention needed in order to reduce systemic risk imply that the formation of a more highly tiered system of interbank relations should be among the priorities of the central bank or the government. Secondly, the *optimal* level of targeted bail-outs which guarantees the stability of the system at minimal cost should be chosen by explicitly taking into account the features of the underlying banking structure. Specifically, if the degree distribution generating the network of interbank connections is scale-free, optimal targeted bail-outs are predetermined by the exponent parameter, or the level of tiering in the scale-free network. The number of these bail-outs declines as the level of tiering increases. Finally, insights about the role of bank connectivity as a shock absorber or a shock amplifier suggest that the close monitoring by the regulator is required for banks whose connectivity is "risky": the smaller the number of incoming links (low level of risk diversification) and the larger the number of outgoing links (high number of obligations to other banks), the higher the potential of a bank to be a key transmitter of shocks in the system.[27]

The model and results presented in this paper suggest some directions for future research. An interesting extension of the paper would be to simulate the model for a large banking system, other than Austrian, using real balance sheets for all banks and calibrating the joint degree distribution to match the observed data. Alternatively, one could think of endogenizing the formation of the banking network so as to demonstrate and explain the incentives of banks in the real world to settle within the tiered banking systems. The framework of the network formation model could loosen the assumprions of the present setting, allowing for distinctions in interbank asset positions of banks and for the possibility of clusters/cycles in the banking network. In addition, to reflect the difference in conditions for borrowing and lending in the interbank market, it could also differentiate between the costs of formation of incoming and outgoing links. These strands of research would add realism to the

[27]If the number of incoming and outgoing links of a bank is the same, as it is assumed in the model, then the highest risk of contagion transmission is posed by the institutions with somewhat average degree: high-degree banks absorb shocks and therefore, are "immune" against contagion, whereas low-degree banks may default but have only a limited scope for spreading the shock further.

model and provide new, potentially valuable, insights.

5 Appendix

Proof of Proposition 3.2.1

$K_{\max}$ is monotonically decreasing in λ as soon as its derivative with respect to λ is strictly negative for all $2 < \lambda < 3$. To calculate the derivative of $K_{\max}$, notice that from (3.14):

$$\ln(K_{\max}) = \frac{1}{3-\lambda} \ln\left(\frac{1}{\lambda-2}\right)$$

so that

$$\frac{1}{K_{\max}} \cdot \frac{dK_{\max}}{d\lambda} = \frac{1}{(3-\lambda)^2} \ln\left(\frac{1}{\lambda-2}\right) - \frac{1}{3-\lambda} \cdot \frac{1}{\lambda-2}$$

From this expression it follows that

$$\frac{dK_{\max}}{d\lambda} = K_{\max} \cdot \frac{1}{3-\lambda} \cdot \frac{-(\lambda-2)\ln(\lambda-2) - (3-\lambda)}{(3-\lambda)(\lambda-2)} \tag{5.1}$$

Since $2 < \lambda < 3$, this derivative is negative as soon as $f(\lambda) = -(\lambda-2)\ln(\lambda-2) - (3-\lambda) < 0$. Notice that $f(\lambda)$ is monotonically increasing in λ for any $2 < \lambda < 3$. Indeed:

$$\frac{df(\lambda)}{d\lambda} = -\ln(\lambda-2) > 0 \quad \forall 2 < \lambda < 3$$

Hence, for all $2 < \lambda < 3$, $f(\lambda) < f(3) = 0$. Therefore, $dK_{\max}/d\lambda$ in (5.1) is strictly negative for any $2 < \lambda < 3$.

■

Proof of Proposition 3.2.2

ρ is monotonically increasing in λ if its derivative with respect to λ is strictly positive for all $2 < \lambda < 3$. I differentiate with respect to λ both ratios on the right-hand side of (3.17). Denote the first ratio by y:

$$y = \left[\frac{1}{K_{\max}}\right]^{\lambda-1} = (\lambda-2)^{\frac{\lambda-1}{3-\lambda}}$$

Then

$$\ln(y) = \frac{\lambda-1}{3-\lambda} \ln(\lambda-2)$$

and

$$\frac{dy}{d\lambda} = (\lambda-2)^{\frac{\lambda-1}{3-\lambda}} \cdot \left(\frac{2}{(3-\lambda)^2} \ln(\lambda-2) + \frac{\lambda-1}{(3-\lambda)(\lambda-2)}\right)$$

Therefore,

$$\frac{d\rho}{d\lambda} = (\lambda-2)^{\frac{\lambda-1}{3-\lambda}} \cdot \left(\frac{2}{(3-\lambda)^2}\ln(\lambda-2) + \frac{\lambda-1}{(3-\lambda)(\lambda-2)}\right) - \left(\frac{1}{\overline{K}}\right)^{\lambda-1} \ln\left(\frac{1}{\overline{K}}\right) =$$

$$= (\lambda-2)^{\frac{\lambda-1}{3-\lambda}} \cdot \frac{1}{3-\lambda} \cdot \frac{2(\lambda-2)\ln(\lambda-2) + (\lambda-1)(3-\lambda)}{(3-\lambda)(\lambda-2)} - \left(\frac{1}{\overline{K}}\right)^{\lambda-1} \ln\left(\frac{1}{\overline{K}}\right)$$

Since $2 < \lambda < 3$ and $\overline{K} \geq 1$, this derivative is positive as soon as $f(\lambda) = 2(\lambda-2)\ln(\lambda-2) + (\lambda-1)(3-\lambda) > 0$. Notice that $f(\lambda)$ is monotonically decreasing in λ for any $2 < \lambda < 3$. Indeed:

$$\frac{df(\lambda)}{d\lambda} = 2(\ln(\lambda-2) - \lambda + 3) < 0 \quad \forall 2 < \lambda < 3$$

where $2(\ln(\lambda-2) - \lambda + 3) < 0$ due to the fact that $2(\ln(\lambda-2) - \lambda + 3)$ is monotonically increasing in λ $\forall 2 < \lambda < 3$ and its maximal value at $\lambda = 3$ is equal to 0.

So, $f(\lambda)$ is decreasing in λ for all $2 < \lambda < 3$. Therefore, for $2 < \lambda < 3$, $f(\lambda) > f(3) = 0$. This implies that $d\rho/d\lambda$ is strictly positive for any $2 < \lambda < 3$.

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