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Abstract: We consider dynamic optimization problems on one-dimensional state spaces. Under standard smoothness and convexity assumptions, the optimal solutions are characterized by an optimal policy function h mapping the state space into itself. There exists an extensive literature on the relation between the size of the discount factor of the dynamic optimization problem on the one hand and the properties of the dynamical system $x_{t+1} = h(x_t)$ on the other hand. The purpose of this paper is to survey some of the most important contributions of this literature and to modify or improve them in various directions. We deal in particular with the topological entropy of the dynamical system, with its Lyapunov exponents, and with its periodic orbits.

Journal of Economic Literature **classification codes:** C61, O41

Key words: dynamic optimization, discounting, topological entropy, Lyapunov exponents, periodic orbits.

1 Introduction

Let $X \subseteq \mathbb{R}$ be an interval on the real line and let $h : X \mapsto X$ be a continuous function. It has been known since the early 1990s that certain *geometric* properties of the function h or certain *dynamic* properties of the difference equation $x_{t+1} = h(x_t)$ impose non-trivial upper bounds on the discount factors of *all* dynamic optimization problems that are defined on the state space X, satisfy standard smoothness and curvature assumptions, and have h as its optimal policy function.[1] In particular, such restrictions exist,

- if the dynamical system $x_{t+1} = h(x_t)$ has positive topological entropy (Montrucchio, 1994, Montrucchio and Sorger, 1996, and Mitra, 1998);
- if a trajectory of the dynamical system $x_{t+1} = h(x_t)$ has a positive Lyapunov exponent (Hewage and Neumann, 1990, and Hewage, 1991);
- or if the dynamical system $x_{t+1} = h(x_t)$ admits a periodic point that is not a power of 2 (Sorger, 1994, Mitra, 1996, Nishimura and Yano, 1996, and Mitra, 1998).

In the present paper we review these findings and present several modifications as well as related results.

The starting point of our analysis is the paper by Mitra and Sorger (1999) which provides necessary and sufficient conditions for the existence of a dynamic optimization problem that admits given functions h and V as its optimal policy function and optimal value function, respectively. Using the necessary rationalizability condition from Mitra and Sorger (1999), we first show how the above mentioned results on the relation between the topological entropy κ of the dynamical system $x_{t+1} = h(x_t)$ and the discount factor ρ can be improved, if one restricts attention to optimal value functions that have a Lipschitz-continuous first derivative. As a matter of fact, it turns out that, instead of the discount factor restriction $\rho \leq e^{-\kappa}$ that has been proved in Montrucchio (1994), Montrucchio and Sorger (1996), and Mitra (1998), one can actually derive the stronger bound $\rho \leq e^{-2\kappa}$. This sheds light on an open problem raised by Mitra (2000).

Then we turn to the Lyapunov exponent $\lambda(x)$ of an optimal path starting at the initial state x. The discount factor restriction $\rho \leq e^{-2\lambda(x)}$ has been derived in an unpublished paper by Hewage and Neumann (1990) and in the PhD dissertation Hewage (1991). The proof of this result, however, depends critically on the assumption that the pair $(x, h(x))$ is in the interior of the transition possibility set. Appealing again to the necessary rationalizability condition from Mitra and Sorger (1999), we show that this interiority assumption can be dropped without affecting the validity of the result.

Finally, we consider the discount factor restrictions imposed by the existence of periodic optimal paths. We first note that all relevant papers mentioned above consider optimal paths with a period that is not a power of 2. This is motivated by well-known results stating that the existence of these periodic orbits implies positive topological entropy. Here we discuss the case of periodic optimal paths with a period of the form 2^ℓ. Using a sufficient rationalizability condition

[1] For a survey of the most important contributions to this literature we refer to Sorger (2006).

from Mitra and Sorger (1999), we first show that no *general* discount factor restriction can be derived from the existence of such periodic orbits. On the other hand, if one has additional information on the topological properties of the periodic orbit, then it becomes possible to obtain such results. We demonstrate this by means of periodic orbits of period 4. There exist four types of such orbits that differ from each other in a topological sense. We show that for one of these types no non-trivial restriction exists, whereas for the other three types such restrictions can be derived along the lines suggested by Mitra (1996). Interestingly, these discount factor restrictions are smaller than the best bounds that have been derived for the case of optimal paths of period 3.

2 Problem formulation

Throughout the paper we consider dynamic optimization problems with the discrete time-domain $\mathcal{I} = \{0, 1, 2, \ldots\}$. The state of the system at the start of period $t \in \mathcal{I}$ is denoted by x_t. The set of all possible states (i.e., the state space of the model) is a non-empty and compact interval on the real line which we denote by X. A transition from state x to state y is feasible if and only if $(x, y) \in \mathbf{T}$, where $\mathbf{T} \subseteq X \times X$ is referred to as the transition possibility set. The following assumption is imposed on $\mathbf{T}$.

A1: The set $\mathbf{T}$ is closed and convex, and $\mathbf{T}_x = \{y \in X \mid (x, y) \in \mathbf{T}\}$ is non-empty for all $x \in X$.

A sequence $(x_t)_{t \in \mathcal{I}}$ is called a feasible path (from x_0) if $(x_t, x_{t+1}) \in \mathbf{T}$ holds for all t. For every $x \in X$, we denote by $F(x)$ the set of all feasible paths from x. Assumption A1 ensures that, for all $x \in X$, the set $F(x)$ is non-empty, convex, and compact in the product topology on X^∞.

A state transition from x to y generates the instantaneous utility $u(x, y)$, where $u : \mathbf{T} \mapsto \mathbb{R}$ is a given function. The time-preference rate is assumed to be constant and the corresponding discount factor will be denoted by ρ. This implies that the total utility generated by a feasible path $(x_t)_{t \in \mathcal{I}}$ is given by

$$J\left[(x_t)_{t \in \mathcal{I}}\right] = \sum_{t=0}^{+\infty} \rho^t u(x_t, x_{t+1}).$$

The following assumptions are imposed on the preferences.

A2: (i) The function $u : \mathbf{T} \mapsto \mathbb{R}$ is continuous and concave. (ii) The discount factor ρ satisfies $\rho \in (0, 1)$.

Assumptions A1 and A2 ensure that $J\left[(x_t)_{t \in \mathcal{I}}\right]$ is a finite number for all feasible paths $(x_t)_{t \in \mathcal{I}}$ and that the functional $J : F(x) \mapsto \mathbb{R}$ is concave and continuous with respect to the product topology. Hence, for every $x \in X$, there exists an optimal path, i.e., a feasible path $(x_t)_{t \in \mathcal{I}} \in F(x)$ such that $J\left[(x_t)_{t \in \mathcal{I}}\right] \geq J\left[(y_t)_{t \in \mathcal{I}}\right]$ holds for all feasible paths $(y_t)_{t \in \mathcal{I}} \in F(x)$. The optimal value function $V : X \mapsto \mathbb{R}$ is defined by

$$V(x) = \sup\left\{ J\left[(x_t)_{t \in \mathcal{I}}\right] \,\Big|\, (x_t)_{t \in \mathcal{I}} \in F(x) \right\}$$

for all $x \in X$. Under assumptions A1 and A2 this function is continuous and concave on X. A path $(x_t)_{t \in \mathcal{I}} \in F(x)$ is an optimal path from x, if and only if $J\left[(x_t)_{t \in \mathcal{I}}\right] = V(x)$.

In order to ensure the uniqueness of an optimal path from any given initial state $x \in X$ we impose a strict convexity assumption. For our purpose it is convenient to formulate this assumption directly in terms of the optimal value function V.

A3: The function $V : X \mapsto \mathbb{R}$ is strictly concave.

Strict concavity of V follows immediately from strict concavity of u. However, it is known that weaker conditions than strict concavity of u are sufficient for strict concavity of V. For a discussion of these conditions see, e.g., Stokey and Lucas (1989), where also the proof of the following proposition can be found.

Proposition 1 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $(\mathbf{T}, u, \rho)$ be a dynamic optimization problem satisfying assumptions A1-A3.*
(i) There exists a unique optimal path from every $x \in X$.
(ii) There exists a unique function $h : X \mapsto X$ such that the following is true. A feasible path $(x_t)_{t \in \mathcal{I}}$ is optimal, if and only if it satisfies the difference equation

$$x_{t+1} = h(x_t) \tag{1}$$

for all $t \in \mathcal{I}$.
(iii) The function h from part (ii) is continuous on X.

This result establishes the existence and uniqueness of optimal paths and shows that these paths can be characterized as the trajectories of the dynamical system (1). The function h is called the optimal policy function of the optimization problem $(\mathbf{T}, u, \rho)$. It is characterized by the equation

$$h(x) = \operatorname{argmax}\{u(x, y) + \rho V(y) \,|\, y \in \mathbf{T}_x\}.$$

Denoting the t-th iterate of h by $h^{(t)}$, it follows that the unique optimal path emanating from a given initial state $x \in X$ is given by $(h^{(t)}(x))_{t \in \mathcal{I}}$.

The purpose of the present paper is to state and discuss a few results that relate the properties of the dynamical system (1) to the size of the discount factor ρ. We shall make use of the following two propositions from Mitra and Sorger (1999), in which $\partial V(z)$ denotes the subdifferential of the concave function V at $z \in X$.

Proposition 2 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $(\mathbf{T}, u, \rho)$ be a dynamic optimization problem satisfying assumptions A1-A3. Moreover, let h and V be the optimal policy function and the optimal value function, respectively, of $(\mathbf{T}, u, \rho)$. For every $x \in X$ such that $\partial V(x) \neq \emptyset$ and for every $p \in \partial V(x)$ there exists $q \in \partial V(h(x))$ such that the inequality*

$$d_V(y; x, p) \geq \rho d_V(h(y); h(x), q) \tag{2}$$

holds for all $y \in X$, where

$$d_V(y; x, p) = V(x) - V(y) + p(y - x).$$

Proposition 3 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ be a Lipschitz-continuous functions with Lipschitz constant L. For every $\rho \leq 1/L^2$ there exists an optimization problem $(\mathbf{T}, u, \rho)$ satisfying assumptions A1-A3 such that h is the optimal policy function of this model.*

3 Topological entropy

In this section we discuss the relationship between the topological entropy of the dynamical system (1) and the discount factor ρ. This relationship has been studied by Montrucchio (1994), Montrucchio and Sorger (1996), and Mitra (1998). Even if these papers use mutually different assumptions, they all derive the very same result, namely

$$\kappa(h) \leq -\ln \rho, \tag{3}$$

where $\kappa(h)$ denotes the topological entropy of (1).

The topological entropy of a dynamical system is one of the most important measures of the complexity of the orbit structure of the dynamical system. To explain its interpretation, suppose that an observer cannot distinguish between two states x and y if $|y - x| \leq \varepsilon$, where ε is a positive number. This means that state observations are possible only with finite precision as measured by ε. Even if two initial states are indistinguishable in this sense, it can be the case that by observing the dynamical system (1) over a finite number of periods, say T periods, the trajectories starting in the two initial states can be distinguished. This will be the case, if and only if there exists an integer $t \in \{0, 1, 2, \ldots, T-1\}$ such that $|h^{(t)}(x) - h^{(t)}(y)| > \varepsilon$. The topological entropy measures the rate at which different trajectories become distinguishable as the number of observations, T, increases. In other words, the topological entropy measures the rate at which information is generated by iterating h. If a dynamical system has positive topological entropy, it is often said that the dynamical system exhibits topological chaos. A result like (3) provides therefore a formal justification for the claim that high impatience (small ρ) is necessary for optimal paths to exhibit very complicated dynamics (large $\kappa(h)$).

The formal definition of topological entropy is as follows; see, e.g., Guckenheimer and Holmes (1983). Let T be a positive integer and ε a positive real number. A subset $B \subseteq X$ is called (T, ε)-separated if, for any two different points x and y in B, there exists $t \in \{0, 1, 2, \ldots, T-1\}$ such that $|h^{(t)}(x) - h^{(t)}(y)| > \varepsilon$. Because of the compactness of X, the number

$$s_{T,\varepsilon}(h) = \max\{\#B \,|\, B \subseteq X \text{ and } B \text{ is } (T, \varepsilon)\text{-separated}\}$$

is finite, where $\#B$ denotes the cardinality of B. The topological entropy of h is defined as

$$\kappa(h) = \lim_{\varepsilon \to 0} \left[\limsup_{T \to \infty} \frac{\ln s_{T,\varepsilon}(h)}{T} \right].$$

Note in particular that $s_{1,\varepsilon}(h)$ is independent of h and that, because X is a real interval,

$$\limsup_{\varepsilon \to 0} \frac{\ln s_{1,\varepsilon}(h)}{-\ln \varepsilon} \leq 1 \tag{4}$$

must hold.

Mitra (2000) shows how one can use relation (3) in order to derive discount factor restrictions that are necessary for the dynamical system (1) to admit a periodic orbit of period 3, or for the dynamical system (1) to exhibit turbulence in the sense of Block and Coppel (1992). He also notes that the best possible discount factor restrictions for these two phenomena are exactly the square roots of those restrictions that can be obtained from (3). He concludes by writing that an "open question which this observation naturally raises is whether there is a more refined relationship [than (3)] between the discount factor and the topological entropy, which would yield the exact discount factor restrictions for period-three cycles and turbulence [...] as special cases" [Mitra (2000), p. 433]. If such a more refined relationship exists, it must have the form

$$\kappa(h) \leq -(1/2)\ln\rho, \tag{5}$$

as this inequality would yield discount factor restrictions that are exactly the square roots of those derived from (3). In the rest of this section we show how (5) can be obtained by imposing an extra smoothness condition on the optimal value function V. More specifically, we assume that V has a non-empty subdifferential at all states $x \in X$ and that there exists a constant $M > 0$ such that

$$d_V(y; x, p) \leq M(y-x)^2 \tag{6}$$

holds for all $(x, y) \in X \times X$ and all $p \in \partial V(x)$, where d_V is defined in proposition 2. The following preliminary lemma shows that this assumption requires the continuous differentiability of V.

Lemma 1 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $V : X \mapsto \mathbb{R}$ be a concave function. If the subdifferential $\partial V(x)$ is non-empty for all $x \in X$ and if (6) holds for all $(x, y) \in X \times X$ and all $p \in \partial V(x)$, then it follows that V is continuously differentiable on X.*[2]

PROOF: Let $x \in X$ and $p \in \partial V(x)$ be given. Since $d_V(y; x, p)$ is non-negative for all $y \in X$ by the concavity of V, it follows from (6) that

$$0 \leq p - \frac{V(y) - V(x)}{y - x} \leq M(y - x)$$

holds for all $y \in X$ that are different from x. This implies that $\lim_{y \to x}[V(y) - V(x)]/(y - x)$ exists and is equal to p. Consequently, the subdifferential $\partial V(x)$ must be a singleton for all x which, in turn, implies the continuous differentiability of V on X. □

Because of this lemma we may assume that V is continuously differentiable and we can write (6) as

$$D_V(y; x) \leq M(y-x)^2, \tag{7}$$

where $D_V(y; x) = V(x) - V(y) + V'(x)(y - x)$ and where V' denotes the derivative of V. Whereas lemma 1 identifies continuous differentiability of V as a necessary condition for (6), the following lemma states three sufficient conditions for (7).

[2] At the two boundary points of X this is to be interpreted in the sense that the one-sided derivative of V at the boundary point exists and that it is the limit of the derivatives of V at z as z approaches the boundary point.

Lemma 2 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $V : X \mapsto \mathbb{R}$ be a concave and continuously differentiable function with derivative V'. Any of the following three conditions is sufficient for the existence of a number $M > 0$ such that the inequality in (7) holds for all $(x, y) \in X \times X$:*
(i) V' is is Lipschitz continuous on X;
(ii) there exists $\beta > 0$ such that V is $(-\beta)$-convex;[3]
(iii) V is twice continuously differentiable.

Proof: Let $(x, y) \in X \times X$ be given. By the continuous differentiability of V there exists a number z in the interval with boundary points x and y such that $V(y) = V(x) + V'(z)(y - x)$. It follows therefore that

$$D_V(y; x) = [V'(x) - V'(z)](y - x). \tag{8}$$

(i) If V' is Lipschitz continuous, then there exists $L > 0$ such that $|V'(x) - V'(z)| \leq L|x - z|$. Using this in (8) one gets $D_V(y; x) \leq L|x - z||y - x| < L(y - x)^2$ so that condition (7) holds with $M = L$.

(ii) If V is $(-\beta)$-convex, then it holds that $V(x) + (\beta/2)x^2$ is a convex function of x. This implies that $V'(x) + \beta x$ is a non-decreasing function of x. Using this in (8) we get $D_V(y; x) = [V'(x) + \beta x - V'(z) - \beta z](y - x) + \beta(z - x)(y - x) \leq \beta(z - x)(y - x) \leq \beta(y - x)^2$, and condition (7) holds with $M = \beta$.

(iii) If V is twice continuously differentiable, then it follows that V' is Lipschitz continuous and we are back in case (i). $\square$

The following lemma, which is the key step in deriving (5), is also of independent interest.

Lemma 3 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ and $V : X \mapsto \mathbb{R}$ be the optimal policy function and the optimal value function, respectively, of a dynamic optimization problem $(\mathbf{T}, u, \rho)$ satisfying assumptions A1-A3. Assume furthermore that V is continuously differentiable with derivative V' and that there exists $M > 0$ such that condition (7) holds for all $(x, y) \in X \times X$. For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that the following condition is satisfied for all $(x, y) \in X \times X$ and all $t \in \mathcal{I}$:*

$$|h^{(t)}(y) - h^{(t)}(x)| > \varepsilon \quad \Rightarrow \quad |y - x| > \rho^{t/2}\delta(\varepsilon).$$

Proof: Let $\varepsilon > 0$ be given and define the set $K(\varepsilon) = \{(x, y) \in X \times X \,|\, |y - x| \geq \varepsilon\}$. This set is a compact subset of $X \times X$. Since V' is continuous, it follows that D_V is also continuous on $K(\varepsilon)$. Hence, it follows that D_V attains its minimum on $K(\varepsilon)$. Moreover, since V is strictly concave, this minimum must be strictly positive. Defining $\tilde{\delta}(\varepsilon) = \min\{D_V(y; x) \,|\, (x, y) \in K(\varepsilon)\}$ it follows therefore that the inequality

$$D_V(y; x) \geq \tilde{\delta}(\varepsilon) > 0 \tag{9}$$

holds for all $(x, y) \in K(\varepsilon)$.

[3]For any given $\alpha \in \mathbb{R}$, the function $V : X \mapsto \mathbb{R}$ is called α-convex, if the mapping $x \mapsto V(x) - (\alpha/2)x^2$ is convex.

Now consider any $t \in \mathcal{I}$ and assume that $|h^{(t)}(y) - h^{(t)}(x)| > \varepsilon$. Iterating (2) t times, it follows that $D_V(y;x) \geq \rho^t D_V(h^{(t)}(y); h^{(t)}(x))$. Combining this with (7) and (9) and noting that $(h^{(t)}(x), h^{(t)}(y)) \in K(\varepsilon)$ we obtain $M(y-x)^2 \geq \rho^t \tilde{\delta}(\varepsilon)$ or, equivalently, $|y-x| \geq \rho^{t/2}\delta(\varepsilon)$, where $\delta(\varepsilon) = [\tilde{\delta}(\varepsilon)/M]^{1/2} > 0$. This completes the proof of the lemma. □

We are now ready to state the main result of the present section.

Theorem 1 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ and $V : X \mapsto \mathbb{R}$ be the optimal policy function and the optimal value function, respectively, of a dynamic optimization problem $(\mathbf{T}, u, \rho)$ satisfying assumptions A1-A3. Assume furthermore that V is continuously differentiable with derivative V' and that there exists $M > 0$ such that condition (7) holds for all $(x, y) \in X \times X$. Then it follows that the topological entropy of h satisfies (5).*

Proof: The general strategy of the proof is the same as in Montrucchio and Sorger (1996). Let $\varepsilon > 0$ and $T \geq 1$ be given and let $B \subseteq X$ be a (T, ε)-separated subset of X. It follows from lemma 3 that B is also $(1, \mu(T, \varepsilon))$-separated, where $\mu(T, \varepsilon) = \rho^{T/2}\delta(\varepsilon)$. Hence, it must hold that

$$\limsup_{T\to\infty} \frac{\ln s_{T,\varepsilon}(h)}{T} \leq \limsup_{T\to\infty} \frac{\ln s_{1,\mu(T,\varepsilon)}(h)}{T}.$$

Together with (4) and the definition of $\mu(T, \varepsilon)$ this implies that

$$\limsup_{T\to\infty} \frac{\ln s_{T,\varepsilon}(h)}{T} \leq \limsup_{\mu\to 0} \frac{\ln s_{1,\mu}(h)}{-\ln \mu} \lim_{T\to+\infty} \frac{-\ln \mu(T,\varepsilon)}{T} \leq \lim_{T\to+\infty} \frac{-\ln \mu(T,\varepsilon)}{T} = -(1/2)\ln \rho.$$

Since this inequality holds for all $\varepsilon > 0$, one obtains (5). □

Let us conclude this section by a few comments on theorem 1. First of all, Montrucchio (1994) and Montrucchio and Sorger (1996) have proved their theorems for state spaces of arbitrary (finite) dimension and this would also be possible for theorem 1. We have restricted ourselves to a one-dimensional state space only because of the simplicity of the presentation.

As a second remark let us note that the only place where assumption (7) is used is in lemma 3, which implies that the exponent of ρ in the definition of $\mu(T, \varepsilon)$ (see proof of theorem 1) is equal to $T/2$ instead of T. Since all that matters is the limit of $-[\ln \mu(T, \varepsilon)]/T$ as T approaches $+\infty$ and ε approaches 0, it is probably not necessary that (7) holds for all $(x, y) \in X \times X$. As the concave function V is known to be differentiable almost everywhere, it can be conjectured that (7) holds for sufficiently many points so that the argument in the proof of theorem 1 remains valid. If this conjecture were true, it would provide an affirmative answer to the question by Mitra (2000) quoted earlier in this section.

4 Lyapunov exponents

This section studies the relation between the Lyapunov exponents of the optimal policy function and the discount factor of the underlying dynamic optimization problem $(\mathbf{T}, u, \rho)$. The

Lyapunov exponent of the optimal policy function h at the state $x \in X$ measures the local rate of divergence from or convergence to the optimal path starting in x. Formally, let $(x_t)_{t\in\mathcal{I}}$ be the trajectory of the dynamical system (1) starting in $x \in X$, that is, $x_t = h^{(t)}(x)$ for all $t \in \mathcal{I}$. The Lyapunov exponent of h at x is defined as

$$\lambda(x; h) = \lim_{T\to+\infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln |h'(h^{(t)}(x))|, \tag{10}$$

provided that the limit on the right-hand side exists; see, e.g., Medio and Lines (2001). The following result can be found in Hewage and Neumann (1990) and Hewage (1991).

Proposition 4 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ and $V : X \mapsto \mathbb{R}$ be the optimal policy function and the optimal value function, respectively, of a dynamic optimization problem $(\mathbf{T}, u, \rho)$ satisfying assumptions A1-A3. Assume furthermore that u is twice continuously differentiable with $u_{xx}(x, y) < 0$, $u_{yy}(x, y) < 0$, and $u_{xx}(x, y)u_{yy}(x, y) - u_{xy}(x, y)^2 > 0$ for all $(x, y) \in \mathbf{T}$. Then the following statements are true:*
(i) h is continuously differentiable and V is twice continuously differentiable.
(ii) For all $x \in X$ such that $(x, h(x))$ is in the interior of $\mathbf{T}$ it holds that

$$\lambda(x; h) \leq -(1/2) \ln \rho. \tag{11}$$

Part (i) of this proposition is due to Santos (1991) and is therefore not proved in Hewage and Neumann (1990) or Hewage (1991). The proof of part (ii), on the other hand, is provided by Hewage and Neumann (1990) and Hewage (1991) and uses the identity

$$[h'(x)]^2 = \frac{w_{xx}(x, h(x))}{w_{yy}(x, h(x))}, \tag{12}$$

where $w(x, y) = u(x, y) - V(x) + \rho V(y)$. For this identity to be valid, Hewage and Neumann (1990) and Hewage (1991) need the assumption that $(x, h(x))$ is in the interior of $\mathbf{T}$. This is illustrated by the following example.

Example: Let the state space be $X = [0, 1]$ and consider the dynamic optimization problem $(\mathbf{T}, u, \rho)$ with $\rho = 1/6$,

$$\mathbf{T} = \{(x, y) \,|\, 0 \leq x \leq 1, 0 \leq y \leq 2x - x^2\},$$

and

$$u(x, y) = -2x + y - 3x^2 + 2xy - y^2/3 - x^2 y.$$

It is straightforward to verify that the optimal policy function of this problem is $h(x) = 2x - x^2$, that the optimal value function is $V(x) = -4x^2$, and that assumptions A1-A3 are satisfied. Note that, for all $x \in X$, the point $(x, h(x))$ is a boundary point of $\mathbf{T}$. Furthermore, one has $[h'(x)]^2 = 4(1 - x)^2$ and $w_{xx}(x, h(x))/w_{yy}(x, h(x)) = -(1 - x)^2$. Consequently, equation (12) is not satisfied. □

We conclude from these observations that the assumption that $(x, h(x))$ is in the interior of $\mathbf{T}$ is essential for the correctness of the proof of inequality (11) that one can find in Hewage and Neumann (1990) and Hewage (1991). The goal of the present section is to provide an alternative proof of (11) that does not require any interiority assumptions. In order to focus on this issue we essentially maintain the other assumptions from proposition 4 except that we directly impose the properties stated in part (ii) of the proposition. As a first step we derive a condition that can be regarded as a local version of inequality (2). The derivation of this condition is the key step in our proof of (11).

Lemma 4 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ and $V : X \mapsto \mathbb{R}$ be given functions. Assume that V is twice continuously differentiable. Let $x \in X$ be given and assume that h is continuously differentiable on an open neighborhood of x.*[4] *Then it follows that a necessary condition for (2) to hold is that*

$$\rho V''(h(x))[h'(x)]^2 \geq V''(x). \tag{13}$$

Proof: Because V is twice continuously differentiable, we have $\partial V(x) = \{V'(x)\}$ for all $x \in X$, where V' denotes the derivative of V. Consequently, we can write condition (2) from proposition 2 as $f_x(y) \geq 0$ for all $y \in X$, where $f_x(y) = d_V(y; x, V'(x)) - \rho d_V(h(y); h(x), V'(h(x)))$. Since $f_x(x) = 0$ this implies that $y = x$ is a global minimum of the function f_x. The first derivative of this function is given by

$$f'_x(y) = V'(x) - V'(y) - \rho[V'(h(x)) - V'(h(y))]h'(y). \tag{14}$$

Note in particular that the necessary first-order condition $f'_x(x) = 0$ for the minimization of f_x at $y = x$ is satisfied. The rest of the proof of lemma 4 employs essentially the necessary second-order condition.

Because of the smoothness assumptions imposed on V and h it holds for all y sufficiently close to x that $V'(x) - V'(y) = -\int_x^y V''(z)\,\mathrm{d}z$ and $V'(h(x)) - V'(h(y)) = -\int_x^y V''(h(z))h'(z)\,\mathrm{d}z$. Substituting these expressions into (14) we obtain

$$f'_x(y) = \int_x^y [\rho V''(h(z))h'(z)h'(y) - V''(z)]\,\mathrm{d}z. \tag{15}$$

The proof is completed by contradiction. Suppose that (13) does not hold. By continuity of h, h', and V'' it follows that the integrand on the right-hand side of (15) is negative whenever y and, hence, also z are sufficiently close to x. This, in turn, implies that f'_x is strictly decreasing around $y = x$ and, as a consequence, that f_x is strictly concave around $y = x$. Together with $f'_x(x) = 0$ this shows that f_x has a strict local maximum at $y = x$. Since this is a contradiction to what has been shown above, the proof of the lemma is complete. □

Condition (13) is only necessary for (2) but, in general, not sufficient. This is demonstrated by the following example.

Example: Let $X = [0, 1]$ and define $h(x) = (1 - x^\alpha)^{1/\alpha}$, where $\alpha > 1$. This function is continuous but neither continuously differentiable nor Lipschitz continuous at $x = 1$. Moreover,

[4] Openness is to be understood relative to X.

the dynamics (1) generated by this policy function has the property that every trajectory except for the unique fixed point $\bar{x} = 2^{-1/\alpha}$ is a neutrally stable periodic orbit of period 2. As a consequence, the topological entropy $\kappa(h)$ as well as all Lyapunov exponents $\lambda(x;h)$, where $x \in (0,1)$, are equal to 0.[5]

Let us try $V(x) = -x^{2\alpha}/[2\alpha(2\alpha-1)]$ as an optimal value function. Since $\alpha > 1$ was assumed, this function is twice continuously differentiable and strictly concave. Before we try to verify the global condition (2), we consider its local version (13). We have $V''(x) = -x^{2(\alpha-1)}$. Because of $h'(x) = -x^{\alpha-1}(1-x^\alpha)^{(1-\alpha)/\alpha}$, we obtain $V''(h(x))[h'(x)]^2 = -x^{2(\alpha-1)} = V''(x)$ for all $x \in (0,1)$. It follows that condition (13) holds for all $\rho \in (0,1)$ and all $x \in (0,1)$.

We now turn to the global condition (2). For all $(x,y) \in (0,1)^2$ we have

$$\begin{aligned} f_x(y) &= d_V(y;x,V'(x)) - \rho d_V(h(y);h(x),V'(h(x))) \\ &= \Big[(2\alpha-1)(1-\rho)x^{2\alpha} + (1-\rho)y^{2\alpha} - 2\alpha x^{2\alpha-1}y - 2\alpha\rho + 2(2\alpha-1)\rho x^\alpha + 2\rho y^\alpha \\ &\quad +2\alpha\rho(1-x^\alpha)^{(2\alpha-1)/\alpha}(1-y^\alpha)^{1/\alpha}\Big]/[2\alpha(2\alpha-1)]. \end{aligned}$$

It is easily seen that this function can be continuously extended to all of $[0,1]^2$. In particular, it holds that $f_0(1) = [1-(2\alpha-1)\rho]/[2\alpha(2\alpha-1)]$ and, hence, $f_0(1) < 0$ whenever the discount factor satisfies $1/(2\alpha-1) < \rho < 1$. Thus, condition (2) is not satisfied for these discount factors. Note that this does not imply that h cannot be an optimal policy function for $\rho > 1/(2\alpha-1)$. It simply says that, for these discount factors, h cannot be an optimal policy function with corresponding optimal value function V. $\square$

We are now ready to prove the main result of the present section.

Theorem 2 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ and $V : X \mapsto \mathbb{R}$ be the optimal policy function and the optimal value function, respectively, of a dynamic optimization problem $(\mathbf{T}, u, \rho)$ satisfying assumptions A1-A3. Assume furthermore that h is continuously differentiable and that V is twice continuously differentiable. Let $x \in X$ be given and assume that 0 is not contained in the closure of $\{V''(h^{(t)}(x)) \,|\, t \in \mathcal{I}\}$. Then it follows that the Lyapunov exponent $\lambda(x;h)$ exists (possibly equal to $-\infty$) and satisfies (11).*

Proof: Because of $V''(h^{(t)}(x)) < 0$ for all $t \in \mathcal{I}$, it follows from lemma 4 that

$$[h'(h^{(t)}(x))]^2 \leq \frac{V''(h^{(t)}(x))}{\rho V''(h^{(t+1)}(x))}$$

holds for all $t \in \mathcal{I}$. Taking logarithms, summing for $t \in \{0,1,\ldots,T-1\}$, and dividing by $2T$ we obtain

$$\frac{1}{T}\sum_{t=0}^{T-1} \ln|h'(h^{(t)}(x))| \leq -\frac{\ln\rho}{2} + \frac{\ln|V''(x)| - \ln|V''(h^{(T)}(x))|}{2T}.$$

Since the sequence $(V''(h^{(T)}(x)))_{T\in\mathcal{I}}$ is bounded away from 0, the limit of the right-hand side as T approaches infinity is equal to $-(1/2)\ln\rho$ and, hence, inequality (11) follows. $\square$

[5]The Lyapunov exponents for $x \in \{0,1\}$ are not defined.

Comparing theorem 2 to theorem 1 we see that both the topological entropy and the Lyapunov exponents are bounded above by $-(1/2)\ln\rho$. The factor $1/2$ in this bound reflects the smoothness conditions that we have imposed on the optimal value function, i.e., condition (7) in the case of theorem 1 and twice continuous differentiability in the case of theorem 2. Whereas we have conjectured at the end of section 3 that theorem 1 may remain valid also without this additional assumption, this is certainly not the case for theorem 2. As a matter of fact, it has been shown by means of an example in Sorger (1995) that the best bound one can hope for without an additional smoothness assumption on the optimal value function is $\lambda(x;h) \leq -\ln\rho$.

We conclude the present section by pointing out several results that are closely related to theorem 2. To this end recall that a probability measure μ on X is said to be invariant under h, if $\mu\{x \in X \mid h(x) \in B\} = \mu(B)$ holds for all measurable sets $B \subseteq X$. Furthermore, the probability measure μ is called ergodic if, for every measurable set B satisfying $\{x \in X \mid h(x) \in B\} = B$, it follows that $\mu(B) \in \{0, 1\}$.

Now suppose that there exists a probability measure μ on X that is invariant under h and that $V''(h(x)) < 0$ holds for μ-almost all $x \in X$. Then it follows from lemma 4 that

$$2\ln|h'(x)| \leq \ln|V''(x)| - \ln|V''(h(x))| - \ln\rho$$

holds for μ-almost all $x \in X$. Integrating the latter inequality with respect to μ and using the fact that μ is invariant under h one obtains

$$\mathbb{E}_\mu \ln|h'(x)| \leq -(1/2)\ln\rho. \tag{16}$$

Birkhoff's ergodic theorem tells us furthermore that the limit in (10) exists μ-almost everywhere and that $\mathbb{E}_\mu\lambda(x;h) = \mathbb{E}_\mu \ln|h'(x)|$; see, e.g., Medio and Lines (2001). Together with (16) this proves

$$\mathbb{E}_\mu\lambda(x;h) \leq -(1/2)\ln\rho. \tag{17}$$

Finally, if μ is ergodic, then it follows that $\lambda(x;h)$ is constant μ-almost everywhere so that $\lambda(x;h) = \mathbb{E}_\mu\lambda(x;h)$. Combining this with (17) is just another way to see that (11) holds for μ-almost all $x \in X$.

5 Periodic orbits

Let us finally turn to the question of which implications the existence of periodic orbits have for the size of the discount factor. In order to get started, let us recall the famous result from Sarkovskii (1964).

Proposition 5 *Let $X \subseteq \mathbb{R}$ be a closed interval on the real line and let $h : X \mapsto X$ be a continuous function. If the dynamical system (1) admits a periodic point of minimal period p, then it admits also a periodic point of minimal period q whenever q satisfies $p \lhd q$. Here, the Sarkovskii order $\lhd$ is defined by*

$$3 \lhd 5 \lhd \ldots \lhd 3 \times 2 \lhd 5 \times 2 \lhd \ldots \lhd 3 \times 2^\ell \lhd 5 \times 2^\ell \lhd \ldots \lhd 2^\ell \lhd 2^{\ell-1} \lhd \ldots \lhd 2 \lhd 1.$$

According to this proposition, every dynamical system that admits a periodic point of minimal period 3 admits periodic points of all periods, and every dynamical system that admits a periodic point of period $k2^\ell$, where $k > 1$ is an odd integer and $\ell \in \mathcal{I}$, has infinitely many periodic points of different periods. It is therefore not surprising that the implications of the existence of periodic points of these periods have been thoroughly studied.

Mitra (1996) and Nishimura and Yano (1996) have shown that the discount factor of any dynamic optimization problem $(\mathbf{T}, u, \rho)$ that satisfies assumptions A1-A3 and that admits an optimal path with minimal period 3 satisfies $\rho < (3 - \sqrt{5})/2 \approx 0.382$. These authors have also shown that this bound is the least upper bound on the set of discount factors that are consistent with the existence of an optimal path of period 3. As for the case of optimal paths with a minimal period of the form $k2^\ell$, where $k > 1$ is an odd integer and $\ell \in \mathcal{I}$, Mitra (1998) has proved that the discount factor has to satisfy $\rho < (1/2)^{1/2^{\ell+1}}$, whereby this is not the best possible bound. The proof of the latter bound is an indirect one. Mitra (1998) uses results on the topological entropy of dynamical systems that admit periodic points with a period that is different from a power of 2 and combines these results with (3). From theorem 1 it follows therefore immediately that $\rho < (1/2)^{1/2^\ell}$ must hold whenever the dynamic optimization problem $(\mathbf{T}, u, \rho)$ admits an optimal path with minimal period $k2^\ell$ and satisfies the smoothness condition (7).

In the remainder of this section we focus on dynamical systems that have periodic points with minimal period 2^ℓ for some $\ell \in \mathcal{I}$. Such dynamical systems do not necessarily have positive topological entropy, and it is therefore not surprising that they can be optimal policy functions for arbitrarily large discount factors $\rho \in (0, 1)$. This will be an immediate corollary of the following proposition.

Proposition 6 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval. For every $L > 1$ and every $\ell \in \mathcal{I}$ there exists a function $h : X \mapsto X$ such that the following two conditions hold:*
(i) The function h is Lipschitz-continuous with Lipschitz constant smaller than or equal to L.
(ii) The dynamical system (1) has a periodic point with minimal period 2^ℓ.

PROOF: Without loss of generality we may assume $X = [0, 1]$. Let $L > 1$ be given. We prove the proposition by induction with respect to ℓ. For $\ell = 0$ and $\ell = 1$ we can choose $h(x) = 1 - x$. Now suppose that the statement is true for ℓ. In other words, we assume that there exists a mapping $h_\ell : X \mapsto X$ that is Lipschitz-continuous with Lipschitz constant smaller than or equal to L and that admits a periodic point of minimal period 2^ℓ.

Given h_ℓ and an arbitrary number $\varepsilon \in (0, 1/2)$ we define the mapping $h_{\ell+1} : X \mapsto X$ by

$$h_{\ell+1}(x) = \begin{cases} 1 - \varepsilon + \varepsilon h_\ell(x/\varepsilon) & \text{if } x \in [0, \varepsilon], \\ \dfrac{[1 - \varepsilon + \varepsilon h_\ell(1)](1 - \varepsilon - x)}{1 - 2\varepsilon} & \text{if } x \in [\varepsilon, 1 - \varepsilon], \\ x - 1 + \varepsilon & \text{if } x \in [1 - \varepsilon, 1]. \end{cases} \tag{18}$$

The construction of $h_{\ell+1}$ is illustrated in figure 1. The first line in (18) implies that the top-left box of the right-hand diagram in figure 1 contains a rescaled version of the graph of h_ℓ, which is shown in the left-hand diagram. The third line of (18) implies that the bottom-right box of the right-hand diagram in figure 1 contains a 45° line. Finally, the second line of (18) simply requires connecting the rescaled version of the graph of h_ℓ in the top-left box with the 45° line in the bottom-right box by a straight line.

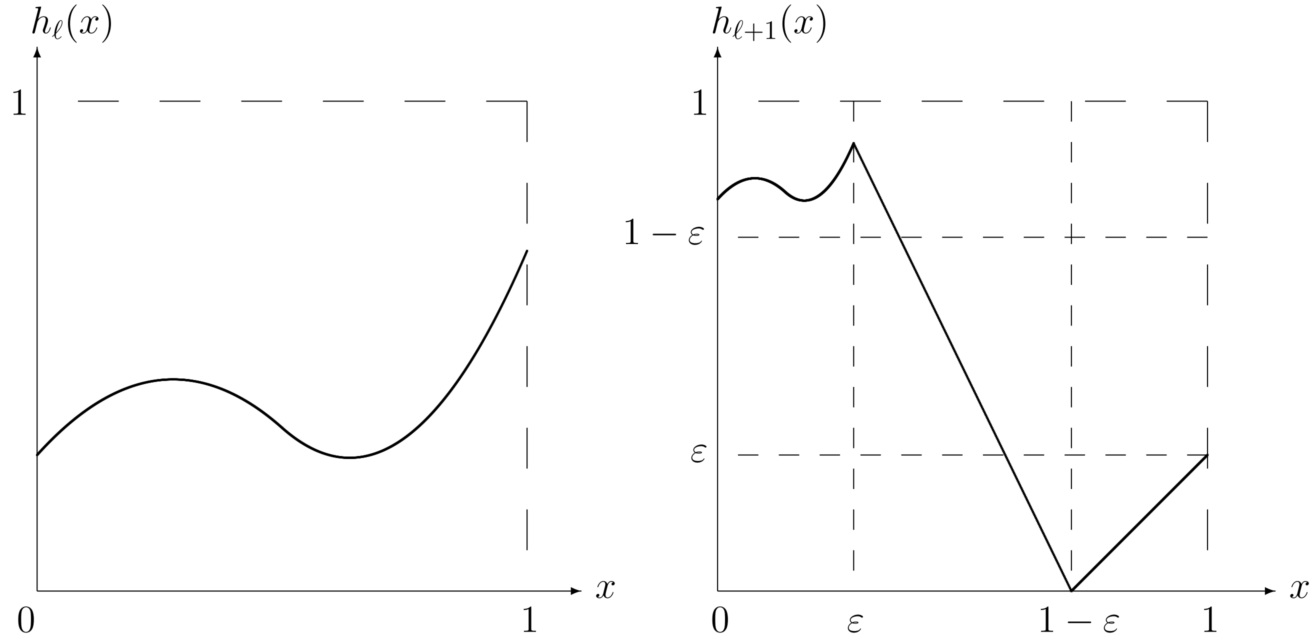

Figure 1: The construction of the function $h_{\ell+1}$.

In the next step we show that we can choose $\varepsilon \in (0, 1/2)$ such that $h_{\ell+1}$ is Lipschitz-continuous with Lipschitz constant smaller than or equal to L. To this end we first note that $h_{\ell+1}$ is continuous by construction. By the induction hypothesis (Lipschitz-continuity of h_ℓ) it follows that $h_{\ell+1}$ is Lipschitz-continuous on $[0, \varepsilon]$ with Lipschitz constant smaller than or equal to L. On the two other intervals $[\varepsilon, 1-\varepsilon]$ and $[1-\varepsilon, 1]$ the function $h_{\ell+1}$ is linear with slope $-[1-\varepsilon+\varepsilon h_\ell(1)]/(1-2\varepsilon)$ and 1, respectively. This shows that, whenever $\varepsilon < (L-1)/(2L)$, the function $h_{\ell+1}$ is also Lipschitz-continuous on X with Lipschitz constant smaller than or equal to L.

Now consider an arbitrary point $x \in X = [0, 1]$. Define $y = \varepsilon x$ and note that $y \in [0, \varepsilon]$ and, hence, $h_{\ell+1}(y) = 1 - \varepsilon + \varepsilon h_\ell(x)$. This, in turn, implies that $h_{\ell+1}(y) \in [1-\varepsilon, 1]$ and, hence, $h_{\ell+1}^{(2)}(y) = h_{\ell+1}(1-\varepsilon+\varepsilon h_\ell(x)) = \varepsilon h_\ell(x)$. We have therefore shown that $h_{\ell+1}^{(2)}(\varepsilon x) = \varepsilon h_\ell(x)$ holds for all $x \in X$. Using this result we obtain $h_{\ell+1}^{(4)}(\varepsilon x) = h_{\ell+1}^{(2)}(h_{\ell+1}^{(2)}(\varepsilon x)) = h_{\ell+1}^{(2)}(\varepsilon h_\ell(x)) = \varepsilon h_\ell^{(2)}(x)$. Repeating this argument it follows that

$$h_{\ell+1}^{(2^{k+1})}(\varepsilon x) = \varepsilon h_\ell^{(2^k)}(x) \tag{19}$$

for all $x \in X$ and all $k \in \mathcal{I}$.

From the induction hypothesis we know that there exists $x_\ell \in X$ such that

$$h_\ell^{(2^\ell)}(x_\ell) = x_\ell \neq h_\ell^{(m)}(x_\ell) \tag{20}$$

for all $m \in \{1, 2, \ldots, 2^\ell - 1\}$. Defining $x_{\ell+1} = \varepsilon x_\ell$ and applying (19) with $x = x_\ell$ and $k = \ell$ we get $h_{\ell+1}^{(2^{\ell+1})}(x_{\ell+1}) = \varepsilon h_\ell^{(2^\ell)}(x_\ell) = \varepsilon x_\ell = x_{\ell+1}$. This proves that $x_{\ell+1}$ is a periodic point of $h_{\ell+1}$ with period $2^{\ell+1}$. If the *minimal* period of $x_{\ell+1}$ were $m < 2^{\ell+1}$, then it would follow that m is a power of 2, say, $m = 2^k$ with $k \in \{0, 1, \ldots, \ell\}$. This, in turn, would imply that $h_{\ell+1}^{(2^\ell)}(x_{\ell+1}) = x_{\ell+1}$. Using the definition of $x_{\ell+1}$ and (19) with $x = x_\ell$ and $k = \ell - 1$ we obtain therefore

$$\varepsilon x_\ell = x_{\ell+1} = h_{\ell+1}^{(2^\ell)}(x_{\ell+1}) = h_{\ell+1}^{(2^\ell)}(\varepsilon x_\ell) = \varepsilon h_\ell^{(2^{\ell-1})}(x_\ell).$$

This implies $h_\ell^{(2^{\ell-1})}(x_\ell) = x_\ell$ which is a contradiction to the inequality in (20). It follows that $2^{\ell+1}$ is the minimal period of $x_{\ell+1}$. $\square$

Combining proposition 6 with proposition 3 we obtain the following corollary.

Corollary 1 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval. For every $\rho \in (0,1)$ and every $\ell \in \mathcal{I}$ there exists a transition possibility set $\mathbf{T} \subseteq X \times X$ and a utility function $u : \mathbf{T} \mapsto \mathbb{R}$ such that the following two conditions hold:*
(i) The dynamic optimization problem $(\mathbf{T}, u, \rho)$ satisfies assumptions A1-A3.
(ii) The optimal policy function h of $(\mathbf{T}, u, \rho)$ has a periodic point with minimal period 2^ℓ.

Although the above corollary proves that there exist periodic optimal paths with a period of the form 2^ℓ for discount factors arbitrarily close to 1, it does not rule out that one can derive non-trivial discount factor restrictions from the existence of periodic points with minimal period $m = 2^\ell$, provided that one has additional information about the periodic orbit. In the remainder of this section we elaborate on this issue by means of the case of $m = 4$.

The dynamical system (1) has a periodic point of minimal period 4, if and only if there exist mutually different states $\{a, b, c, d\} \subset X$ such that $\{h^{(t)}(x) \,|\, t \in \mathcal{I}\} = \{a, b, c, d\}$ for all $x \in \{a, b, c, d\}$. Without loss of generality we may assume that $a < b < c < d$. It is easy to see that there are four topologically different types of periodic orbits of period 4. These types are illustrated in figure 2 below, whereby the four dots indicate the states a, b, c, and d, respectively, and the arrows describe the action of h.

If one follows the instructions in the proof of proposition 6 to construct a Lipschitz-continuous function with a periodic point of minimal period 4, one obtains an orbit of type 4. We can therefore not hope for a non-trivial discount factor restriction for this type. However, as we shall see below, it is possible to derive non-trivial discount factor restrictions for optimal policy functions which have orbits of type 1, 2, or 3. We deal with these three cases in lemmas 6-7 below. Before that, however, we state a preliminary result.[6]

Lemma 5 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $(\mathbf{T}, u, \rho)$ be a dynamic optimization problem satisfying assumptions A1-A3. Moreover, let h and V be the optimal policy function and the optimal value function, respectively, of $(\mathbf{T}, u, \rho)$. Suppose that there exists a periodic optimal path of period $m \geq 3$, that is, there exists a set $\{x_1, x_2, \ldots, x_m\} \subseteq X$*

[6]See corollary 1 in Mitra (1996) for a similar result under somewhat different assumptions.

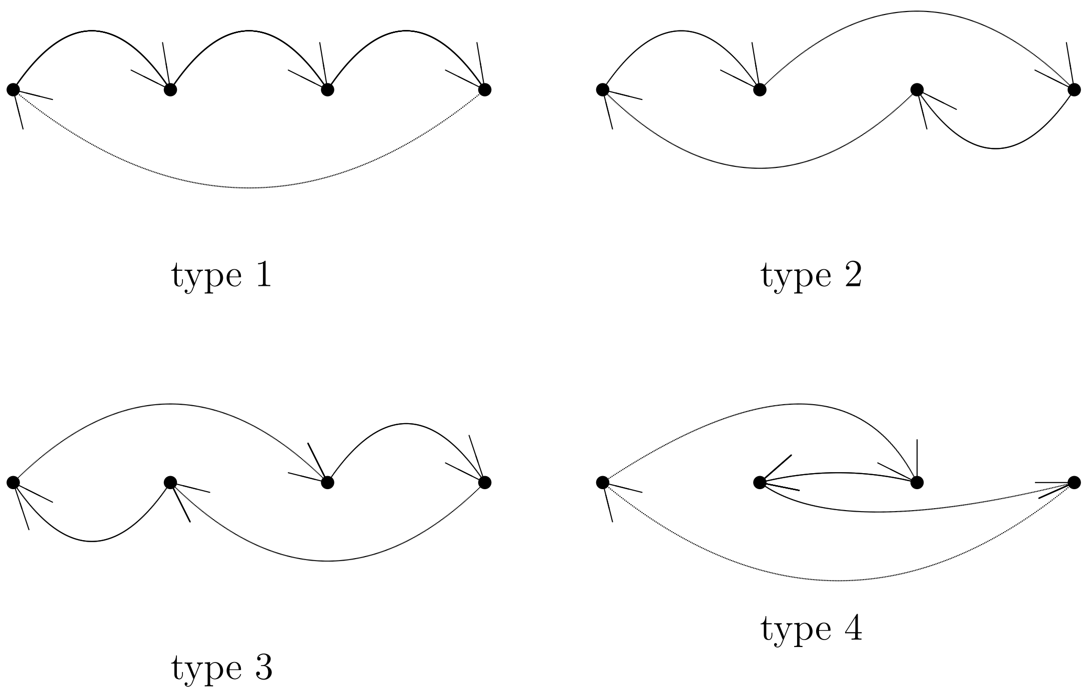

Figure 2: Periodic orbits of period 4.

such that $h(x_i) = x_{i+1}$ *for all* $i \in \{1, 2, \ldots, m-1\}$ *and* $h(x_m) = x_1$. *Then there exist support prices* $\{p_1, p_2, \ldots, p_m\}$ *with* $p_i \in \partial V(x_i)$ *such that*[7]

$$(p_i - p_j)(x_j - x_i) \geq \rho(p_{i+1} - p_{j+1})(x_{j+1} - x_{i+1}) \tag{21}$$

holds for all $(i, j) \in \{1, 2, \ldots, m\}$.

PROOF: Because of $m \geq 3$ it must be the case that one element of the periodic orbit is in the interior of X. Without loss of generality we may assume this element to be x_1 and it follows that $\partial V(x_1) \neq \emptyset$. Choosing an arbitrary subgradient $q_1 \in \partial V(x_1)$ it follows from proposition 2 that there exists $q_2 \in \partial V(x_2)$ such that (2) holds. Proceeding in that way we obtain a sequence of support prices $(q_t)_{t \in \mathcal{I}}$ such that for all $s \in \mathcal{I}$, all $i \in \{1, 2, \ldots, m\}$, and all $y \in X$ it holds that $q_{sm+i} \in \partial V(x_i)$ and

$$V(x_i) - V(y) + q_{sm+i}(y - x_i) \geq \rho[V(x_{i+1}) - V(h(y)) + q_{sm+i+1}(h(y) - x_{i+1})]. \tag{22}$$

For any fixed $i \in \{1, 2, \ldots, m\}$ consider the sequence $(q_{sm+i+1})_{s \in \mathcal{I}}$. If x_{i+1} is in the interior of X, then it follows that $\partial V(x_{i+1})$ is compact and $(q_{sm+i+1})_{s \in \mathcal{I}}$ must therefore contain a convergent subsequence. If x_{i+1} is not in the interior of X, then it must be one of the two boundary points of X. If it is the lower boundary point, then it follows that $\partial V(x_{i+1})$ is bounded below. Choosing $y = x_{i-1}$, where x_0 is interpreted as x_m, inequality (22) becomes

$$V(x_i) - V(x_{i-1}) + q_{sm+i}(x_{i-1} - x_i) \geq \rho[V(x_{i+1}) - V(x_i) + q_{sm+i+1}(x_i - x_{i+1})].$$

[7]Here and in what follows we shall interpret x_{m+1} and p_{m+1} as x_1 and p_1, respectively.

Because x_i is in the interior of X, the left-hand side of this inequality is uniformly bounded in s. Because x_{i+1} is the lower boundary point of X, the term $x_i - x_{i+1}$ is positive. Using these two observations in the above inequality it follows that the sequence $(q_{sm+i+1})_{s\in\mathcal{I}}$ must be bounded from above. Since it is bounded from below as well (recall that $\partial V(x_{i+1})$ is bounded from below), it is a bounded sequence and must therefore contain a convergent subsequence. If x_{i+1} is the upper boundary point of X, an analogous argument applies and we can therefore conclude that there exists an infinite set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\lim_{s\in\mathcal{I}_0} q_{sm+i}$ exists for all $i \in \{1, 2, \dots, m\}$. Let us denote this limit by p_i. Since $\partial V(x_i)$ is closed, it holds that $p_i \in \partial V(x_i)$. Passing over to the limit and setting $y = x_j$ in (22) yields therefore

$$V(x_i) - V(x_j) + p_i(x_j - x_i) \geq \rho[V(x_{i+1}) - V(x_{j+1}) + p_{i+1}(x_{j+1} - x_{i+1})].$$

Interchanging the roles of i and j in this inequality and adding the two inequalities we obtain (21). □

We are now in a position to derive discount factor restrictions for period-4 cycles of type 1.

Lemma 6 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval, and let $h : X \mapsto X$ be the optimal policy function of a dynamic optimization problem $(\mathbf{T}, u, \rho)$ satisfying assumptions A1-A3. Assume furthermore that there exists a set $\{a, b, c, d\} \subset X$ such that*

$$h(d) = a < h(a) = b < h(b) = c < h(c) = d. \qquad \textit{(type 1)}$$

Then it follows that $\rho \leq r^2 \approx 0.296$, where r is the unique real number in $(0, 1)$ that satisfies $r + r^2 + r^3 = 1$.

Proof: From lemma 5 it follows that there exist subgradients $p_a \in \partial V(a)$, $p_b \in \partial V(b)$, $p_c \in \partial V(c)$, and $p_d \in \partial V(d)$ such that the following inequalities hold:

$$\begin{aligned}
\rho &\leq \frac{[(p_a - p_b)(b - a)}{(p_b - p_c)(c - b)}, \\
\rho &\leq \frac{(p_a - p_c)(c - a)}{(p_b - p_d)(d - b)}, \\
\rho &\leq \frac{(p_a - p_d)(d - a)}{(p_a - p_b)(b - a)}, \\
\rho &\leq \frac{(p_b - p_c)(c - b)}{(p_c - p_d)(d - c)}, \\
\rho &\leq \frac{(p_b - p_d)(d - b)}{(p_a - p_c)(c - a)}, \\
\rho &\leq \frac{(p_c - p_d)(d - c)}{(p_a - p_d)(d - a)}.
\end{aligned}$$

Note that $a < b < c < d$ implies $p_a > p_b > p_c > p_d$. This shows that the right-hand side of the third inequality above is greater than 1 such that it does not yield a non-trivial discount factor restriction. Defining

$$P = \frac{p_a - p_b}{p_a - p_d}\,,\ Q = \frac{p_c - p_d}{p_a - p_d}\,,\ Y = \frac{b - a}{d - a}\,,\ Z = \frac{d - c}{d - a}$$

we can write the remaining five inequalities as $\rho \leq \min\{f_i(P,Q)f_i(Y,Z) \,|\, i \in \{1,2,3,4,5\}\}$, where

$$f_1(y,z) = \frac{y}{1-y-z}\,,\ f_2(y,z) = \frac{1-z}{1-y}\,,\ f_3(y,z) = \frac{1-y-z}{z}$$
$$f_4(y,z) = \frac{1-y}{1-z}\,,\ f_5(y,z) = z.$$

Note furthermore that $(P,Q) \in S$ and $(Y,Z) \in S$, where $S = \{(y,z) \,|\, y > 0, z > 0, y+z < 1\}$. It follows therefore that

$$\rho \leq \max\{\min\{f_i(P,Q)f_i(Y,Z) \,|\, i \in \{1,2,3,4,5\}\} \,|\, (P,Q) \in S\,,\ (Y,Z) \in S\}.$$

Since $f_i(y,z) \geq 0$ holds for all $i \in \{1,2,3,4,5\}$ and all $(y,z) \in S$, this inequality can also be written as

$$\rho \leq [\max\{\min\{f_i(y,z) \,|\, i \in \{1,2,3,4,5\}\} \,|\, (y,z) \in S\}]^2\,. \tag{23}$$

It remains to be shown that the right-hand side of this inequality is given by r^2. To this end, define $y = r^3$ and $z = r$ and note that $(y,z) \in S$. Furthermore, it holds that $f_1(y,z) = f_2(y,z) = f_3(y,z) = f_5(y,z) = r$ and $f_4(y,z) = 1/r > 1$. Consequently, we have $\min\{f_i(y,z) \,|\, i \in \{1,2,3,4,5\}\} = r$, and it follows that the right-hand side of (23) is greater than or equal to r^2. Suppose that it is strictly greater than r^2. Then it follows that there exists $(\bar{y},\bar{z}) \in S$ such that $r < f_i(\bar{y},\bar{z})$ holds for all $i \in \{1,2,3,4,5\}$. From $r < f_5(\bar{y},\bar{z})$ we get $\bar{z} > r$. Together with $r < f_2(\bar{y},\bar{z})$ we obtain $r < (1-\bar{z})/(1-\bar{y}) < (1-r)/(1-\bar{y})$. Together with the definition of r this implies $\bar{y} > r^3$. Using these results in $r < f_3(\bar{y},\bar{z})$ we obtain $r < (1-\bar{y}-\bar{z})/\bar{z} < (1-r^3-r)/r = r$, where the last inequality follows from the definition of r. This contradiction completes the proof of the lemma. $\square$

Using a similar proof we can deal with periodic orbits of types 2 and 3.

Lemma 7 *Let $X \subseteq \mathbb{R}$ be a non-empty and compact interval and let $h : X \mapsto X$ be the optimal policy function of a dynamic optimization problem $(\mathbf{T}, u, \rho)$ satisfying assumptions A1-A3. Assume furthermore that there exists a set $\{a,b,c,d\} \subseteq X$ such that either*

$$h(c) = a < h(a) = b < h(d) = c < h(b) = d \qquad \textit{(type 2)}$$

or

$$h(b) = a < h(d) = b < h(a) = c < h(c) = d. \qquad \textit{(type 3)}$$

Then it follows that $\rho \leq 3 - 2\sqrt{2} \approx 0.172$.

Proof: Suppose that there exists a periodic orbit of type 2. As in the proof of lemma 6 it follows that there exist numbers p_a, p_b, p_c, and p_d such that $p_a > p_b > p_c > p_d$ and such that

$$\rho \leq \frac{(p_a - p_b)(b-a)}{(p_b - p_d)(d-b)},$$
$$\rho \leq \frac{(p_a - p_c)(c-a)}{(p_a - p_b)(b-a)},$$

$$\rho \leq \frac{(p_a - p_d)(d-a)}{(p_b - p_c)(c-b)},$$
$$\rho \leq \frac{(p_b - p_c)(c-b)}{(p_a - p_d)(d-a)},$$
$$\rho \leq \frac{(p_b - p_d)(d-b)}{(p_c - p_d)(d-c)},$$
$$\rho \leq \frac{(p_c - p_d)(d-c)}{(p_a - p_c)(c-a)}.$$

The right-hand side of the third inequality above is greater than 1 and can therefore be omitted. Defining P, Q, Y, Z, and S in exactly the same way as in the proof of lemma 6 we can write the remaining five inequalities as $\rho < \min\{f_i(P,Q) f_i(Y,Z) \,|\, i \in \{1,2,3,4,5\}\}$, where

$$f_1(y,z) = \frac{y}{1-y}\,,\ f_2(y,z) = \frac{1-z}{y}\,,\ f_3(y,z) = 1-y-z$$
$$f_4(y,z) = \frac{1-y}{z}\,,\ f_5(y,z) = \frac{z}{1-z},$$

and where $(P,Q) \in S$ and $(Y,Z) \in S$. As in the proof of lemma 6 this can be written as

$$\rho \leq \left[\max\{\min\{f_i(y,z) \,|\, i \in \{1,2,3,4,5\}\} \,|\, (y,z) \in S\}\right]^2. \tag{24}$$

It remains to be shown that the right-hand side of this inequality is given by $3-2\sqrt{2}$. To this end, define $y = z = (2-\sqrt{2})/2$ and note that $(y,z) \in S$. Then it follows that $f_1(y,z) = f_3(y,z) = f_5(y,z) = \sqrt{2}-1 < 1$ and $f_2(y,z) = f_4(y,z) = 1/(\sqrt{2}-1) > 1$. Consequently, we have $\min\{f_i(y,z) \,|\, i \in \{1,2,3,4,5\}\} = \bar{r} := \sqrt{2}-1$, and it follows that the right-hand side of (24) is greater than or equal to $\bar{r}^2 = 3-2\sqrt{2}$. Suppose that it is strictly greater than $\bar{r}^2$. Then it follows that there exists $(\bar{y},\bar{z}) \in S$ such that $\bar{r} < f_i(\bar{y},\bar{z})$ holds for all $i \in \{1,2,3,4,5\}$. From $\bar{r} < f_1(\bar{y},\bar{z})$ we get $\bar{y} > \bar{r}/(1+\bar{r})$. Analogously, from $\bar{r} < f_5(\bar{y},\bar{z})$ it follows that $\bar{z} > \bar{r}/(1+\bar{r})$. Using these results in $\bar{r} < f_3(\bar{y},\bar{z})$ we obtain $\bar{r} < 1-2\bar{r}/(1+\bar{r}) = \bar{r}$. Since this is a contradiction, the proof for the case of a periodic orbit of type 2 is complete.

For a periodic orbit of type 3 we obtain the inequalities

$$\rho \leq \frac{(p_a - p_b)(b-a)}{(p_a - p_c)(c-a)},$$
$$\rho \leq \frac{(p_a - p_c)(c-a)}{(p_c - p_d)(d-c)},$$
$$\rho \leq \frac{(p_a - p_d)(d-a)}{(p_b - p_c)(c-b)},$$
$$\rho \leq \frac{(p_b - p_c)(c-b)}{(p_a - p_d)(d-a)},$$
$$\rho \leq \frac{(p_b - p_d)(d-b)}{(p_a - p_b)(b-a)},$$
$$\rho \leq \frac{(p_c - p_d)(d-c)}{(p_b - p_d)(d-b)},$$

which can also be written in the form of (24) provided that

$$\begin{aligned} f_1(y,z) &= \frac{y}{1-z}\,,\ f_2(y,z) = \frac{1-z}{z}\,,\ f_3(y,z) = 1-y-z \\ f_4(y,z) &= \frac{1-y}{y}\,,\ f_5(y,z) = \frac{z}{1-y}. \end{aligned}$$

As in the case of type-2 orbits one can show that the right-hand side of (24) is greater than $\bar{r}^2$, where $\bar{r} = \sqrt{2} - 1$. Suppose that it is strictly greater than $\bar{r}^2$. Then it follows that there exists $(\bar{y}, \bar{z}) \in S$ such that $\bar{r} < f_i(\bar{y}, \bar{z})$ holds for all $i \in \{1, 2, 3, 4, 5\}$. From $\bar{r} < f_3(\bar{y}, \bar{z})$ we get $1 - \bar{z} > \bar{r} + \bar{y}$. Together with $\bar{r} < f_1(\bar{y}, \bar{z})$ this implies $\bar{y} > \bar{r}^2/(1 - \bar{r})$. Analogously, from $\bar{r} < f_3(\bar{y}, \bar{z})$ and $\bar{r} < f_5(\bar{y}, \bar{z})$ it follows that $\bar{z} > \bar{r}^2/(1 - \bar{r})$. Using these results in $\bar{r} < f_3(\bar{y}, \bar{z})$ we obtain $\bar{r} < 1 - 2\bar{r}^2/(1 - \bar{r}) = \bar{r}$. Since this is a contradiction, the proof for the case of a periodic orbit of type 3 is complete. $\square$

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