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Michael Greinecker

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Payoff-Relevant States in Dynamic Games with Infinite Action Spaces*

Michael Greinecker†

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Abstract

Maskin and Tirole have defined payoff-relevant states in discrete time dynamic games with observable actions in terms of a partition of the set of histories. Their proof that this partition is unique cannot be applied, when action spaces are infinite or when players are unable to condition on calendar time. This note provides a unified proof of existence and uniqueness for these cases. The method of proof is useful for problems other than the one treated here. To illustrate this, a well known characterization of common knowledge is generalized.

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1 Introduction

Markov perfect equilibrium and stationary Markov perfect equilibrium are among the most popular solution concepts for dynamic games in applied work. Such strategies depend only on time and states in the case of Markov perfect equilibrium or on states alone in the case of stationary Markov perfect equilibrium. Players do not condition on general histories or on extrinsic variables like the number of sunspots. The underlying philosophy is

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†Institut für Wirtschaftswissenschaften, Universität Wien, Hohenstaufengasse 9, A-1010 Wien, Austria. E-mail: michael.greinecker@univie.ac.at

that only payoff relevant states should matter. Payoffs depend only on the actions taken by all players, possibly including nature. So states are a derived concept in terms of payoff relevance. Maskin and Tirole (2001) have therefore given a definition of the payoff relevant states (Markov states) as cells in a partition of histories. Histories that do neither differ in a payoff relevant way nor are of different length belong to the same partition cell. The relevant partitions are those that are maximally coarse (there is no coarser partition satisfying a certain consistency condition). They show that such a partition exists and is unique as long as all action spaces are finite. Their proof for uniqueness in the case of finite action spaces depends on the fact that only a finite number of different histories of a given length is possible. Therefore, their proof does not generalize to the case of infinite action spaces. If we allow for histories of different length to be in the same partition cell (the appropriate state space for stationary Markov perfect equilibria), the same problem emerges. In this note, a unified proof for existence and uniqueness of the set of payoff-relevant states for arbitrary action spaces with or without dependence on calendar time is given. The proof is based on a result from the theory of partition lattices, originally due to Ore (1942), and allows one to use arguments for the finite case directly. Our proofs are for a deterministic framework. In a short section, we explain how one can extend them to a framework with mixed strategies and stochastic payoffs. At the end we discuss a further application of the method and illustrate its usefulness by generalizing a well-known characterization of common knowledge.

2 The Framework

There are n players. There is either a finite or countably infinite number of periods. The final period is $T \in \mathbb{Z}_+ \cup \{\infty\}$. Let A be the set of actions that can be played by any player at some time. In period 0, agent i has the *action space* $A_i(0) \subset A$ and the initial set of histories is $H(0) = \{\emptyset\}$. For $t > 0$ feasible actions for player i are given by a function $A_i(t) : H(t) \to 2^A$, where $A_i(h_t)$ is the set of actions available to player i at t given the history $h_t \in H(t)$. Define $H(1) = A(0)$ and for $t > 1$ define $H(t)$ to be the set of all finite sequences $(a(0), \dots, a(t-1))$ such that $(a(0), \dots, a(t-2)) \in H(t-1)$ and $a_i(t-1) \in A_i(t-1)(a(0), \dots, a(t-2))$ for all i. A *play* is a sequence $(a(0), \dots, a(T))$ such that $(a(0), \dots, a(t)) \in H(t+1)$

for all $t < T$. A (pure) *strategy* s_i for player i maps each history $h_t \in H(t)$ to an element of $A_i(t)(h_t)$. A *continuation strategy* for history h_t, written $s_i|h_t$, is the restriction of a strategy s_i to the set of all histories of the form $(h_t, a_1, \dots, a_m)$ for some finite sequence[1] $a_1, \dots, a_m$ such that $(h_t, a_1, \dots, a_m) \in H(t+m)$. A continuation strategy tells a player what to do for every history that extends history h_t. Finally, each player i has a complete and transitive preference ordering on the set of plays.

Let X be a nonempty set. Denote the set of all partitions (disjoint covers) on X by $\Pi(X)$. The elements of a partition are called *cells*. For $\Pi_1, \Pi_2 \in \Pi(X)$, say that Π_1 is *coarser* than Π_2, or Π_2 *finer* than Π_1, if every cell in Π_2 is the subset of a cell in Π_1 and write $\Pi_2 \sqsubseteq \Pi_1$ in that case.

Let $H = \bigcup_t H(t)$ be the set of all histories. The set of payoff-relevant states will be a maximally coarse partition of H satisfying a certain consistency requirement. Since what is payoff relevant may be different for different players, different players will in general have different partitions. So we will work with lists of partions of H.[2] A function with domain H is *measurable* with respect to a partition of H if it is constant on all partition cells. A *collection* $\mathbb{H} = \{\mathbb{H}_1, ..., \mathbb{H}_n\}$ is an element of $\Pi(H)^n$. Say that one such collection is coarser than another if each agent's partition in the former is coarser than the corresponding partition in the latter. Say that the collection $(\mathbb{H}_1, \dots, \mathbb{H}_n)$ is *consistent* if (i) for any player i and any two histories $h_t, h_{t'}$ in the same cell of $\mathbb{H}_i$ and any finite sequence $(a_1, \dots, a_m)$ one has that $(h_t, a_1, \dots, a_m) \in H(t+m)$ and $(h_{t'}, a_1, \dots, a_m) \in H(t'+m)$ imply $A_i(t+m)(h_t, a_1, \dots, a_m) = A_i(t'+m)(h_{t'}, a_1, \dots, a_m)$ and (ii) whenever all other players $j \neq i$ play strategies measurable with respect to $\mathbb{H}_j$, and h_t and $h_{t'}$ are in the same cell of $\mathbb{H}_i(t)$, then player i is indifferent between continuation strategies $s_i|h_t$ and $s_i'|h_{t'}$ such that $s_i|h_t(h_t, a_1, \dots, a_m) = s_i'|h_{t'}(h_{t'}', a_1, \dots, a_m)$ for every finite sequence of action profiles $a_1, \dots, a_m$ satisfying $(h_t, a_1, \dots, a_m) \in H(t+m)$. Since, given the strategies of all other players, any continuation strategy determines a play, preferences over continuation strategies are well defined. A consistent collection is *time-dependent* if for every agent all elements in the same partition cell are of the same length. This means that every agent knows calendar time. *Markov states* are the cells in a maximally coarse time-dependent consistent collection. *Stationary Markov states* are the cells in a maximally coarse

[1] Finite sequences include sequences of zero length.
[2] Maskin and Tirole give conditions under which the partitions of all players coincide.

consistent collection. That these collections actually exist and are unique is established in the next section.

3 Existence and Uniqueness of Markov States and Stationary Markov States

A *complete lattice* is a partially odered set such that each subset S has a least upper bound, the *join* $\bigvee S$, and a greatest lower bound, the *meet* $\bigwedge S$. The following useful theorem is from Ore (1942):

Theorem 1 *Let A be a nonempty set. Then $(\Pi(A), \sqsubseteq)$ is a complete lattice. Let $S \subset \Pi(A)$. Two elements in A are in the same cell of $\bigwedge S$ if they are in the same cell for every partition in S. Two elements $a, a' \in A$ are in the same cell of $\bigvee S$ if and only if there are elements $a_1, \ldots, a_{n+1} \in A$ with $a_1 = a, a_{n+1} = a'$ and partitions $\Pi_1, \ldots, \Pi_n \in S$ such that a_i, a_{i+1} are in the same cell of Π_i for $i = 1, \ldots, n$.*

A relatively straightforward proof can be found in Roman (2008). The basic idea is that there is an order isomorphism between $(\Pi(A), \sqsubseteq)$ and the set of all equivalence relations on A ordered by set inclusion. For that space it is easy to show that the meet of a family of equivalence relations is its intersection and its join is the transitive closure of its union. No finiteness assumption is needed. With Theorem 1 proving existence and uniqueness of Markov states and stationary Markov states is easy.

Theorem 2 *The set of Markov states and the set of stationary Markov states exist and are unique.*

Proof: Let $\mathbb{M}$ the set of all time-dependent and consistent collections and $\mathbb{M}_S$ be the set of all consistent collections. Markov states will be the cells in $\bigvee \mathbb{M}$ and stationary Markov states the cells in $\bigvee \mathbb{M}_S$. If these are Markovian or respectively stationary Markovian collections, they are obviously maximally coarse. They exist by Theorem 1. It remains to show that $\bigvee \mathbb{M}$ is Markovian (the proof that $\bigvee \mathbb{M}_S$ is stationary Markovian is completely analogous).

Define $\mathbb{M}_i = \{\mathbb{H}_i : (\mathbb{H}_1, \ldots, \mathbb{H}_i, \ldots, \mathbb{H}_n) \in \mathbb{M}\}$. Let h, h' be in the same cell of agent i's partition in $\bigvee \mathbb{M}$. By Theorem 1, there are histories $h_1, \ldots, h_{m+1}$ with $h_1 = h$ and $h' = h_{m+1}$ and partitions $\Pi_1, \ldots, \Pi_m$ for agent i in $\mathbb{M}_i$ so that h_l, h_{l+1} are in the same cell of Π_l for $l = 1, \dots, m+1$. Hence h, h' have the same length, allow for the same continuation strategies and induce the same payoffs over continuation strategies, since for $j \neq i$, a strategy is $\bigvee \mathbb{M}_j$-measurable only if it is measurable for each partition in $\mathbb{M}_j$. ■

4 Randomized strategies

The argument above depends in no way on allowing only pure strategies. What is necessary to apply it to expected utility (or a variant thereof) and mixed strategies is that all strategies induce a distribution on plays, so that expected utilities are associated to every outcome.[3] The definition of mixed strategies for uncountable action spaces poses some subtle difficulties in general. Aumann (1964) has shown that when action spaces are Borel subsets of some Euclidean space, mixed and behavior strategies can be well defined as certain random variables for a large class of extensive form games including the games treated here. Aumann has also given a version of Kuhns theorem on the equivalence of mixed and behavior strategies for games with perfect recall. Behavior strategies are the appropriate strategies for applying the present method, for they allow a comparison of continuation strategies. When we allow for mixed strategies, introducing nature as a player allows us to introduce stochastic payoffs. Most of the discrete time[4] dynamic games with observable actions arising in applied work are covered by this framework.

5 Discussion

Partitions or the closely related equivalence relations plays an important role in economics. Partitions are used for representing knowledge and information, or for symmetries that a solution has to satisfy. In classical game

[3]When this is the case has been characterized by Alós-Ferrer and Ritzberger (2008) for general extensive form games.

[4]Strategy profiles in differential games may fail to induce plays (see the discussion by Alós-Ferrer and Ritzberger (2008)). For this reason, the payoff-relevant states cannot be defined for differential games using the present method.

theory, we want solutions not to change when strategically irrelevant parameters, like the specific extensive form representation of a given strategic situation, change. This may be done by requiring a solution concept to be constant on the cells of a partition of the set of extensive forms. In normative social choice theory, one may want to exclude morally irrelevant information such as the names of agents or alternatives. This can be done by requiring social choice rules to be constant on the cells of a partition of all preference profiles.

Theorem 1 allows one to reason with the partitions of an infinite set for many purposes as if the set were finite. What follows is an application to a different problem.

Let Ω be a nonempty set of *states of the world*. The knowledge possibilities of an agent are modelled by a partition of Ω. If the state ω occurs, agent i with *knowledge partition* Π_i only knows that some state in the cell of Π_i that includes ω has occured. Denote this cell by $\pi_i(\omega)$. Agent i *knows* an event (a subset of Ω) E at $\omega \in \Omega$ if $\pi_i(\omega) \subseteq E$. Aumann (1976) defined the partition Π_K corresponding to *common knowledge* as the meet of the individuals partitions. To see that this coincides with the intuitive notion of common knowledge (that everyone knows that everyone knows that...), he proved the following theorem for a finite set of states of the world Ω and a finite set of agents I.

Theorem 3 *An event E is common knowledge at ω for all agents in I if and only if for any state ω' such that there are m agents $1, \dots, m$ and $m+1$ states $\omega_1, \dots, \omega_{m+1}$ with $\omega_1 = \omega$ and $\omega_{m+1} = \omega'$ satisfying $\omega_, \omega_{i+1} \in \pi_i(\omega_i)$ for $i = 1, \dots, m$ one has $\omega' \in E$.*

This is a trivial consequence of the join characterization from Theorem 1.

References

[1] Alós-Ferrer, C., and K. Ritzberger (2008): “Trees and extensive forms”, *Journal of Economic Theory* **143**, 216-250.

[2] Aumann, R. (1964): “Mixed and Behavior Strategies in Infinite Extensive Games”, in *Advances in Game Theory*, New Jersey: Princeton University Press, pp. 627–650.

[3] Aumann, R. (1976): "Agreeing to Disagree", *The Annals of Statistics* **4**, 1236-1239.

[4] Maskin, E., and J. Tirole (2001): "Markov Perfect Equilibrium: I. Observable Actions", *Journal of Economic Theory* **100**, 191-219.

[5] Ore, O. (1942): "Theory of equivalence relations", *Duke Mathematics Journal* **9**, 573-627.

[6] Roman, S. (2008): *Lattices and Ordered Sets*, New York: Springer.