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Constrained Interactions and Social Coordination

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Constrained Interactions and Social Coordination *

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Abstract

We consider a co-evolutionary model of social coordination and network formation where agents may decide on an action in a 2×2- coordination game and on whom to establish costly links to. We find that a payoff dominant convention is selected for a wider parameter range when agents may only support a limited number of links as compared to a scenario where agents are not constrained in their linking choice. The main reason behind this result is that constrained interactions create a tradeoff between the interactions an agent has and those he would rather have. Further, we discuss convex linking costs and provide sufficient conditions for the payoff dominant convention to be selected in $m \times m$ coordination games.

Keywords: Coordination Games, Equilibrium Selection, Learning, Network Formation.
JEL Classification Numbers: C72, D83.

1 Introduction

In many situations people can benefit from coordinating on the same action. Typical examples include common technology standards (e.g. Blue-ray Disc vs. HD DVD), the choice of a telecommunication provider in the presence of discriminatory pricing, the choice of common legal standards (e.g. driving on the left versus the right side of the road), or a common social norm (e.g. the affirmative versus the disapproving meaning of shaking one's head in different parts of the world.) These situations give rise to coordination games with multiple strict Nash equilibria.

A broad range of global and local interaction models (see e.g. Kandori, Mailath, and Rob (1993), Young (1993), Blume (1993, 1995), Ellison (1993, 2000), or Alós-Ferrer and Weidenholzer (2007)) finds that in coordination games (potentially) inefficient risk dominant conventions will emerge in the long run when agents use myopic best response rules and occasionally make

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mistakes.[1] The main reason behind this result is that risk dominant strategies perform well in a world of uncertainty, where there is the possibility of coordination error, and will eventually take over the entire population.

In the present paper, we present a model where agents in addition to their action choice in a 2×2 coordination game may directly choose the set of their opponents.[2] We model this by assuming that agents decide on whom to maintain (costly) links to, thereby giving rise to a model of non-cooperative network formation à la Bala and Goyal (2000). Our main analysis turns around the comparison of two scenarios.

In our first scenario, which is a slight modification of Goyal and Vega-Redondo's (2005) model, we discuss the case when agents may in principle support links to everybody in the population. In line with Goyal and Vega-Redondo (2005), we find that for relatively low costs of link formation the risk dominant complete network is selected whereas for relatively high costs of link formation the payoff dominant convention is selected. The main reason for this result is that if costs are low agents obtain a positive payoff from linking to other players irrespective of their strategy. Hence, the complete network will always form and we are basically back in the framework of global interactions where the risk dominant strategy is uniquely selected. If costs of forming links are however high agents may not want to support links to agents using a different strategy, which renders the advantage of the risk dominant strategy obsolete.

In our second scenario we discuss a setting where agents may only maintain a limited number of links.[3] The underlying idea is that in many situations the set of interaction partners a typical economic agent is linked to is fairly small compared to the overall population.[4] For instance, there is hardly anybody who is linked to everybody on facebook. In this constrained link scenario agents will have to carefully decide on whom to establish a link to. In this sense, constrained interactions create a tradeoff between the links an agent has and those he would rather have. This in turn implies that if there is a (relatively) small number of agents choosing the payoff dominant strategy all agents will want to establish links with those agents and switch to (or remain at) the payoff dominant action. On the contrary, it becomes very difficult to leave the payoff dominant convention as it will always spread back from a relatively small number of agents using it. This creates a fairly strong force allowing agents to reach efficient outcomes.

In the present paper we provide a full characterization of the set of long run outcomes depending on the cost of link formation *and* on the number of links agents may support. We show that regardless of the costs of link formation the payoff dominant convention is selected

[1]See Eshel, Samuelson, and Shaked (1998) and Alós-Ferrer and Weidenholzer (2008) for local interaction models showing that imitation learning may lead to efficient outcomes.

[2]See also Jackson and Watts (2002) and Goyal and Vega-Redondo (2005), or Hojman and Szeidl (2006).

[3]The implications of such a setting have already been discussed by Jackson and Watts (2002) in a model where action- and link- choice are not simultaneous (see the discussion below.)

[4]See e.g. Goyal, van der Leij, and Moraga-Gonzalez (2006) who study the coauthor network in economics and find that on average economists have less than two coauthors. Further, note that in *small world networks* the number of links supported by individual agents is also small, (see e.g. Watts and Strogatz (1998)).

if agents may only support relatively few links. In contrast, we find that the risk dominant convention is only selected if agents may support many links and the costs of link formation are low. Interestingly, we find that if agents may only support links to less than half of the overall population the payoff dominant convention will always be selected. The main reason behind this insight is that moving into the basin of attraction of the payoff dominant convention takes less than half of the population to switch whereas leaving the basin of attraction of the payoff dominant convention takes more than half.

We further provide two extensions to our model. First, we consider a setting where the costs of link formation are convex in the number of links. In this setting the number of links an agent maximally supports arises endogenously. Here we find that if the costs of link formation are "high" so that agents will not link to everybody in the population the payoff dominant convention is selected whereas for relatively low linking costs agents will link to everybody in the population and the selection of the risk dominant convention remains. Further, as a robustness check of our model, we revisit the (exogenously) constrained linking scenario, to show that in fact payoff dominant conventions will be selected in general $m \times m$ coordination games if agents interact with less than half of the population.

The present work is closely related to the recent literature on social coordination and network formation (see Jackson and Watts (2002), Goyal and Vega-Redondo (2005), and Hojman and Szeidl (2006)). As in the present paper, in Goyal and Vega-Redondo (2005) and Hojman and Szeidl (2006)[5] agents may unilaterally decide on whom to maintain links to. In Jackson and Watts (2002) the consent of both parties is needed to form a link, which stipulates the use of Jackson and Wolinsky's (1996) concept of pairwise stability. However, the crucial difference between the first approach and the second approach is that in Goyal and Vega-Redondo (2005) and Hojman and Szeidl (2006) agents may change strategies and links at the same time whereas in Jackson and Watts (2002) an updating agent may either decide on his action or on a link.

The only difference between our unconstrained link scenario and the work by Goyal and Vega-Redondo (2005) is that in their paper the payoff received by agents is on both sides (the active and the passive one) of the link whereas in our setup the payoff is only on the active side.[6] Goyal and Vega-Redondo (2005) shortly discuss the implications of having only active links and put forward the interpretation of such a network as *peer network*, i.e. a network where influence is uni-directional and if k regards j as his peer than j need not be in influenced by the action of k. Jackson and Watts (2002) show that for low linking costs the risk dominant convention is

[5]The setup of Hojman and Szeidl (2006) is very similar to the one of Goyal and Vega-Redondo (2005). The payoff structure of their model extends to a situations where agents also obtain payoffs form path connected agents.

[6]Qualitatively, our results are the same, though. The main difference lies in the magnitude of the thresholds for equilibrium selection we identify. See also the discussion in the extensions section of Goyal and Vega-Redondo (2005).

selected whereas for high linking costs the payoff dominant and the risk dominant convention are selected. Goyal and Vega-Redondo (2005) demonstrate that the fact that Jackson and Watts' (2002) model does not uniquely select the payoff dominant strategy for high linking costs is inherent in the assumption that agents may not choose links and actions in the base game at the same time. In particular, the nature of transition from one convention to another is different. In Jackson and Watts (2002) this transition is stepwise: starting with a connected component of size two other players mutating will join one-by-one and we gradually reach the other convention, whereas in Goyal and Vega-Redondo's (2005) model and in the present paper, once a sufficiently large number of players plays one action all other players will immediately follow.

Jackson and Watts (2002) also discuss the implications of constrained interactions in their model. They do however not find any relevant effect of constrained interactions on the predictions of their model (with of course the exception being the number of links agents form). The reason why constrained links do not play an important role lies again in the assumption that agents may not adjust actions and links at the same time, thereby giving rise again to a step-by-step transition from one convention to another.

A different branch in the literature has presented us with models where agents in addition to their strategy choice may choose among several locations where the game is played (see e.g. Oechssler (1997, 1999) Dieckmann (1999), Ely (2002), or Bhaskar and Vega-Redondo (2004)). In these models, the most likely scenario will be the emergence of payoff dominant conventions. The reason behind this result is that agents using risk dominant strategies may no longer prompt their interaction partners to switch strategies but instead to simply move away. In this sense, agents can *vote by their feet* which allows them to coordinate at efficient outcomes.

If, however, one is prepared to identify free mobility with low linking costs, this leaves a puzzle to explain: Ely (2002) selects the payoff dominant convention, while in Goyal and Vega-Redondo (2005) and in our unconstrained interactions the risk dominant convention is selected. The main reason for this discrepancy lies in the fact that Ely (2002) considers average payoffs whereas Goyal and Vega-Redondo (2005) and we consider additive payoffs.[7] In the framework of Goyal and Vega-Redondo (2005) and in our unconstrained interactions scenario the additive payoff structure implies that all links are valuable for sufficiently low linking costs, giving rise to the risk dominant convention. On the contrary, in Ely's (2002) model the number of potential opponents does not matter and players will always prefer to interact with a small number of players choosing the payoff dominant strategy than with a large number choosing the inefficient strategy. Note that in our constrained links scenario a similar mechanism is at work. If the number of permitted links is relatively small only a small fraction of agents using

[7] Ely (2002) also considers a variation with an additive payoff structure. He shows that there exists a parameter region such that the risk dominant strategy is selected. In addition, he shows in Theorem 2 that the simultaneous choice of locations and actions is essential for the selection of the efficient convention.

the payoff dominant action will prompt other agents to link up with them and switch to the payoff dominant action.

The rest of the paper is organized as follows. Section 2 introduces the model and discusses the techniques used. In Section 3 we present our main results. Section 4 spells out two extensions of the model and Section 5 concludes.

2 Model Setup

2.1 The game

Our model is set in the following environment. We consider a set I of agents with $|I| = N$ who play a 2×2 symmetric coordination game against each other. In addition to choosing an action in the coordination game agents can decide on the set of their interaction partners.

Each player i can choose an action $a_i \in \{A, B\}$ in the coordination game. We denote by $u(a_i, a_j)$ the payoff agent i receives from interacting with agent j. The following table describes the payoffs of the coordination game.

	A	B
A	a	c
B	d	b

We assume that the payoffs satisfy the following ordering:

$$b > a > c > d > 0$$

Note that as $a > d$ and $b > c$ we have that both, (A, A) and (B, B), are strict Nash equilibria, NE, where the latter is Pareto-dominant. Further, we assume that

$$a + c > d + b$$

so that the equilibrium (A, A) is risk dominant in the sense of Harsanyi and Selten (1988), i.e. A is the unique best response against an opponent playing both strategies with equal probability. We let

$$q^* = \frac{b - c}{a - d + b - c}$$

denote the critical mass placed on A in the mixed strategy equilibrium. Note that risk dominance of (A, A) translates into $q^* < \frac{1}{2}$.

We assume that in addition to their choice in the coordination game agents can decide on whom to link to. If player i forms a link to player j we write $g_{ij} = 1$ and we write $g_{ij} = 0$ if player i decides not to form a link to player j. We assume that players may not be linked to themselves, i.e. $g_{ii} = 0$ for all $i \in I$. The linking decision of agent i can be summarized by the n-tuple $g_i = (g_{i1}, g_{i2}, \ldots, g_{in}) \in \mathcal{G}_i = \{0,1\}^n$ where g_{ii} is always zero. We denote by $g = (g_i)_{i \in I}$ the network induced by the link decisions of all agents. A pure strategy of an agent consists of her action choice in the coordination game, $a_i \in \{A, B\}$, *and* of her linking decisions, i.e. $s_i = (a_i, g_i) \in \mathcal{S}_i = \{A, B\} \times \mathcal{G}_i$. A strategy profile is a tuple $s = (s_i)_{i \in I} \in \prod_{i \in I} \mathcal{S}_i = \mathcal{S}$. For a given strategy profile s we denote by $n(s)$ the number of A-players given strategy profile s. Conversely, the number of B-players under strategy profile s is given by $N - n(s)$. We denote by $N(i) = \{j \in I | g_{ij} = 1\}$ the set of neighbors of an agent i, i.e. the set of agents agent i has established a link to. We refer to the number $d_i = |N(i)| = \sum_j g_{ij}$ as the *out-degree* of player i.

We assume that the payoff of an agent is given by the sum of payoffs she receives from interacting with each of her neighbors minus a cost incurred from linking up to these neighbors. In particular, we assume that the cost of linking to d_i other agents is given by the cost function $\phi(d_i)$. So, given a strategy profile $s = (s_i)_{i \in I}$ the total payoff for player i is given by

$$U_i(s_i, s_{-i}) = \sum_{j=1}^{N} g_{ij} u(a_i, a_j) - \phi(d_i).$$

Note that, in contrast to Jackson and Watts's (2002) model of two sided links, in our model the link decision of agents is unilateral, i.e. it does not take the consent of the other party to form a link. Thus, the networks arising in our model are directed graphs. Further, note that we also assume that the payoff from interaction is only on the active side.[8] Note that this implies that the action of the active party does not influence the payoff of the passive party. Goyal and Vega-Redondo (2005) put forward the interpretation of such networks as a model of peer groups and fashion, where asymmetric flow of influence seems a natural feature.

A model of one sided links has the advantage that the passive party does not benefit nor suffer from a link and thus has no incentives to reject it. If (negative) payoff was also generated on the passive side then the passive side should also have the possibility to terminate a link, giving rise to a model of two sided passive links.

2.2 Learning

We consider a model of *noisy best response learning* in discrete time à la Kandori, Mailath, and Rob (1993), Young (1993). Each period $t = 0, 1, 2, \ldots$ an agent receives the opportunity to update her strategy with independent probability $\lambda \in (0,1)$. When such a revision opportunity arises we assume that each agent chooses a strategy (i.e. an action in the base game *and* the set

[8] This is actually the only difference to the setup chosen by Goyal and Vega-Redondo (2005). In fact Goyal and Vega-Redondo (2005) shortly discuss a model of one sided active links in their extensions section.

of agents she links to) that would have maximized her payoff in the previous period.[9,10] More formally, in period t agent i chooses

$$s_i(t) \in \arg\max_{s_i \in \mathcal{S}_i} U_i(s_i, s_{-i}(t-1))$$

where $s_{-i}(t-1)$ is the strategy profile used by all other agents except i in the previous period. If multiple strategies are suggested by the adjustment process we assume that agents choose one at random. We refer to this adjustment process as the *unperturbed process.*

With independent probability $\varepsilon \in (0,1)$ the updating agent ignores the prescription of the adjustment process and chooses a strategy at random. Following the previously received literature, we call such unintentional choices trembles, mistakes or mutations. If $\varepsilon = 0$ we obtain a best-reply process without mistakes, called the *unperturbed process.* We refer to the process with mistakes as the perturbed process.

The process defined above gives rise to a finite state time–homogenous Markov chain with stationary transition probabilities. We are interested in the limit invariant distribution of the perturbed process as the probability of mistakes tends to zero. In the following we give a brief summary of the techniques employed in the main section of the paper.[11]

2.3 Review of Techniques

An *absorbing set* of the unperturbed process is a minimal subset, in the sense of set inclusion, of states which, once entered, is never abandoned. An *absorbing state* is a singleton absorbing set. States that are not in any absorbing set are called *transient.*

Every absorbing set of a Markov chain induces an *invariant distribution*, i.e. a distribution over states $\mu \in \Delta(\Omega)$ which, if taken as initial condition, is reproduced in probabilistic terms in the next period, i.e. $\mu \cdot P = \mu$. The invariant distribution induced by an absorbing set W has support W. Since experiments make transitions between any two states possible, under the perturbed process the only absorbing set is the whole state space (i.e. the perturbed process is irreducible and aperiodic). We denote the (unique) invariant distribution of the perturbed process by $\mu(\varepsilon)$.

The *limit invariant distribution* (as the rate of experimentation tends to zero) $\mu^* = \lim_{\varepsilon \to 0} \mu(\varepsilon)$ exists and is an invariant distribution *of the unperturbed process* P (see e.g. Freidlin and Wentzell (1988), Kandori, Mailath, and Rob (1993), Young (1993), or Ellison (2000)). It singles out a stable prediction of the original process, in the sense that, for any ε small enough, the play approximates that described by μ^* in the long run. The states in the support of μ^*, $\{\omega \in \Omega \mid \mu^*(\omega) > 0\}$

[9]In this sense agents do not attempt to conduct a forecast of future behavior but rather base their decision on the current pattern of play.

[10]Note that unlike the process defined in Jackson and Watts (2002) in our setup agents may choose the action in the base game and decide on their links at the same time.

[11]For textbook treatments of these concepts see e.g. Samuelson (1997), Young (1998), or Sandholm (2009).

are called *Long Run Equilibria (LRE)* or stochastically stable states. The set of stochastically stable states is a union of absorbing sets of the unperturbed process P.

Ellison (2000) presents a powerful method to determine the set of LRE which is based on a characterization by Freidlin and Wentzell (1988). Let X and Y be two absorbing sets and let $c(X,Y) > 0$ be the minimal number of mistakes needed for a direct transition from X to Y (i.e. the cost of transition). Define a path from X to Y as a finite sequence of absorbing sets $P = \{X = S_0, ..., S_K = Y\}$ and let $S(X,Y)$ be the set of paths from X to Y. Given a path P, define its length $l(P)$ as the number of elements of the sequence minus 1, so that $P = \{X = S_0, ..., S_{l(P)} = Y\}$. The cost of a path is given by the sum of its transition costs

$$c(P) = \sum_{k=1}^{l(P)} c(S_{k-1}, S_k).$$

The minimal number of mistakes required for a (possibly indirect) transition from X to Y

$$C(X,Y) = \min_{P \in S(X,Y)} c(P).$$

The *Radius* of an absorbing set X is defined as

$$R(X) = \min\{C(X,Y) | Y \text{ is an absorbing set, } Y \neq X\},$$

i.e. the minimal number of mistakes needed to leave X. The *coradius* of X is defined as

$$CR(X) = \max\{C(Y,X) | Y \text{ is an absorbing set, } Y \neq X\},$$

i.e. the maximal number of mistakes needed to reach X. Ellison (2000) shows that

Lemma 1. *(Ellison 2000). If $R(X) > CR(X)$ the only long run equilibria (LRE) are contained in X.*

Note that $R(X) > CR(X)$ simply expresses the idea that for X to be LRE X should be easier to reach than to leave by simultaneous mutations.

3 Network Formation and Social Coordination

As a benchmark, we will first consider a scenario where the costs of link formation are linear in the number of links and players may initiate all links, similar to the model of Goyal and Vega-Redondo (2005). We will then move on to discuss a scenario where each agent may only have a limited number of links. Before we begin with our main analysis of these two scenarios we will however devote some more time to a discussion of the best response of an agent when actions and links may be adjusted at the same time.

3.1 Link Optimized Payoffs and Optimal Actions

Note that in our model agents may choose both their actions and the set of agents they want to link up to at the same time. Consequently, when analyzing under which conditions an agent will choose a particular action one has to take into account her optimal linking decision. The decision procedure can therefore be split in two parts: First, to determine the optimal set of links for both actions, A and B given the distribution of play in the population. And second, to decide which of the two actions to play, given the optimal set of links. We solve the first part of this problem by introducing the concept of a *link optimized payoff function*, for short LOP. The LOP, which we denote by $v_i(a_i, n(s))$, of an agent i using action $a_i \in \{A, B\}$ is given by the maximally attainable payoff when linking up to other agents given that $n(s)$ agents play A (and $N - n(s)$ agents play B), i.e.

$$v_i(a_i, n(s)) = \max_{g_i \in \mathcal{G}_i} U_i\left((a, g_i), n(s)\right).$$

Given the optimal set of links for agent i, we can then solve the second part of our problem, which consists of finding the optimal action. Here, we follow Sandholm (1998) and consider the following best response rule which takes into account that agent i's decision today will influence tomorrow's distribution of play.[12]

- If $a_i = A$ switch to B if $v_i(B, n(s) - 1) > v_i(A, n(s))$. Otherwise stay with A.
- If $a_i = B$ switch to A if $v_i(A, n(s) + 1) > v_i(B, n(s))$. Otherwise stay with B.

As mentioned above, we assume that eventual ties are broken at random.

3.2 Unconstrained Interactions

In the first scenario, we consider the case where the cost of linking to d_i other agents is given by $\phi(d_i) = \kappa d_i$ with $\kappa \geq 0$. As we are interested in situations where there is a conflict between the two conventions, we focus on cases where the cost of forming links is smaller than the smallest of the two equilibrium payoffs of the base game, i.e. $\kappa \leq a$.[13] In the linking decision of the agents the magnitude of the linking cost will turn out to play a crucial role. For relatively low linking costs ($0 \leq \kappa \leq d$) agents wish to form all links to all other agents regardless of the action they are choosing in the base game. For intermediate linking cost ($d \leq \kappa \leq c$), A-players will still link up to all other agents regardless of their strategy whereas B-players only wish to link up to agents of their own kind. And for high linking costs, $c \leq \kappa \leq a$, agents only want to link

[12] Kandori, Mailath, and Rob's (1993) original process was actually based on imitation.

[13] If $a < \kappa \leq b$ it will only pay off for B-players to form links to other B-players. Hence, in this scenario the fully connected network where everybody chooses B and the empty network where everybody chooses A are the only Nash equilibria. Note that as the latter can be upset with only mutation to B the former will be LRE. If $\kappa > b$ no links will form and every action configuration where no links has formed is a Nash equilibrium.

up to other agents of their own kind and interaction between groups choosing different actions is completely shut down. Consequently, the LOPs of an A-agent and a B-agent are given by

$$v(A,n) = [a-\kappa]_+(n-1) + [c-\kappa]_+(N-n)$$
$$v(B,n) = [d-\kappa]_+ n + [b-\kappa]_+(N-n-1)$$

where $[x]_+ = \max\{0,x\}$.

We denote by $\overrightarrow{A}$ the state where everybody chooses action A and the network is complete, i.e. all possible links are present, i.e. $\overrightarrow{A} = \{s \in \mathcal{S} | a_i = A \text{ and } d_i = N-1 \quad \forall i \in I\}$. Likewise $\overrightarrow{B}$ denotes the complete monomorphic network where everybody chooses action B, i.e. $\overrightarrow{B} = \{s \in \mathcal{S} | a_i = B \text{ and } d_i = N-1 \quad \forall i \in I\}$. Our first result is the following.

Lemma 2. *For a large enough population, $\overrightarrow{A}$ and $\overrightarrow{B}$ are the only Nash equilibria. Further, from any other state the unperturbed process converges to either $\overrightarrow{A}$ or $\overrightarrow{B}$.*

Proof. The proof proceeds by considering the low cost-, the intermediate cost-, and the high cost scenario in turn.

i) Suppose $0 \le \kappa \le d$. In this case all links will form and we essentially obtain a scenario of global interactions just as in Kandori, Mailath, and Rob (1993). An A-player will switch to B if

$$(n-1)d + (N-n)b - (N-1)\kappa \ge (n-1)a + (N-n)c - (N-1)\kappa.$$

Solving for n yields that an A-player will switch to B if

$$n \le (N-1)q^* + 1 \tag{1}$$

and will keep on using A otherwise. A B-player will switch to A if

$$na + (N-n-1)c - (N-1)\kappa \ge nd + (N-n-1)b - (N-1)\kappa$$

i.e. if

$$n \ge (N-1)q^*. \tag{2}$$

and will remain a B-player otherwise.

ii) Now assume that $d < \kappa < c$. In this case A-players want to link to all other players, whereas B-players only want to link up to other B-players. An A-player will switch to B if and only if

$$(N-n)b - (N-n)\kappa \ge (n-1)a + (N-n)c - (N-1)\kappa$$

i.e. if

$$n \le (N-1)q' + 1 \tag{3}$$

where

$$q' = \frac{b-c}{a-c+b-\kappa}.$$

Note that since $\kappa > d$ we have $q' > q^*$, i.e. if B-players do not link to A-players, then it requires more A-players for A to be a best response as in the previous scenario. Likewise, one can show that B-players will switch to A if

$$n \geq (N-1)q' + 1 \tag{4}$$

iii) If $c \leq \kappa \leq a$ both A- and B-players will only link to agents of their own kind. Here an A-player will switch to B if

$$(N-n)b - (N-n)\kappa \geq (n-1)a - (n-1)\kappa.$$

if we denote by

$$q'' = \frac{b-\kappa}{a+b-2\kappa}$$

(with $q'' > \frac{1}{2} > q^*$) the previous inequality translates into

$$n \leq (N-1)q'' + 1. \tag{5}$$

Likewise it can be shown that a B-player will switch to A if

$$n \geq (N-1)q''. \tag{6}$$

To see that in all three scenarios the unperturbed process converges to either of the two complete monomorphic networks consider an arbitrary state with $n \in \{1, N-1\}$ A-players. In all three scenarios, (by (1) and (2) in the low cost scenario, by (3) and (4) in the intermediate cost scenario, and by (5) and (6) in the high cost scenario) we have that if an agent prefers to stick to her action all agents using the other action will switch. Assume an A-player is presented with the opportunity to revise her strategy. We know that if she decides to stick with her action then i) all other A agents will also stick with their action as they have the same LOP and ii) all B agents will follow. On the contrary, assume that our A-player has decided to switch. As before, this implies that all other A-players offered revision opportunity will also switch. Further, for a sufficiently large population (such that the best response regions do not overlap) also B-player will stick to their action. It follows that for a large enough population the process will converge to either $\overrightarrow{A}$ or $\overrightarrow{B}$. ∎

With the help of the previous lemma we are able to prove the following proposition:

Theorem 1. *In the unconstrained links scenario for a large enough population:*

a) for low linking costs, $0 \leq \kappa \leq d$, $\overrightarrow{A}$ *is LRE.*

b) for intermediate linking costs, $d \leq \kappa \leq c$, $\overrightarrow{A}$ is LRE if

$$\kappa < a + c - b$$

and $\overrightarrow{B}$ is LRE otherwise.

c) for high linking costs, $c \leq \kappa \leq a$, $\overrightarrow{B}$ is LRE.

Proof. First, consider the case when $0 \leq \kappa \leq d$. In order to move from the state $\overrightarrow{B}$ into the basin of attraction of $\overrightarrow{A}$ at least $\lceil (N-1)q^* \rceil$ players have to mutate to A establishing that $R(\overrightarrow{B}) = CR(\overrightarrow{A}) = \lceil (N-1)q^* \rceil$.[14] Likewise, in order to move from $\overrightarrow{A}$ into the basin of attraction of the efficient complete network, $\overrightarrow{B}$ at least $\lceil N-(N-1)q^*-1 \rceil = \lceil (N-1)(1-q^*) \rceil$ players have to mutate to B, establishing $R(\overrightarrow{A}) = CR(\overrightarrow{B}) = \lceil (N-1)(1-q^*) \rceil$. For large enough N, we have $CR(\overrightarrow{A}) < R(\overrightarrow{A})$ as $q^* < \frac{1}{2}$ establishing that under low linking cost $\overrightarrow{A}$ is the unique LRE.

Second, consider the case of intermediate linking costs. $d \leq \kappa \leq c$. Here, we have $R(\overrightarrow{B}) = CR(\overrightarrow{A}) = \lceil (N-1)q' \rceil$ and $R(\overrightarrow{A}) = CR(\overrightarrow{B}) = \lceil (N-1)(1-q') \rceil$. Reconsidering q' reveals that $q' > \frac{1}{2}$ if $\kappa > a+c-b$ and that $q' < \frac{1}{2}$ otherwise. Hence, for $\kappa < a+c-b$ the risk dominant complete network is unique LRE and for $\kappa > a+c-b$ the efficient complete network is unique LRE

Finally consider $c \leq \kappa \leq a$. Here we have $R(\overrightarrow{B}) = CR(\overrightarrow{A}) = \lceil (N-1)q'' \rceil$ and $R(\overrightarrow{A}) = CR(\overrightarrow{B}) = \lceil (N-1)(1-q'') \rceil$. As $q'' > \frac{1}{2}$ the efficient complete network is uniquely selected. ■

Taking a closer look at the conditions identified in Theorem 1 and noting that, by assumption, $d < c < a$ and $a+c > d+b$, we have that if $\kappa < a+c-b$ the risk dominant convention is selected and if $\kappa > a+c-b$ the efficient convention is selected. The main reason behind this result is that if costs are low agents obtain a positive payoff from linking to other agents irrespective of their action. Hence, the complete network will always form and we obtain global interaction where the risk dominant strategy is uniquely selected due to the standard uncertainty considerations. If however costs of forming links are high the agents do not wish to form all links anymore, which gives the efficient strategy a decisive advantage.

3.3 Constrained Interactions

We now turn to a scenario where each agent may only support a limited number of links. We model this situation by assuming that the costs of links exceed the largest payoff in the base

[14] Where $\lceil x \rceil$ denotes the smallest integer larger than x.

game once a certain number of $k < N-1$ links have been established.[15] Formally,

$$\phi(d_i + 1) - \phi(d_i) = \begin{cases} \kappa & \text{if } d_i \leq k \\ \bar{\kappa} & \text{otherwise.} \end{cases}$$

with $\phi(0) = 0$, $\kappa \leq a$, and $\bar{\kappa} > b$. Note that this implies that in equilibrium every player will maximally support k links. It will turn out that considering this setup will significantly alter the results as compared to the unconstrained link scenario.

In the following we denote by $\overrightarrow{A}[k] = \{s \in \mathcal{S} | a_i = A \text{ and } d_i = k \quad \forall i \in I\}$ the set of states where all agents choose action A and each agent supports k links. $\overrightarrow{B}[k]$ is defined in the same way.

Consider first the case of low linking costs, $0 \leq \kappa \leq d$. In this scenario agents will first connect to other agents of their own kind and will only then fill up the remaining slots with agents using different actions. Consequently, the LOPs of an A-player and of a B-player, when confronted with a distribution of play $(n, N-n)$, are given by

$$\begin{aligned} v(A, n) &= a \min\{k, n-1\} + c(k - \min\{k, n-1\}) - \kappa k \\ v(B, n) &= b \min\{k, N-n-1\} + d(k - \min\{k, N-n-1\}) - \kappa k. \end{aligned}$$

For intermediate linking costs, $d \leq \kappa \leq c$, B-players will only link up to other B-players whereas A- player will first link up to all other A-players and will then also link up to B-players, yielding

$$\begin{aligned} v(A, n) &= a \min\{k, n-1\} + c(k - \min\{k, n-1\}) - \kappa k \\ v(B, n) &= (b - \kappa) \min\{k, N-n-1\}. \end{aligned}$$

For high linking costs $d \leq \kappa \leq c$ agents will only interact with agents using the same action and we obtain

$$\begin{aligned} v(A, n) &= (a - \kappa) \min\{k, n-1\} \\ v(B, n) &= (b - \kappa) \min\{k, N-n-1\}. \end{aligned}$$

We can nest the low-, the intermediate, and the high- cost scenario in the following LOPs.

$$\begin{aligned} v(A, n) &= [a - \kappa]_+ \min\{k, n-1\} + [c - \kappa]_+ (k - \min\{k, n-1\}) \\ v(B, n) &= [b - \kappa]_+ \min\{k, N-n-1\} + [d - \kappa]_+ (k - \min\{k, N-n-1\}) \end{aligned}$$

Lemma 3. *For a large enough population N, the states in the sets $\overrightarrow{A}[k]$ and $\overrightarrow{B}[k]$ are the only Nash equilibria. Further, from any other state the process converges to either the absorbing set*

[15] The unconstrained links scenario corresponds to the case where $k = N-1$.

$\overrightarrow{A}[k]$ *or the absorbing set* $\overrightarrow{B}[k]$.

Proof. In a first step, we will provide thresholds when an A-player will switch to B and thresholds when a B-player will switch to A for the the low, the intermediate-, and the high- cost scenario. Depending on the relationship between n, N, and k, we have to analyze four subcases for each of these scenarios.[16] For each of these 12 cases we have to identify conditions on the distribution of play in the population, $(n, N-n)$, under which an A-player will switch to B, i.e. when $v(B, n-1) \geq v(A, n)$, and conditions under which a B-player will switch to A, i.e. when $v(A, n+1) > v(B, n)$. We report our findings in Table 1 and have relegated the straightforward derivation of these thresholds into the appendix.

Switching thresholds for A-players				
$v(B,n-1) > v(A,n)$	$k \geq n-1$ $k \geq N-n$	$k \geq n-1$ $k < N-n$	$k < n-1$ $k \geq N-n$	$k < n-1$ $k < N-n$
$0 \leq \kappa \leq d$	$n < (N-1)q^* + f(k,d) + 1$	$a.s.$	$n < N - \frac{a-d}{b-d}k$	$a.s.$
$d < \kappa \leq c$	$n < (N-1)q' + f(k,\kappa) + 1$	$a.s.$	$n < N - \frac{a-\kappa}{b-\kappa}k$	$a.s.$
$c < \kappa \leq a$	$n < (N-1)q'' + 1$	$a.s.$	$n < N - \frac{a-\kappa}{b-\kappa}k$	$a.s.$
Switching thresholds for B-players				
$v(A,n+1) > v(B,n)$	$k \geq n$ $k \geq N-n-1$	$k \geq n$ $k < N-n-1$	$k < n$ $k \geq N-n-1$	$k < n$ $k < N-n-1$
$0 \leq \kappa \leq d$	$n > (N-1)q^* + f(k,d)$	$n.s$	$n > N-1-\frac{a-d}{b-d}k$	$n.s.$
$d < \kappa \leq c$	$n > (N-1)q' + f(k,\kappa)$	$n.s$	$n > N-1-\frac{a-\kappa}{b-\kappa}k$	$n.s.$
$c < \kappa \leq a$	$n > (N-1)q''$	$n.s$	$n > N-1-\frac{a-\kappa}{b-\kappa}k$	$n.s.$

Table 1: Where $f(k,x) \equiv \frac{(N-k-1)(c-x)}{(a-c+b-x)}$, "a.s." means that an A-player always switches to B, and "n.s." means that a B-player never switches to A.

In each of the above cases, we find that if it is optimal for an agent to remain at her action then it is optimal for players using a different action to switch. For a large enough population, the best response regions of the two actions do not overlap and we have that if it is optimal for an agent to switch actions then it is also optimal for an agent with the other action to remain. Now, note that in situations where everybody chooses the same action all agents will support k links, establishing that the only NE are those states in $\overrightarrow{A}[k]$ and $\overrightarrow{B}[k]$. Further, note that from any state $s \notin \overrightarrow{A}[k] \cup \overrightarrow{B}[k]$ the process will converge to either the set $\overrightarrow{A}[k]$ or the set $\overrightarrow{B}[k]$. Finally, note that under our best response process ties are broken randomly. As agents do not care about the identity of their opponents they are indifferent between having links to, say,

[16] In the first subcase, with $k \geq n-1$ and $k \geq N-n$ neither A- nor B- players may fill not all their slots with agents of their own kind. In the low cost case they will fill them up with agents of the other kind. In the intermediate case, A-agents will fill them up and B agents leave them empty and in the high cost case A- and B-players leave them empty. In the second scenario, we have $k \geq n-1$ and $k \leq N-n$ implying that A-players do not find enough A-players to fill up all their slots whereas B-players can fill up all their slots with other B-players. In the low and intermediate cost case A-players will fill up the remaining slots with B-players whereas in the high cost scenario they will leave them empty. In the third case, with $k \leq n-1$ and $k \geq N-n$, A-players will link only to other A players whereas B-players can not fill up all their slots with agents choosing the same action. In the low cost scenario they will also link up to B-players whereas in the intermediate- and high- cost scenario they will leave them empty. In the remaining case,with $k \leq n-1$ and $k \leq N-n$ both A- and B- players will link up only to agents of their own kind.

agent i and agent j. It follows that for each pair of states $s, s' \in \overrightarrow{A}[k]$ (and also for pair in $\overrightarrow{B}[k]$) that there is positive probability of moving from s to s' without mistakes, i.e. all states in $\overrightarrow{A}[k]$ (and all sates in $\overrightarrow{B}[k]$) form an absorbing set.

■

We are now able to state our main theorem which characterizes the set of long run equilibria in the constrained link scenario.

Theorem 2. *In the constrained links scenario for a large enough population:*

a) for low linking costs $0 \leq \kappa \leq d$ the set $\overrightarrow{B}[k]$ is LRE provided

$$k \leq \frac{N-1}{2}\left(\frac{b-a}{c-d}+1\right) \tag{7}$$

and the set $\overrightarrow{A}[k]$ is LRE otherwise,

b) for intermediate linking costs $d \leq \kappa < c$ the set $\overrightarrow{B}[k]$ is LRE provided

$$k \leq \frac{N-1}{2}\left(\frac{b-a}{c-\kappa}+1\right) \tag{8}$$

and the set $\overrightarrow{A}[k]$ is LRE otherwise,

c) for high linking costs $c \leq \kappa \leq a$ the set $\overrightarrow{B}[k]$ is LRE.

Proof. Using the switching thresholds reported in Table 1 we can study the low-, the intermediate-, and the high- cost scenario in turn. First, consider the low cost scenario $0 \leq \kappa \leq d$ and the set $\overrightarrow{A}[k]$. Depending on whether A-players will link up to only A-players or to both kinds of players after the mutations have occurred we distinguish two cases. In the first case, after the mutations have occurred A-players will only link up to other A-players, i.e. $k \leq n-1$. In this case, once we have less than $\lceil N - \frac{a-d}{b-d}k \rceil$ A-players, A-players given revision opportunity will switch to B. This requires $\lceil \frac{a-d}{b-d}k \rceil$ agents mutating from A to B, establishing $R(\overrightarrow{A}[k]) = CR(\overrightarrow{B}[k]) = \lceil \frac{a-d}{b-d}k \rceil$. On the contrary to move from $\overrightarrow{B}[k]$ to $\overrightarrow{A}[k]$ we need at least $\lceil N - 1 - \frac{a-d}{b-d}k \rceil$ agents to mutate to A, implying $R(\overrightarrow{B}[k]) = CR(\overrightarrow{A}[k]) = \lceil N - 1 - \frac{a-d}{b-d}k \rceil$. Hence, all states in $\overrightarrow{B}[k]$ are LRE provided that

$$k \leq \frac{N-1}{2}\frac{b-d}{a-d}. \tag{9}$$

Let us now consider the question when it can be the case that after the mutations have occurred A-players only link up to other A-players, i.e.

$$k \leq n-1 = N-1-\frac{a-d}{b-d}k.$$

Rearranging terms yields

$$k \leq (N-1)\frac{b-d}{a+b-2d} \tag{10}$$

Note that once the previous inequality is fulfilled also inequality (9) holds, implying that if after the necessary number of mutations have occurred A-players will link up only to A-players, the set $\overrightarrow{B}[k]$ is LRE.

Let us now consider the case when after the mutations have occurred there are not sufficiently many A-players to fill up all slots and A-players will link up to both kinds of players, i.e. $k \geq n-1$. In this case we have $R(\overrightarrow{A}[k]) = CR(\overrightarrow{B}[k]) = \lceil (N-1)(1-q^*) - f(k,d) \rceil$ and $R(\overrightarrow{B}[k]) = CR(\overrightarrow{A}[k]) = \lceil (N-1)q^* + f(k,d) \rceil$. Now the set $\overrightarrow{B}[k]$ is LRE if (7) holds and the set $\overrightarrow{A}[k]$ is LRE otherwise.

Consider now the intermediate cost case with $d \leq \kappa \leq c$. Here B- players will only link up to agents of their own kind whereas A-players may still link up to agents of both kinds. Again, to move into the basin of attraction of $\overrightarrow{B}[k]$ we need less (or equal) than k players to mutate to B. As above, we distinguish two cases: In the first case, $k \leq n-1$, after the mutations have occurred A-players can fill up all their slots with other A-players whereas in the second case, $k \geq n-1$, A-players will also connect to B-players. Consider first the case where $k \leq n-1$ and the set $\overrightarrow{A}[k]$: In order to move into the basin of attraction of $\overrightarrow{B}[k]$ at least $\lceil \frac{a-\kappa}{b-\kappa} k \rceil$ agents have to mutate from A to B, establishing $R(\overrightarrow{A}[k]) = CR(\overrightarrow{B}[k]) = \lceil \frac{a-\kappa}{b-\kappa} k \rceil$. Moving from $\overrightarrow{B}[k]$ to $\overrightarrow{A}[k]$ requires at least $\lceil N-1-\frac{a-\kappa}{b-\kappa} k \rceil$ mutations, implying $R(\overrightarrow{B}[k]) = CR(\overrightarrow{A}[k]) = \lceil N-1-\frac{a-\kappa}{b-\kappa} k \rceil$. Consequently, all states in $\overrightarrow{B}[k]$ are LRE provided that

$$k \leq \frac{N-1}{2}\frac{b-\kappa}{a-\kappa}. \tag{11}$$

As in the low cost case, we check when it can be the case that after the mutations A-players will only link up to agents of their own kind, i.e.

$$k \leq n-1 = N-1-\frac{a-\kappa}{b-\kappa}k.$$

Rearranging terms we arrive at the following inequality.

$$k \leq (N-1)\frac{b-\kappa}{a+b-2\kappa} \tag{12}$$

Note that if the previous inequality holds also inequality (11) holds. Hence, if after the necessary number of mutations have occurred A-players will link up to only A-players and $\overrightarrow{B}[k]$ is LRE.

Let us now consider the case when after the mutations have taken place A-players link up to agents of other agents irrespective of their chosen action, i.e. $k \geq n-1$. Here we have $R(\overrightarrow{A}[k]) = CR(\overrightarrow{B}[k]) = \lceil (N-1)(1-q') - f(k,d) \rceil$ and $R(\overrightarrow{B}[k]) = CR(\overrightarrow{A}[k]) = \lceil (N-1)q' + f(k,d) \rceil$. Now the set $\overrightarrow{B}[k]$ is LRE if (8) holds and the set $\overrightarrow{A}[k]$ is LRE otherwise.

Finally, consider the high cost scenario, $c \leq \kappa \leq a$. Now all agents only wish to link up to other agents of their own kind. First, consider the case, when A-players fill up all their slots after the mutations, $k \leq n-1$. As A-players only connect to A-players this is essentially the same situation as under intermediate costs and we have $R(\overrightarrow{A}[k]) = CR(\overrightarrow{B}[k]) = \lceil \frac{a-\kappa}{b-\kappa} k \rceil$ and $R(\overrightarrow{B}[k]) = CR(\overrightarrow{A}[k]) = \lceil N - 1 - \frac{a-\kappa}{b-\kappa} k \rceil$. Consequently, the set $\overrightarrow{B}[k]$ is LRE provided that

$$k \leq \frac{N-1}{2} \frac{b-\kappa}{a-\kappa}. \tag{13}$$

As in the intermediate cost case we find that A-players will fill up all their slots provided that

$$k \leq (N-1) \frac{b-\kappa}{a+b-2\kappa}.$$

As above, if the previous inequality holds also inequality (13) holds and all states in $\overrightarrow{B}[k]$ are LRE.

Consider now the case when after the mutations have occurred both A- and B- players prefer not to fill up all their slots, i.e $k \geq n-1$. In this case we have $R(\overrightarrow{A}[k]) = CR(\overrightarrow{B}[k]) = (N-1)(1-q'')$ and $R(\overrightarrow{B}[k]) = CR(\overrightarrow{A}[k]) = (N-1)q''$. As $q'' > \frac{1}{2}$ the set $\overrightarrow{B}[k]$ is LRE in this case.

■

The intuition behind this result is the following: If agents may only support a limited number of connections agents will first try to fill up their slots with agents using the same action as they do. If the number of available links is relatively small already a small number of agents using the efficient action will cause other agents to switch to the efficient action. Upsetting the efficient convention is also harder (as compared to the unconstrained interactions scenario) as the efficient action may spread back from relatively small subgroups using it.

It is insightful to reconsider the bound established for low linking costs in (7):

$$k \leq \frac{N-1}{2} \left(\frac{b-a}{c-d} + 1 \right).$$

As $b > a$, $c > d$ and $a + c > b + d$ we have that $\frac{b-a}{c-d} + 1 \in (1, 2)$ implying that (7) holds if $k < \frac{N-1}{2}$. This implies that if agents only may support links to less than half of the population we will always observe efficient outcomes in the long run. This is in sharp contrast to the results obtained in the unconstrained links scenario where the risk dominant convention is always selected for low linking costs. Likewise, in the case of intermediate linking costs (8) holds for $k < \frac{N-1}{2}$. Hence, also in the intermediate cost case we will observe coordination at the efficient action if agents may only interact with less than half of the population.

In Figure 1 we plot the parameter combinations under which either of the two conventions is LRE for general linking costs $0 \leq \kappa a$ and $1 \leq k \leq N-1$ permitted links. Note that the right border of figure 1 corresponds to the unconstrained interaction scenario. In contrast to this unconstrained interaction case the efficient convention is selected for a quite large range of parameter combinations. Further, note that for each level of linking costs κ there exists a number of permitted links k such that the efficient convention is selected.

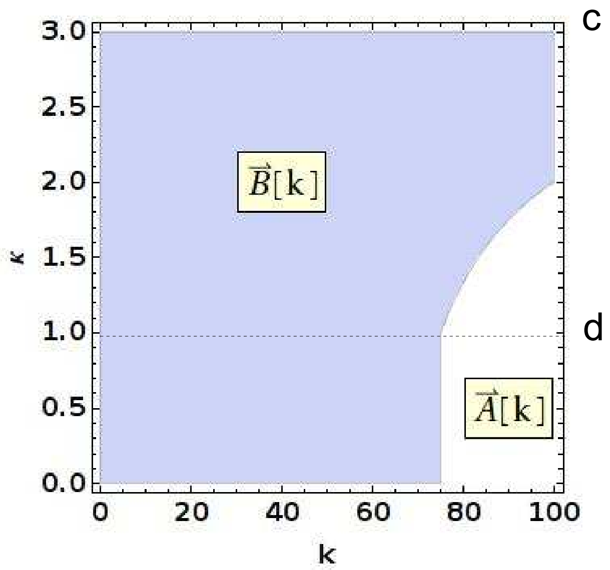

Figure 1: LRE in the game $[a, c, d, b] = [4, 3, 1, 5]$ with $N = 101$.

It might be tempting to think that a population of agents can be made better off by constraining the maximal number of allowed links. For instance, consider a benevolent "network designer" who can influence the maximally allowed number of links and seeks to maximize agents utility in the long run equilibrium. Clearly, as B is the efficient action it earns a higher per interaction payoff to agents than A does. However, if one was to constrain interaction in order to achieve coordination at the efficient equilibrium, there will also be fewer interaction per agents. So it would only pay off to constrain the number of maximally allowed interactions if

$$(b-\kappa)k^* > (a-\kappa)(N-1) \tag{14}$$

where k^* is the integer k that fulfills (7) in the low linking cost scenario (respectively (8) in the intermediate linking cost scenario). Thus we have that constraining interactions increases welfare if

$$\frac{b-\kappa}{a-\kappa} \geq \frac{2(c-d)}{b-a+c-d}$$

in the low linking costs scenario, and if

$$\frac{b-\kappa}{a-\kappa} \geq \frac{2(c-\kappa)}{b-a+c-\kappa}$$

in the intermediate linking cost scenario. The following example highlights that the welfare effects of constraining interactions are ambiguous.

Example 1. Consider a population of $N = 101$, assume $\kappa = 0$, and consider the following two games.

$$u = \begin{array}{c|c|c|} & A & B \\ \hline A & 4 & 3 \\ \hline B & 1 & 5 \\ \hline \end{array} \qquad \tilde{u} = \begin{array}{c|c|c|} & A & B \\ \hline A & 7 & 6 \\ \hline B & 1 & 11 \\ \hline \end{array}$$

In the game u we have $k^* = 75$. In the LRE under unconstrained interactions agents would earn a payoff of $a(N-1) = 400$. In the constrained scenario (with $k = 75$) they could maximally earn $bk = 375$. Thus agents are better off in the unconstrained scenario. Now consider the game $\tilde{u}$. Here we have that $k^* = 90$. In the unconstrained links scenario agents earn 700. However, in the constrained links scenario (with $k = 90$) they would earn 990. In fact, it can be verified from equation (14) that agents would be better off under constrained interactions for any $k^* \in [63, 90]$.

Thus, whether agents are better off under constrained interactions or unconstrained interactions depends on the particular nature of the game.

4 Extensions

4.1 Convex Linking Costs

We now consider an extension where linking cost functions are convex in the number of links supported by an agent. It will turn out that this formulation automatically leads to a version of an constrained interaction model. To be specific we assume that the cost functions are of the form

$$\phi(d_i) = \beta d_i^2. \tag{15}$$

The parameter $\beta \geq 0$ sets an upper bound on the number of links an individuals will want to support in equilibrium. Further, we assume $\beta \leq a$, so that agents will (regardless of their action choice) support at least one link.[17]

We are interested in an *optimal linking strategy*, defining the optimal number of A- and B-links an agent will want to form, given his own strategy a_i and the global action profile $(n, N-n)$. Let us denote by $\vec{d}^*_A(n) := (d^*_{A|A}(n), d^*_{A|B}(n))$ the optimal linking strategy for A-players and by $\vec{d}^*_B(n)$ the optimal linking strategy for B-players, given the action distribution $(n, N-n)$. The following Lemma characterizes these optimal linking strategies for A- and B-players.

Lemma 4. *In the convex linking cost scenario the optimal linking strategies for A and B players*

[17]If $b \leq \beta < a$ only B agents will link to other B agents and consequently the efficient convention is the unique LRE. Further, if $\beta < b$ no agent will have an incentive to form a link.

are given by:

$$d^*_{A|A}(n) = \min\{n_a(\beta), n-1\}, \; d^*_{A|B}(n) = \min\left\{N-n, [n_c(\beta)-(n-1)]_+\right\},$$
$$d^*_{B|A}(n) = \min\left\{n, [n+1-n_d(\beta)]_+\right\}, \; d^*_{B|B}(n) = \min\{N-n_b(\beta), N-n-1\}$$

where

$$n_a(\beta) = \left\lfloor \frac{a-\beta}{2\beta} \right\rfloor, \; n_c(\beta) = \left\lfloor \frac{c-\beta}{2\beta} \right\rfloor, \; n_d(\beta) = N - \left\lfloor \frac{d-\beta}{2\beta} \right\rfloor, \; n_b(\beta) = N - \left\lfloor \frac{b-\beta}{2\beta} \right\rfloor.$$

Proof. We only provide the derivation for A-players and remark that the optimal linking strategies of B-players can be derived analogously. An A-player who links to $d_{A|A}$ other A-players and to $d_{A|B}$ B-players will obtain a payoff of

$$U_i((A, \vec{g}_i), s_{-i}) = a d_{A|A} + c d_{A|B} - \beta d_A^2,$$

where $d_A = d_{A|A} + d_{A|B}$. First, note that as $a > c$, A-players will first establish links to other A-players and will only then consider linking to B-players. If there are n A-players the optimal number of A links $d^*_{A|A}(n)$ has to be such that creating an additional link will not increase the utility of the agent

$$\begin{aligned}
d^*_{A|A}(n) &= \max\left\{ d_{A|A} \in \{0,1,\ldots,n-1\} \,\middle|\, a(d_{A|A}+1) - \beta(d_{A|A}+1)^2 - a d_{A|A} + \beta d^2_{A|A} \leq 0 \right\} \\
&= \max\left\{ d_{A|A} \in \{0,1,\ldots,n-1\} \,\middle|\, d_{A|A} \leq \frac{a-\beta}{2\beta} \right\} \\
&= \min\{n_a(\beta), n-1\}.
\end{aligned}$$

Now, let's consider the question how many links an A-player will establish to B-players. As in the previous case, the optimal number of links an A-player establishes to B-players has to be such that creating an additional link will not increase his utility. Hence, we are searching for the largest integer $d_{A|B} \in \{0,1,2,\ldots,N-n\}$ that satisfies

$$c(d_{A|B}+1) - \beta(d^*_{A|A}(n) + d_{A|B} + 1)^2 - c d_{A|B} + \beta(d^*_{A|A}(n) + d_{A|B})^2 \geq 0.$$

Note that $n_c(\beta) - d^*_{A|A}(n)$ is the largest $d_{A|B}$ that fulfills this inequality. Taking into account that there may in fact not be sufficiently many B-players and that $d_{A|B}$ may not be negative, we have

$$d^*_{A|B}(n) = \min\left\{ N-n, \max\left\{ \left\lfloor \frac{c-\beta}{2\beta} \right\rfloor - d^*_{A|A}(n), 0 \right\} \right\}.$$

If $d^*_{A|A}(n) = \left\lfloor \frac{a-\beta}{2\beta} \right\rfloor = n_a(\beta)$, then $\left\lfloor \frac{c-\beta}{2\beta} \right\rfloor \leq \left\lfloor \frac{a-\beta}{2\beta} \right\rfloor$, and consequently $d^*_{A|B}(n) = 0$. Hence, the optimal number of B links for an A-player, $d^*_{A|B}(n)$,can be written as

$$d^*_{A|B}(n) = \min\left\{N - n, [n_c(\beta) - (n-1)]_+\right\}.$$

■

This Lemma completely characterizes the structure of equilibrium networks, given the distribution of actions $(n, N-n)$. We now move on to discuss which action will be adopted by the agents, given this optimal linking strategy. Thus, we consider the LOPs:

$$\begin{aligned} v(A,n) &= ad^*_{A|A}(n) + cd^*_{A|B}(n) - d^*_A(n)^2 \\ v(B,n) &= bd^*_{B|B}(n) + dd^*_{B|A}(n) - d^*_B(n)^2. \end{aligned}$$

Consider first the optimal linking strategy for an A-player. Note that if $n_a(\beta) \geq N-1$, or equivalently $\beta < \frac{a}{2N-1}$, an A-player will establish all links with other A-players, i.e.

$$d^*_{A|A}(n) = n - 1 \quad \forall n \in \{1, 2, \ldots, N\}.$$

For $\beta \geq \frac{a}{2N-1}$, A-players do not necessarily want to interact with all other A-players and we have that

$$d^*_{A|A}(n) = \begin{cases} n-1 & \text{if } n \in \{1, 2, \ldots, n_a(\beta)\}, \\ n_a(\beta) & \text{if } n \in \{n_a(\beta)+1, \ldots, N\}. \end{cases}$$

Hence, once there are more than $n_a(\beta)$ A-players in the population A-players will not have an incentive to support more connections to other A-players than $n_a(\beta)$. Now consider the question how many links an A-player will establish to B-players. If $\beta < \frac{c}{2N-1}$ A-players wish to link up to all B-players, i.e.

$$d^*_{A|B}(n) = N - n \quad \forall n \in \{1, 2, \ldots, N-1\}.$$

If $n_c(\beta) < N-1$, A-players may not wish to establish links to all other B-players and we have that the optimal number of A to B links is given by

$$d^*_{A|B}(n) = \begin{cases} n_c(\beta) - (n-1) & \text{if } n \in \{1, \ldots, n_c(\beta)\}, \\ 0 & \text{if } n \in \{n_c(\beta)+1, \ldots, N\}. \end{cases}$$

Hence, an A-players will only link up to B players if there are less than $n_c(\beta)$ A-players. Now, let us consider the optimal linking strategy of B-players. If $\beta < \frac{b}{2N-1}$, B-players wish to form all links to other B-players and we have that

$$d^*_{B|B}(n) = N - n - 1, \ \forall n \in \{0, 1, \ldots, N-1\}.$$

If $\beta \geq \frac{b}{2N-1}$ we have that

$$d^*_{B|B}(n) = \begin{cases} N - n_b(\beta) & \text{if } n \in \{0, 1, \dots, n_b(\beta) - 1\}, \\ N - n - 1 & \text{if } n \in \{n_b(\beta), \dots, N\} \end{cases}$$

Let us now turn to the question how many links a B-player will establish to A- players. For $\beta < \frac{d}{2N-1}$ we have that a B-player will link up to all other A-players, i.e.

$$d^*_{B|A}(n) = n, \ \forall n \in \{0, 1, \dots, N-1\},$$

and for $\beta \geq \frac{d}{2N-1}$ we have that

$$d^*_{B|A}(n) = \begin{cases} 0 & \text{if } n \in \{0, 1, \dots, n_d(\beta) - 1\}, \\ n + 1 - n_d(\beta) & \text{if } n \in \{n_d(\beta), \dots, N\}. \end{cases}$$

We are now able to state our first result for the convex linking cost scenario.

Proposition 5. *In the convex linking cost scenario, for* $0 \leq \beta \leq \frac{d}{2N-1}$, $\overrightarrow{A}$ *is the unique LRE.*

Proof. Note that for $\beta \leq \frac{d}{2N-1}$ all agents wish link up to to all other agents regardless of their action choice. Thus, we essentially have a global interactions model and it follows from Theorem 1 that $\overrightarrow{A}$ is unique LRE. ■

The main reason behind this result is that low values of β do not influence the interaction pattern of agents, implying that we are back in Kandori, Mailath, and Rob's (1993) model of global interactions where the risk dominant convention is selected.

We will focus on situations where the linking cost influences the interaction pattern of agents, i.e. we consider $\beta > \frac{d}{2N-1}$. In this case the LOP function of action A can be written piecewise as

$$v(A, n) = \begin{cases} a(n-1) + c(n_c(\beta) - (n-1)) - \beta n_c(\beta)^2 & \text{if } n \in \{1, 2, \dots, n_c(\beta)\}, \\ a(n-1) - \beta(n-1)^2 & \text{if } n \in \{n_c(\beta) + 1, \dots, n_a(\beta)\}, \\ a n_a(\beta) - \beta n_a(\beta)^2 & \text{if } n \in \{n_a(\beta) + 1, \dots, N\}. \end{cases}$$

Note that, by definition of the thresholds $n_c(\beta)$ and $n_a(\beta)$, we have that $v(A, \cdot) > 0$ and that the LOP is a non-decreasing function of n. Further, we see that once there are more than $n_a(\beta)$ A-players in the population the LOP of an A-player is constant. Similarly, for a B-player we have that

$$v(B, n) = \begin{cases} b(N - n_b(\beta)) - \beta(N - n_b(\beta))^2 & \text{if } n \in \{0, 1, \dots, n_b(\beta) - 1\}, \\ b(N - n - 1) - \beta(N - n - 1)^2 & \text{if } n \in \{n_b(\beta), \dots, n_d(\beta) - 1\}, \\ b(N - n - 1) + d(n + 1 - n_d(\beta)) - \beta(N - n_d(\beta))^2 & \text{if } n \in \{n_d(\beta), \dots, N - 1\}. \end{cases}$$

As above, it follows by the definitions of the thresholds $n_d(\beta)$ and $n_b(\beta)$ that $v(B,\cdot) > 0$ and that the LOP is non-increasing in n. Once there are strictly less than $n_b(\beta)$ A-players, the group of B-players will only interact with other B-players.

In the following, we denote by $\vec{A}[d_A^*(N)]$ the set of states where everybody chooses action A and has established $d_A^*(N)$ links, i.e.

$$\vec{A}[d_A^*(N)] = \{s \in \mathcal{S} | a_i = A, \text{ and } \vec{d}_i^*(N) = \vec{d}_A^*(N) \ \forall i \in I\}.$$

The set $\vec{B}[d_B^*(0)]$ is defined accordingly as,

$$\vec{B}[d_B^*(0)] = \{s \in \mathcal{S} | a_i = B, \text{ and } \vec{d}_i^*(0) = \vec{d}_B^*(0) \ \forall i \in I\}.$$

Proposition 6. *In the convex linking cost scenario, for $\beta > \frac{b}{N}$, the set $\vec{B}[d_B^*(0)]$ is the unique LRE.*

Proof. If $\beta > \frac{b}{N}$ we have that $n_b(\beta) > \frac{N+1}{2}$ and $n_a(\beta) < \frac{N-1}{2}$. Hence, the maximal value of the LOP of an A-player is attained at a point below $\frac{N-1}{2}$. For any $n \geq n_a(\beta)$ it is constant (at its maximal value) and given by

$$v^*(A) = a n_a(\beta) - \beta n_a(\beta)^2 = a \left\lfloor \frac{a-\beta}{2\beta} \right\rfloor - \beta \left\lfloor \frac{a-\beta}{2\beta} \right\rfloor^2 = \left(a - \beta \left\lfloor \frac{a-\beta}{2\beta} \right\rfloor \right) \left\lfloor \frac{a-\beta}{2\beta} \right\rfloor.$$

The LOP function for action B is constant at its maximal value for $n \leq n_b(\beta)$ and given by

$$v^*(B) = b(N - n_b(\beta)) - \beta(N - n_b(\beta))^2 = b \left\lfloor \frac{b-\beta}{2\beta} \right\rfloor - \beta \left\lfloor \frac{b-\beta}{2\beta} \right\rfloor^2 = \left(b - \beta \left\lfloor \frac{b-\beta}{2\beta} \right\rfloor \right) \left\lfloor \frac{b-\beta}{2\beta} \right\rfloor.$$

Note that since $b > a$ we have that $v^*(B) > v^*(A)$, i.e. B-players earn a higher maximum payoff than A-players (as expected). Consequently, once there are more than $N - n_b(\beta)$ B-players all agents given revision opportunity will either remain at B or switch to A, establishing $CR(\vec{B}[d^*(0)]) \leq N - n_b(\beta) < (N-1)/2$. Conversely, to leave the basin of attraction of $\vec{B}[d^*(0)]$ we need more than $n_b(\beta)$ B-agents to switch from B to A, establishing $R(\vec{B}[d_B^*(0)]) \geq n_b(\beta) > (N+1)/2$. ■

The main intuition behind this result is that for sufficiently high linking costs agent do not wish to form all links allowing the efficient agents to play out its advantage. We remark that, in contrast to the constrained linking scenario, in the convex linking cost scenario the number of links agents support in equilibrium arises endogenously. Even though we have provided a partial characterization of the set of LRE in the convex linking scenario it seems to us that is very difficult to obtain a full characterization of the set of LRE for $\beta \in [\frac{d}{2N-1}, \frac{b}{N}]$. We are however able to provide the following result for a particular class of coordination games.

Proposition 7. *Suppose the base game payoffs satisfy the ordering* $\frac{d}{c} > \frac{a+d-b}{d}$*, and that* $\beta \in (\frac{d}{2N-1}, \frac{c}{2N-1}]$.

(a) There exists an $\varepsilon_A > 0$ *so that for all* $\beta \in (\frac{d}{2N-1}, \frac{d}{2N-1} + \delta)$ *and* $\delta \in (0, \varepsilon_A(N))$*, the set* $\overrightarrow{A}[d_A^*(N)]$ *is the unique LRE for* N *sufficiently large.*

(b) There exists an $\varepsilon_B > 0$ *so that for all* $\beta \in (\frac{c}{2N-1} - \delta, \frac{c}{2N-1}]$ *and* $\delta \in (0, \varepsilon_B(N))$*, the set* $\overrightarrow{B}[d_B^*(0)]$ *is the unique LRE for* N *sufficiently large.*

Proof. We want to assess when an A-player will switch to B and when B-players will not switch to A. So, we are searching for two thresholds $n_1^*(\beta)$ and $n_2^*(\beta)$, such that

$$v(B, n-1) - v(A, n) \left\{ \begin{array}{c} \geq \\ < \end{array} \right\} 0 \text{ if } n \left\{ \begin{array}{c} \leq \\ < \end{array} \right\} n_1^*(\beta) \tag{16}$$

and

$$v(B, n) - v(A, n+1) \left\{ \begin{array}{c} \geq \\ < \end{array} \right\} 0 \text{ if } n \left\{ \begin{array}{c} \leq \\ > \end{array} \right\} n_2^*(\beta). \tag{17}$$

We are seeking for a solution of the thresholds $n_1^*(\beta), n_2^*(\beta)$ in the range $\{n_d(\beta), \ldots, N-1\}$ and $\beta \in (\frac{d}{2N-1}, \frac{c}{2N-1})$. These two numbers will give us accurate values of the radius and coradius of the absorbing sets, since $CR(\vec{B}[d_B^*(0)]) = R(\overrightarrow{A}[d_A^*(N)]) = N - n_1^*(\beta)$ and $R(\vec{B}[d_B^*(0)]) = CR(\overrightarrow{A}[d_A^*(N)]) = n_2^*(\beta)$.

For these values of marginal linking costs A-players wish to form links to everybody in the population. Consequently, the LOP of an A-player is given by

$$v(A, n) = a(n-1) + c(N-n) - \beta(N-1)^2.$$

If an A-player would switch to B he would calculate his payoffs according to the function

$$v(B, n-1) = b(N-n) + d(n - n_d(\beta)) - \beta(N - n_d(\beta))^2.$$

One can show that $v(B, n-1) \geq v(A, n)$ if

$$n \leq \left\lceil \frac{a - dn_d(\beta) + bN - cN + \beta - n_d(\beta)^2\beta - 2N\beta + 2Nn_d(\beta)\beta}{a+b-c-d} \right\rceil = n_1^*(\beta). \tag{18}$$

It can be verified that $n_1^*(\beta)$ is increasing in β and that $n_1^*(d/(2N-1)) = \left\lceil \frac{a-d+bN-cN}{a+b-c-d} \right\rceil = \lceil (N-1)q^* + 1 \rceil$. Similarly, one can show that $v(B, n) \geq v(A, n+1)$, so that a B-player will not switch to A, if

$$n \leq \left\lceil \frac{c + b(N-1) - cN - (n_d(\beta) - 1)(d + \beta + n_d(\beta)\beta - 2N\beta)}{a+b-c-d} \right\rceil = n_2^*(\beta). \tag{19}$$

Note that $n_2^*(\beta)$ is also increasing in β and $n_2^*(d/(2N-1)) = \lceil (N-1)q^* \rceil$. We see that $n_1^*(d/2N-1)$ and $n_2^*(d/2N-1)$ coincide with the points of separation of the basins of attraction of the two equilibria found in the low costs scenario with unconstrained interactions, as it should be in view of Proposition 5. From Theorem 1 we know that there is an N_0 such that for all $N \geq N_0$ the absorbing set with all agents playing A is the unique LRE. Fix such a population size $N \geq N_0$ and consider

$$\varepsilon_A(N) := \arg\max_{\delta>0}\{n_1^*(d/(2N-1)+\delta) + n_2^*(d/(2N-1)+\delta) - N \leq 0\}.$$

For all $\delta < \varepsilon_A(N)$ the radius of $\overrightarrow{A}[d_A^*(N)]$ exceeds the coradius of the same set. Consequently, we have that for all $\delta \in (0, \varepsilon_A(N))$ the set $\overrightarrow{A}[d_A^*(N)]$ is selected as unique LRE, which proves part (a).

To show part (b) of the proposition, observe that

$$R(\overrightarrow{B}[d_B^*(0)]) - CR(\overrightarrow{B}[d_B^*(0)]) = n_1^*(\beta) + n_2^*(\beta) - N$$

is an increasing function of β. Therefore we consider the value of the thresholds at the higher cost level $\beta = c/(2N-1)$. Direct substititution and some algebraic manipulations gives us

$$n_1^*(c/2N-1) = \left\lceil \frac{4ac - d^2 - 3c^2}{4c(a+b-c-d)} + N\tilde{q} \right\rceil$$

where

$$\tilde{q} = \frac{2bc - c^2 - 2cd + d^2}{2c(a+b-c-d)}.$$

We have $\tilde{q} \in (1/2, 1)$ if and only if [18]

$$\frac{d}{c} > \frac{a+d-b}{d}.$$

Hence, $n_1^*(c/(2N-1)) = \lceil 1 + (N-1)\tilde{q} \rceil$. Similarly, we obtain

$$n_2^*(c/2N-1) = \left\lceil \frac{(c-d)(c+d) - 4c(b-d)}{4c(a+b-c-d)} + N\tilde{q} \right\rceil.$$

Since $\frac{d^2+4c(b-d)-c^2}{4c(a+b-c-d)} \in (-1, 0)$ we get $n_2^*(c/(2N-1)) = \lceil (N-1)\tilde{q} \rceil$. Hence, for $\beta = c/(2N-1)$

[18] That $\tilde{q} < 1$ follows from the fact that

$$1 - \tilde{q} > 0 \Leftrightarrow 2ac > d^2 + c^2$$

which holds since $a > c > d$, and consequently $2ac = ac + ac > d^2 + c^2$. Moreover observe that $0 < \frac{4ac-d^2-3c^2}{4c(a+b-c-d)} < 1 - \tilde{q}$ and consequently $n_1^*(c/(2N-1)) \leq 1 + (N-1)\tilde{q}$. In fact, we have $n_1^*(c/(2N-1)) \in [N\tilde{q}, 1 + (N-1)\tilde{q}]$, so that $n_1^*(c/(2N-1)) = \lceil 1 + (N-1)\tilde{q} \rceil \geq \lceil (N+1)/2 \rceil$.

we get that

$$R(\overrightarrow{B}[d_B^*(0)]) - CR(\overrightarrow{B}[d_B^*(0)]) = \lceil (N-1)\tilde{q} \rceil + \lceil 1 + (N-1)\tilde{q} \rceil - N > 0$$

for N sufficiently large so that $(2\tilde{q}-1)(N-1) \geq 1$ and $\tilde{q} > 1/2$. Now we can invoke a continuity argument; Pick N sufficiently large so that $B[d_B^*(0)]$ is LRE at $\beta = c/(2N-1)$. Then consider

$$\varepsilon_B(N) := \arg\max_{\delta > 0} \{ n_1^*(c/(2N-1) - \delta) + n_2^*(c/(2N-1) - \delta) - N \geq 0 \}.$$

It follows that for all $\delta \in (0, \varepsilon_B(N))$ and $\beta \in (\frac{c}{2N-1} - \delta, \frac{c}{2N-1}]$ the radius of $\overrightarrow{B}[d_B^*(0)]$ exceeds its coradius, and so this set will be the unique LRE. ■

We remark that the proof of Proposition 7 relied on the piecewise definition of the LOPs, and that the case treated in the proposition is certainly not the only possible parameter constellation for which the set $\vec{B}[d_B^*(0)]$ is LRE. However, we think that a general characterization of the set of LRE under convex linking costs is very difficult to obtain and leave this question to further research.

4.2 General $m \times m$ Coordination Games

We will now discuss general $m \times m$ strict symmetric coordination games. Although, it might be difficult to obtain a general characterization of the set of LRE as provided by Theorem 2 we are nevertheless able to provide sufficient conditions under which the efficient convention is LRE under the constrained interaction scenario outlined in Section 3.3.

The action set in the base game $\mathcal{A} = \{a^1, a^2, \ldots, a^m\}$ now contains m actions. We concentrate on strict coordination games, i.e. (symmetric) games where coordination on any pure action is a strict Nash equilibrium. Formally, $u(a,a) > u(a',a)$ for all $a, a' \in \mathcal{A}$ with $a \neq a'$. Morris, Rob, and Shin (1995) have introduced the concept of p-dominance which is a generalization of risk dominance to $m \times m$ games. An action a is said to be p-dominant if it is a best response to any mixed action profile σ with $\sigma(a) \geq p$.

Without loss of generality, we assume that the action a^1 is payoff dominant, i.e. $u(a^1, a^1) > u(a,a)$ for all $a \in \mathcal{A}$. Further, we denote by p, the smallest probability weight for which a^1 is still a best response against any profile with mass p on a^1. Note that as (a^1, a^1) is a strict Nash equilibrium we have that $p < 1$. We denote by $\vec{a}^1[k] = \{s \in \mathcal{S} | a_i = a^1 \text{and } d_i = k \quad \forall i \in I\}$ the set of states where everybody chooses action a^1 and supports k links.

We are interested in two scenarios: In the first scenario the linking costs κ are smaller than all payoffs in the base game, i.e. $\kappa \leq u(a,a')$ for all $a, a' \in \mathcal{A}$ and in the second scenario the linking costs are higher than the smallest payoff in the game but smaller than either of the equilibrium payoffs, i.e. $\kappa > u(a,a')$ for some $a, a' \in \mathcal{A}$ and $\kappa \leq u(a,a)$ for all $a \in \mathcal{A}$. We are now able to state the following result.

Proposition 8. *In the constrained links scenario for a large enough population:*

a) for low linking costs the set $\overrightarrow{a}^1[k]$ is LRE provided that

$$k < \frac{N-1}{2p} \tag{20}$$

b) for high linking costs the set $\overrightarrow{a}^1[k]$ is LRE provided that

$$k < \frac{N-1}{2}. \tag{21}$$

Proof. First, consider the low linking cost case. Note that all agents will establish k links. Further, note that the a^1- agents will first establish links to other a^1- agents and will only then link to other agents. Now, note that a^1 is a best response whenever there is a fraction of p agents of the k permitted links adopting it. Consequently, we have that if pk agents use action a^1 agents using other actions will follow, establishing $CR(\overrightarrow{a}^1[k]) \leq \lceil pk \rceil$. In order to leave the basin of attraction of $\overrightarrow{a}^1[k]$ we need less than pk agents in the overall population to adopt it, establishing $R(\overrightarrow{a}^1[k]) \geq \lceil N-1-pk \rceil$. Hence, for a large enough population we have $R(\overrightarrow{a}^1[k]) > CR(\overrightarrow{a}^1[k])$ if (20) holds.

Second, consider the high linking cost case. Now certain interactions between groups of agents using different actions are shut down. However, as a^1 is payoff dominant, we know that once we have k agents adopting it all other agents will follow. Hence, $CR(\overrightarrow{a}^1[k]) \leq k$. Conversely, to leave the basin of attraction of $\overrightarrow{a}^1[k]$ we at least need $N-1-k$ agents to mutate to something else, establishing $R(\overrightarrow{a}^1[k]) \geq N-1-k$. If (21) holds we have $R(\overrightarrow{a}^1[k]) > CR(\overrightarrow{a}^1[k])$. ∎

Note that the previous result holds also in the presence of alternative q-dominant even if we consider $q \to 0$. Thus, in general coordination games payoff dominance remains the main criterion for equilibrium selection under (sufficiently) constrained interactions.

5 Conclusion

We have presented a model of social coordination and network formation where agents may only support a limited number of links. In many cases, where agents would have been stuck in risk dominant and possibly inefficient conventions, constrained interactions allowes societies to coordinate at efficient convention. The main reason behind this result is that under constrained interactions agents carefully have to decide on whom to establish one of their precious links to, thereby giving the efficient convention the decisive advantage.

We remark that in certain situations it might be more plausible that also the passive party receives some payoff from the interaction, just as in the models by e.g. Jackson and Watts (2002) or Goyal and Vega-Redondo (2005). Such an extension would however drastically complicate the analysis. For, when deciding on their strategy agents will not only consider the action distribution in the overall population but will also have to take into account the actions chosen by the agents they are passively linked to. However, we think that also in this setup constrained interactions might foster the emergence of efficient convention but leave the analysis of this scenario to future research.

6 Appendix

Derivation of the switching thresholds reported in Table 1

We only provide the computations for the switching thresholds of A-players. The switching thresholds of B-players can be computed analogously.

Case 1 First, consider the case of low linking costs $0 \leq \kappa \leq d$. An A-player will switch to B if

$$a \min\{k, n-1\} + c(k - \min\{k, n-1\}) > b \min\{k, N-n\} + d(k - \min\{k, N-n\})$$

Depending on the relationship between N, n, and k we obtain four subcases.

(1i) If $k \geq n-1$ and $k \geq N-n$ neither A nor B players may fill up all their slots with other agents of their own kind. An A-player will switch to B if

$$a(n-1) + c(k-n+1) \leq b(N-n) + d(k-N+n)$$

i.e. if

$$n \leq (N-1)q^* + 1 + f(k, d)$$

where $f(k,x) = \frac{(N-k-1)(c-x)}{(a-c+b-x)}$.

(1ii) If $k \geq n-1$ and $k < N-n$, A-players do not find sufficiently many other A-players to fill up all their slots whereas B-players can fill up all their slots with other B-players. As b is the highest payoff in the base game B-players will always earn the highest payoff whenever they may fill up all their slots, and so A-players always switch to B.[19]

(1iii) If $k < n-1$ and $k < N-n$, A-players will link only to other A players whereas B-players can not fill up all their slots with agents of their own kind. An A-player will switch to B whenever

$$ak > b(N-n) + d(k-N+n)$$

[19]Formally $v(A,n) = a(n-1) + c(k-n+1) - \kappa k$ and $v(B,n) = (b-\kappa)k$. A-players will switch to B iff $b > a\frac{n-1}{k} + (1 - \frac{n-1}{k})c$. On the right-hand side we have a convex combination in $[c,a]$. Since $b > a \geq c$ the claim follows.

i.e. if

$$n > N - k\frac{a-d}{b-d}.$$

(1iv) In the remaining case with $k < n-1$ and $k < N-n$ both A- and B- players will link up only to agents of their own kind. Here we find that A-players always have an incentive to switch to B.

Case 2 For intermediate linking costs $d \le \kappa \le c$ B-players will no longer interact with A-players whereas A-players will still interact with B-players. Consequently, an A-player will switch to B whenever

$$a\min\{k, n-1\} + c(k - \min\{k, n-1\}) - \kappa k > (b-\kappa)\min\{k, N-n\}$$

Let us again consider our four subcases.

(2i) When $k \ge n-1$ and $k \ge N-n$ an A-player will switch to B if

$$n \le (N-1)q' + 1 + f(k,d).$$

(2ii) If $k \ge n-1$ and $k < N-n$ there are sufficiently many B-players so that choosing B always gives the highest payoff.

(2iii) Whenever $k < n-1$ and $k \ge N-n$, A-player will switch to B if

$$n \le N - \kappa\frac{a-\kappa}{b-\kappa}.$$

(2iv) If $k < n-1$ and $k < N-n$ then A-players as well as B-players can completely isolate. Since the B-players earn always a higher payoff, all As will switch to B.

Case 3 Finally we consider the case of high linking costs $c \le \kappa \le a$. In this case any interaction between groups of agents choosing different actions is completely shut down. In this scenario, an A-player will switch to B whenever

$$(a-\kappa)\min\{k, n-1\} \le (b-\kappa)\min\{k, N-n\}.$$

(3i) If $k \ge n-1$ and $k \ge N-n$ we find that an A-player will switch to B if

$$n < (N-1)q'' + 1.$$

(3ii) The same applies as in cases (1ii) or (2ii).

(3iii) A-players will switch to B whenever

$$n \leq N - \frac{a-\kappa}{b-\kappa}k.$$

(3iv) The same applies as in cases (1iv) or (2iv).

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