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Strategic Exploitation of a Common-Property Resource under Uncertainty

Abstract

We study the impact of uncertainty on the strategies and dynamics of symmetric non-cooperative games among players who exploit a non-excludable resource that reproduces under uncertainty. We focus on a particular class of games that deliver a unique Nash equilibrium in linear-symmetric strategies of resource exploitation. We show that, for this class of games, the tragedy of the commons is always present. For various changes in the riskiness of the random primitives of the model we provide general characterizations of features of the model that explain links between the degree of riskiness and strategic exploitation decisions. Finally, we provide a specific example that demonstrates the usefulness of our general results and, within the specific example, we study cases where increases in risk amplify or mitigate the tragedy of the commons.

Keywords: resource exploitation, stochastic non-cooperative dynamic games, tragedy of the commons, stochastic dominance

JEL classification: C73, C72, C61, Q20, O13, D90, D43

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1. Introduction

Games of common-property renewable resource exploitation have the feature that each player chooses their current level of exploitation while (partly) controlling the future evolution of the resource, given the strategies of other players. Models in which there is a dynamic element and in which resources are shared, play an important role in economics, e.g., industrial organization models or models with natural resources. The fundamental, infinite-horizon setup, where all players have full information about the economic environment, has been studied in the economics literature almost exclusively within the deterministic framework. The main finding of this literature is that the equilibrium is characterized by a 'tragedy of the commons.' Namely, the higher the number of players, the higher the aggregate exploitation rate, so the resource shrinks to a lower level in the long run.[1]

The tragedy of the commons results after a complex strategic accounting by players in a perfect-foresight environment. In particular, it is transparent to each player that over-exploitation implies future trajectories of the common resource that increase their present and future individual costs. Yet, given what other players do, and what they will be doing throughout the infinite horizon, it is often individually optimal to support a strategic path of aggregate overexploitation of the resource, when the number of players increases. How does such a strategic behavior change if, instead of acting in a deterministic environment, players act in an environment of uncertainty and rational expectations? Our study contributes some answers to this general question, focusing on infinite-horizon games where the law of reproduction of the resource is subject to random shocks and players have rational expectations.[2]

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[1] See, for example, Mirman (1979) and Levhari and Mirman (1980), Levhari, Michener and Mirman (1981), Benhabib and Radner (1992), Dockner and Sorger (1996), Sorger (1998, 2005) and Koulovatianos and Mirman (2007).

[2] Our focus on randomness in resource reproduction is a natural starting point for the study of uncertainty in

The scope and aim of our analysis is to contribute to the understanding of the mechanics of common-property resource games under uncertainty. Yet, stochastic dynamic games can be particularly complex and very difficult to characterize with general primitives, i.e. with general payoff functions of players, general resource or renewal laws of motion and general distributions of random disturbances.[3] At the same time, the task of characterizing decisions in the presence of uncertainty in a general framework can be very demanding even when there is only one player.[4] We first discuss several technical problems that arise in general dynamic games. In particular, in resource games, a part of the objective function of each player is the other players' strategies. When the Markov strategies of other players are strictly concave, a player's objective function may lose key properties, such as concavity, differentiability, or continuity. These technical difficulties, discussed by Mirman (1979), justify our focus on a class of games with a unique Nash solution in linear-symmetric Markov strategies.[5] Our paper is devoted to the study of this tractable class of games, a natural starting point for understanding players' strategic behavior under uncertainty.

We deal with two basic issues concerning all models in the class of dynamic games with a unique Nash solution in linear-symmetric Markov strategies. The first is whether the

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resource games. In the real world, resources evolve according to stochastic laws of motion. Especially when in the context of natural resources, as is the case with biological populations such as forests and fish species, these evolve subject to the existence of predators or climate, that are affected by random disturbances. In cases of governmental provisions of infrastructure for companies, such as railroads, electricity grids, telecommunication networks, etc., financing and maintenance is also subject to random shocks, such as business cycles or political cycles.

[3] An example of a study examining the link between extraction decisions and uncertain reproduction outcomes under perfect competition, and also optimal resource preservation policies is the fishery application of Mirman and Spulber (1985). For a paper studying uncertainty and games see Amir (1996). For studies pointing out technical issues in *deterministic* differential resource games, such as multiplicity of equilibrium strategies, arising even in setups with some simplifying assumptions on primitives, see Dockner and Sorger (1996), and Sorger (1998), while for fundamental proofs of equilibrium existence see Sundaram (1989) and Dutta and Sundaram (1992).

[4] For example, Mirman (1971) analyzes uncertainty in a model with a single controller, providing a general result about the role of uncertainty on decisions in two-period models, and discussing issues arizing in the infinite-horizon setup.

[5] A game that falls in this class is the parametric example of Levhari and Mirman (1980).

tragedy of the commons holds in this class of games. Secondly, we study the effect of the riskiness of the random primitives of the model on the players' strategies. In particular, we highlight those features of the model that explain the links between the degree of riskiness and strategic exploitation decisions. We then provide a parametric example within the class of games with linear strategies, that extends (and also nests) the Levhari-Mirman (1980) example. Apart from demonstrating the applicability of our general results concerning games with linear strategies, this example allows us to study whether the degree of riskiness can amplify or mitigate the tragedy of the commons.

Our first result is that all games, stochastic or deterministic, that have a unique Nash equilibrium in linear-symmetric Markov strategies, *always* imply a tragedy of the commons. This means, in this class of games, adding one more player *always* leads to a higher aggregate exploitation rate in a symmetric equilibrium.

The next result is about the applicability of a recursive procedure for calculating linear-symmetric strategies. Levhari and Mirman (1980) used a method for solving dynamic games that started from the one-period problem and continued recursively for the n-period problem, in order to obtain the infinite-horizon solution asymptotically. We find that for the class of games that lead to linear symmetric strategies the calculation method used by Levhari and Mirman (1980) always converges to the infinite-period solution, as long as linear-symmetric solutions for the n-period problem exist.[6]

Next we examine the link between the degree of riskiness and strategic decisions. For games with the same payoff functions and the same natural law of resource reproduction, we examine how different shocks that are linked through the concepts of second- or first-order stochastic dominance affect strategies. Second-order stochastic dominance implies an

[6] The standard Inada condition on the payoff (utility) functions and a simple condition on the distribution function of the random production shock are sufficient to guarantee the existence of solutions to the associated n-period problem.

increase in riskiness, whereas first-order stochastic dominance implies a change in the mean of the shock. In all cases we identify simple conditions on the payoff function of players that explain the direction of the change in exploitation rates as the stochastic structure of the shocks changes.

We then study specific parametric models in order to show the usefulness of our results about games with linear symmetric strategies. In particular, we show that the Levhari-Mirman (1980) model is a knife-edge case where the presence of uncertainty has no impact on the strategies. So, we present an extended example that nests the Levhari-Mirman (1980) model and provides a unique Nash equilibrium in linear-symmetric strategies in which uncertainty plays a crucial role. This example is a key contribution of this paper, indicating a new benchmark for initiating interesting questions about uncertainty in games. One question that becomes easy to analyze with this example, is to look at the effect of changes in risk on the intensity of the 'tragedy of the commons.' In the context of our example, we show that, indeed, there are cases where the 'tragedy of the commons' is mitigated, and cases where it is amplified, as the random parameter becomes more risky.

In Section 2 we describe the general framework of resource exploitation games and point out the technical difficulties that arise for nonlinear strategies. In Section 3 we provide results for general games with a unique Nash equilibrium in linear-symmetric Markovian strategies. We devote Section 4 to showing that the Levhari-Mirman (1980) model is a knife-edge case where the presence of uncertainty does not influence the players' strategies. In Section 5, we provide our analysis of our extended parametric model.

2. The general framework

Time is discrete and the horizon is infinite, i.e. $t = 0, 1, \ldots$. Let the state variable, x, evolve naturally (when no consumption occurs) according to the law of motion,

$$x_{t+1} = \theta_t f(x_t) \ . \tag{1}$$

Here $f' > 0$, $f'' \leq 0$. The random variable θ_t is i.i.d., independent of x_t and,

$$\theta_t \sim \Theta(\theta_t) \ , \qquad t = 0, 1, \ldots \ .$$

Here Θ is the distribution function of θ_t, for all t, with support $S_\theta \subseteq \mathbb{R}_+$ and $E(\theta_t) < \infty$.

We consider $N \geq 1$ identical players. In period t, each player $j \in \{1, ..., N\}$ consumes $c_{j,t} \geq 0$ units of the available stock, and then a realization of the random shock takes place. Next period's level of x is given by,

$$x_{t+1} = \theta_t f\left(x_t - \sum_{i=1}^{N} c_{i,t}\right) \ .$$

Each player $j \in \{1, ..., N\}$ maximizes his expected discounted utility over an infinite-period horizon,

$$E_0\left[\sum_{t=0}^{\infty} \delta^t u(c_{j,t})\right] \ .$$

Here $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is twice continuously differentiable with $u' > 0$, $u'' < 0$. All players have the same lifetime utility.[7] Moreover, $\delta \in (0, 1)$ is the discount factor.

We study a Markov-Nash equilibrium, i.e. an equilibrium with strategies of the form $\{c_i = C_i(x)\}_{i=1}^N$. The problem of player $j \in \{1, ..., N\}$ in Bellman equation form is,

$$V_j(x) = \max_{c_j \geq 0}\left\{u(c_j) + \delta E\left[V_j\left(\theta f\left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_i(x)\right)\right)\right]\right\} \ . \tag{2}$$

[7] Throughout the paper the functions u and f are assumed to have the properties: $u' > 0$, $u'' < 0$, $f' > 0$ and $f'' \leq 0$.

We suppose, for the moment, that the value function is twice continuously differentiable.[8]

The first-order condition, from the dynamic program (2), is

$$u'(c_j) = \delta f'(y)\, E\left[\theta V_j'(\theta f(y))\right] , \qquad \text{with} \qquad y = x - c_j - \sum_{\substack{i=1 \\ i\neq j}}^{N} C_i(x) \ . \tag{3}$$

Using the envelope theorem on (2) yields,

$$V_j'(x) = \delta \left[1 - \sum_{\substack{i=1 \\ i\neq j}}^{N} C_i'(x)\right] f'(y)\, E\left[\theta V_j'(\theta f(y))\right] \ . \tag{4}$$

Combining (4) with (3),

$$V_j'(x) = u'(c_j)\left[1 - \sum_{\substack{i=1 \\ i\neq j}}^{N} C_i'(x)\right] \ . \tag{5}$$

A significant technical difficulty follows from (5). While $u(\cdot)$ is strictly concave, the strategies $C_i(\cdot)$ of the other players need not be convex, so the value function $V_j(\cdot)$ need not be concave. In fact, Mirman (1979, pp. 65-72) provides several examples of games using functions u and f, which meet our general assumptions, but lead to value 'functions' that are not concave, not differentiable, or even not continuous, in fact, they may be correspondences. What is intriguing about the examples in Mirman (1979, pp. 65-72) is that, if the functions u and f were used in a single-controller optimization problem, all the resulting value functions would be twice continuously differentiable and strictly concave.

3. Games with a unique Nash equilibrium in linear-symmetric Markovian strategies

The difficulty with the nonconcavity of the value function in games with 'standard' primitives u and f, suggest an attack on the problem. We investigate problems that deliver a *unique*

[8] This is not a valid assumption for the general problem, as we discuss below.

Markov Nash equilibrium in linear strategies, i.e. problems where $C_i'(x) = \omega$, for all $i \in \{1, ..., N\}$, or unique symmetric linear strategies of the form,

$$C_i(x) = \omega x \ , \qquad i \in \{1, ..., N\} \ , \quad \text{with} \ \ \omega \in (0, 1/N) \ . \tag{6}$$

For games, u, f and Θ, for which a unique strategy of the form in (6) exists, condition (5) becomes,

$$V'(x) = [1 - (N-1)\,\omega]\, u'(\omega x) \ . \tag{7}$$

Under these assumptions, $V' > 0$, while V'' exists and is strictly negative. In other words, a Nash equilibrium with linear symmetric strategies ensures that V is twice continuously differentiable, strictly increasing and strictly concave. Notice that, since we focus on symmetric strategies, the index j is dropped from the value function.

One example of a model with a unique, linear-symmetric, strategy is the model of Levhari and Mirman (1980), where $u(c) = \ln(c)$, $f(x) = x^{\alpha}$, and $\alpha \in (0, 1)$.[9] In this study, we present an extended pair of functions u and f that nests the Levhari-Mirman (1980) example.[10] Since models that deliver unique, linear-symmetric, Markov strategies are identified, we proceed with a general analysis for this class of models. We focus on two issues: (a) the strategic behavior of players, in particular, the "tragedy of the commons," and (b) the effect of uncertainty, or increases in risk, on consumption strategies. We study the effects of "increasing risk" using the concepts of first- and second order stochastic dominance.

In order to achieve our second goal, we compare decisions made in the stochastic model with decisions made in a version of the deterministic model in which the shock θ is always equal to its mean, $E(\theta) = \bar{\theta}$.[11] In the deterministic model, player j's problem, given the

[9] Levhari and Mirman (1980) also examine cases of non-symmetric strategies by allowing the discount factors of players to differ.

[10] This pair of u and f is presented in Section 5. We prove that the linear-strategy symmetric equilibrium in the extended example is unique. This, implies that the linear-strategy symmetric equilibrium in the Levhari-Mirman model (1980) is unique as well.

[11] See Hahn (1970), Stiglitz (1970) and Mirman (1971).

strategies of all other players is,

$$V_j(x) = \max_{c_j \geq 0} \left[u(c_j) + \delta V_j \left(\bar{\theta} f \left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_i(x) \right) \right) \right] . \tag{8}$$

We distinguish between the stochastic model carrying the subscript "s," and the deterministic model with subscript "d," and we formally define the sets of strategies we are focusing on.

Definition 1 For the game $\langle u, f, \Theta \rangle$,[12] the set of interior linear symmetric Markov-Nash strategies is,

$$\mathbb{S}_s = \Big\{ \{C_{s,i}(x)\}_{i=1}^{N} \mid C_{s,i}(x) = C_s(x) = \omega_s x \ , \ i = 1, ..., N, \text{ with } \omega_s \in (0, 1/N) \, ,$$
$$\text{for all } x > 0, \text{ where } C_s(x) \text{ solves problem (2) for all } j \in \{1, ..., N\} \, ,$$
$$\text{and } C_i(x) = C_s(x) \text{ for all } i \in \{1, ..., N\} \text{ with } i \neq j \Big\} \ ,$$

for the stochastic model, and

$$\mathbb{S}_d = \Big\{ \{C_{d,i}(x)\}_{i=1}^{N} \mid C_{d,i}(x) = C_d(x) = \omega_d x \ , \ i = 1, ..., N, \text{ with } \omega_d \in (0, 1/N) \, ,$$
$$\text{for all } x > 0, \text{ where } C_d(x) \text{ solves problem (8) for all } j \in \{1, ..., N\} \, ,$$
$$\text{and } C_i(x) = C_d(x) \text{ for all } i \in \{1, ..., N\} \text{ with } i \neq j \Big\} \ ,$$

for the deterministic model.

Our analysis pertains to games where the triple $\langle u, f, \Theta \rangle$, implies unique, linear-symmetric, strategies for both the deterministic and the stochastic game, i.e., that the sets $\mathbb{S}_s$ and $\mathbb{S}_d$ are both singletons.[13]

[12] Since players are identical and we focus on state-dependent (Markov) strategies, it suffices to denote a game only by $\langle u, f, \Theta \rangle$ for simplicity.

[13] Amir (1996) shows that, for some games, an equilibrium does not exist in the deterministic case, while there is at least one equilibrium for the stochastic version of the same model. In our extended example, presented in Section 5, linear symmetric strategies in both the deterministic and in the stochastic case exist and are unique.

Fix $x > 0$, consider the symmetric equilibrium based on (3). This necessary condition implies

$$\psi_s(\omega) = -u'(\omega x) + \delta f'((1 - N\omega)x) E[\theta V_s'(\theta f((1 - N\omega)x))] = 0 \ , \tag{9}$$

in the stochastic case.[14] The necessary condition from (8) implies

$$\psi_d(\omega) = -u'(\omega x) + \delta \bar{\theta} f'((1 - N\omega)x) V_d'\left(\bar{\theta} f((1 - N\omega)x)\right) = 0 \ , \tag{10}$$

for the deterministic case. Notice that

$$\psi_s'(\omega) > 0 \qquad \text{and} \qquad \psi_d'(\omega) > 0 \qquad \text{for all } \omega \in \left(0, \frac{1}{N}\right), \tag{11}$$

so there can be at most one solution in each case.[15] A solution exists in both cases, i.e., there exist unique ω_s and ω_d such that

$$\psi_s(\omega_s) = \psi_d(\omega_d) = 0 \ . \tag{12}$$

3.1 The tragedy of the commons

Theorem 1 demonstrates a global result about the strategic behavior of players. We show that for all games with unique linear symmetric strategies the tragedy of the commons always holds.

Theorem 1 For any game such that $\mathbb{S}_s$ and $\mathbb{S}_d$ are both singletons, as the number of players, N, increases, the aggregate exploitation rates, $\Omega_s \equiv N\omega_s$ and $\Omega_d \equiv N\omega_d$ increase.

[14] Notice that since (9) must be met for all $x > 0$ in the case of linear-symmetric strategies, the function $\psi_s(\cdot)$ does not depend on x in equilibrium (i.e., when $\omega = \omega_s$). Even if the left-hand side of (9) depends on x whenever (9) is not met with equality (i.e., when $\omega \neq \omega_s$), this potential dependence on x does not affect our analysis, so we discard x for the sake of simplicity.

[15] A *sufficient* condition for the existence of one solution is the Inada condition $\lim_{c \to 0} u'(c) = \infty$ on the utility function.

Proof

For any $x > 0$, a player's necessary condition is,

$$\psi_s(\omega) = -u'(\omega x) + \delta f'((1 - N\omega)x) E[\theta V_s'(\theta f((1 - N\omega)x))] = 0 \ .$$

We can express the first order condition as a function of the aggregate exploitation rate $\Omega \equiv N\omega$, so,

$$\hat{\Psi}^s(\Omega, N) = -u'\left(\frac{\Omega}{N}x\right) + h_s(\Omega) \ ,$$

where

$$h_s(\Omega) = \delta f'((1 - \Omega)x) E[\theta V_s'(\theta f((1 - \Omega)x))] \ .$$

Given that $V_s'' < 0$, and $f'' \leq 0$, $h_s'(\Omega) > 0$. Applying the implicit function theorem on the equilibrium condition

$$-u'\left(\frac{\Omega_s}{N}x\right) + h_s(\Omega_s) = 0 \ ,$$

i.e.,

$$\frac{d\Omega_s}{dN} = \frac{-u''\left(\frac{\Omega_s}{N}x\right)}{-u''\left(\frac{\Omega_s}{N}x\right) + h_s'(\Omega_s)} \frac{\Omega_s x}{N^2} > 0 \ , \tag{13}$$

which proves the result. The argument for the deterministic case is the same.□

Our result is sharp for the class of games that we examine. Nevertheless, Theorem 1 is an indication that the 'tragedy of the commons' is robust for games of joint exploitation.[16]

3.2 Uncertainty

3.2.1 A general characterization

We analyze the relationship between risk and strategic behavior by comparing outcomes of a stochastic game to outcomes of its deterministic analogue. Theorem 2 is similar to Theorem 2 in Mirman (1971), adjusted to accommodate a symmetric equilibrium in linear strategies.

[16] Other studies that examine the issue of the tragedy of the commons include Dutta and Sundaram (1993), Sorger (1998, 2005), and Dockner and Sorger (1996).

Theorem 2 For any game $\langle u, f, \Theta \rangle$ such that $\mathbb{S}_s$ and $\mathbb{S}_d$ are both singletons,

$$\omega_s \gtreqless \omega_d \Longleftrightarrow E\left[\Lambda_s\left(\theta \rho_d\right)\right] \lesseqgtr \Lambda_d\left(\bar{\theta} \rho_d\right) \tag{14}$$

where,

$$\Lambda_s(z) \equiv z V_s'(z) \ , \quad \Lambda_d(z) \equiv z V_d'(z) \quad \text{and} \quad \rho_d \equiv f\left(\left(1 - N\omega_d\right) x\right) \ ,$$

for all $x > 0$.

Proof

Fix $x > 0$. Given that, by assumption, both the deterministic and the stochastic strategies are interior ($\omega_s, \omega_d \in (0\,, 1/N)$), then $\psi_d(\omega_d) = \psi_s(\omega_s) = 0$. Yet,

$$\omega_s \gtreqless \omega_d \Longleftrightarrow \psi_s(\omega_d) \lesseqgtr 0 \Longleftrightarrow \psi_s(\omega_d) \lesseqgtr \psi_d(\omega_d) \ , \tag{15}$$

since ψ_s is strictly increasing on $(0\,, 1/N)$ (see (11)). Moreover, from (9),

$$\psi_s(\omega_d) = -u'(\omega_d x) + \delta \frac{f'\left(\left(1 - N\omega_d\right) x\right)}{f\left(\left(1 - N\omega_d\right) x\right)} E\left[\theta f\left(\left(1 - N\omega_d\right) x\right) V_s'\left(\theta f\left(\left(1 - N\omega_d\right) x\right)\right)\right] \ ,$$

or

$$\psi_s(\omega_d) = -u'(\omega_d x) + \delta \frac{f'\left(\left(1 - N\omega_d\right) x\right)}{\rho_d} E\left[\Lambda_s\left(\theta \rho_d\right)\right] \ . \tag{16}$$

Similarly, from (10),

$$\psi_d(\omega_d) = -u'(\omega_d x) + \delta \frac{f'\left(\left(1 - N\omega_d\right) x\right)}{\rho_d} \Lambda_d\left(\bar{\theta} \rho_d\right) \ . \tag{17}$$

Combining (16) and (17) with (15), the relationship given by (14) is proved for all $x > 0$.□

From Theorem 2 the general features of the value functions, of both the stochastic and the deterministic problems, emerge that are *both necessary and sufficient* for explaining the direction of the impact of uncertainty on consumption. However, the characteristics of value functions must be specified in order to make use of Theorem 2. Yet, within the class of models that deliver unique linear strategies, explicit value functions are easily derived, making Theorem 2 the key to deriving our results.[17] Nevertheless, as the two value functions, V_s and V_d, generally differ, Theorem 2 does not provide a handy linkup between the primitives of a game and strategic behavior under risk. To find that link, notice that from (7)

$$V_s'(x) = [1 - (N-1)\,\omega_s]\, u'(\omega_s x) \ , \tag{18}$$

and

$$V_d'(x) = [1 - (N-1)\,\omega_d]\, u'(\omega_d x) \ . \tag{19}$$

So,

$$E[\Lambda_s(\theta\rho_d)] = [1 - (N-1)\,\omega_s]\, f'((1-N\omega_d)\,x)\, E[\theta u'(\omega_s f((1-N\omega_d)\,x)\,\theta)] \ , \tag{20}$$

while

$$\Lambda_d\left(\bar{\theta}\rho_d\right) = [1 - (N-1)\,\omega_d]\, f'((1-N\omega_d)\,x)\,\bar{\theta}u'\left(\omega_d f((1-N\omega_d)\,x)\,\bar{\theta}\right) \ . \tag{21}$$

In other words, the connection between the primitives of the game, $\langle u, f, \Theta \rangle$ and the value function can be determined in games with linear strategies. This connection, between the value functions and the primitives of the model proves useful in identifying the role of the model's primitives in the mechanics behind the effects of uncertainty on the players' strategies. Our proofs rely upon the existence of a well-behaved recursive mapping for calculating the equilibrium. In particular, this calculation procedure is the solution technique suggested by Levhari and Mirman (1980). We next present some key results about this procedure.

[17]In our parametric examples in Section 5, we provide explicit formulas for the value functions. The necessity part of Theorem 2 proves useful in characterizing parameter choices that lead to the effect of uncertainty on strategies.

3.2.2 The Levhari-Mirman (1980) recursive procedure

Levhari and Mirman (1980) start from the static symmetric equilibrium, where the consumption rates of players are equal to $1/N$. They use this strategy in order to form the value function and then they continue with the two-period problem, calculate the symmetric strategies again, generalizing the process to the n-period problem. In general, if a linear symmetric strategy exists for the n-period problem, then a tractable recursive mapping on the consumption rates $\omega^{(n)}$ can be constructed, where $\omega^{(n)}$ denotes the symmetric-equilibrium consumption strategy of the n-period problem. Characterizing the evolution of $\omega^{(n)}$, as n increases, is sufficient to characterize the evolution of the symmetric consumption functions in our class of models.

For $n = 2, 3, ...$, the n-period problem of a player $j \in \{1, ..., N\}$ is

$$V_{s,j}^{(n)}(x) = \max_{c_j \geq 0} \left\{ u(c_j) + \delta E \left[V_{s,j}^{(n-1)} \left(\theta f \left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_{s,i}^{(n)}(x) \right) \right) \right] \right\} , \tag{22}$$

for the stochastic case, where $V_{s,j}^{(n)}$ and $V_{s,j}^{(n-1)}$ are the n- and $(n-1)$-period value functions of player j, and $C_{s,i}^{(n)}$ is the n-period strategy of player i. Similarly, in the deterministic case, for $n = 2, 3, ...$, the n-period problem of a player $j \in \{1, ..., N\}$ is

$$V_{d,j}^{(n)}(x) = \max_{c_j \geq 0} \left[u(c_j) + \delta V_{d,j}^{(n-1)} \left(\bar{\theta} f \left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_{d,i}^{(n)}(x) \right) \right) \right] . \tag{23}$$

Again, we focus on linear-symmetric Markov-Nash strategies.

Definition 2 For any game, $\langle u, f, \Theta \rangle$, the set of interior linear symmetric Markov-Nash strategies for the n-period problem, $n = 2, 3, ...$, is,

$$\mathbb{S}_s^{(n)} = \left\{ \left\{ C_{s,i}^{(n)}(x) \right\}_{i=1}^{N} \;\middle|\; C_{s,i}^{(n)}(x) = C_s^{(n)}(x) = \omega_s^{(n)} x \ , \ i = 1, ..., N, \text{ with } \omega_s^{(n)} \in (0, 1/N) , \right.$$

for all $x > 0$, where $C_s^{(n)}(x)$ solves problem (22) for all $j \in \{1, ..., N\}$,

given $C_{s,i}^{(n)}(x) = C_s^{(n)}(x)$ for all $i \in \{1, ..., N\}$ with $i \neq j\Big\}$,

for the stochastic model, and

$$\mathbb{S}_d^{(n)} = \left\{ \left\{ C_{d,i}^{(n)}(x) \right\}_{i=1}^{N} \middle| \; C_{d,i}^{(n)}(x) = C_d^{(n)}(x) = \omega_d^{(n)} x \;,\; i = 1, ..., N, \text{ with } \omega_d^{(n)} \in (0, 1/N), \right.$$

for all $x > 0$, where $C_d^{(n)}(x)$ solves problem (23) for all $j \in \{1, ..., N\}$,

given $C_{d,i}^{(n)}(x) = C_d^{(n)}(x)$ for all $i \in \{1, ..., N\}$ with $i \neq j\Big\}$,

for the deterministic model.

We focus on games $\langle u, f, \Theta \rangle$ such that $\mathbb{S}_s$, $\mathbb{S}_d$, $\mathbb{S}_s^{(n)}$ and $\mathbb{S}_d^{(n)}$, $n = 1, 2, ...$, are all singletons. In this class of games, for any given $\omega^{(n)}$, the necessary condition of the $(n+1)$-period problem of player $j \in \{1, ..., N\}$ is,

$$-u'(c_j) + \delta \left[1 - (N-1)\,\omega_s^{(n)}\right] \times$$

$$\times f'\left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_i(x)\right) E\left[\theta u'\left(\omega_s^{(n)} \theta f\left(x - c_j - \sum_{\substack{i=1 \\ i \neq j}}^{N} C_i(x)\right)\right)\right] = 0 \;,$$

where $C_i(x)$ is the strategy of a player $i \neq j$.[18] So, if linear symmetric equilibrium strategies, $\omega_s^{(n+1)}$, exist for the $(n+1)$-period problem, then $\omega_s^{(n+1)}$ solves,[19]

$$\Psi^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) = 0 \;, \tag{24}$$

[18]This necessary condition is derived from (22) following the same steps as in the infinite-horizon case, i.e., after applying the envelope theorem on (22).

[19]As above, with function $\psi_s(\cdot)$, (24) must be met for all $x > 0$ in the case of linear-symmetric strategies, so the function $\Psi^s(\cdot)$ does not depend on x in equilibrium (i.e., when the expression given by (25) is evaluated at $\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right)$, $n = 1, 2, ...$). Even if the expression given by (25) depends on x whenever (24) is not met with equality (i.e., when this expression is evaluated at some $\left(\omega, \omega_s^{(n)}\right)$, with $\omega \neq \omega_s^{(n+1)}$, $n = 1, 2, ...$), this potential dependence on x does not affect our analysis, so we discard x for the sake of simplicity.

where

$$\Psi^s\left(\omega,\omega_s^{(n)}\right)=$$

$$=-u'(\omega x)+\delta\left[1-(N-1)\,\omega_s^{(n)}\right]f'((1-N\omega)\,x)\,E\left[\theta u'\left(\omega_s^{(n)}\theta f((1-N\omega)\,x)\right)\right]\ . \qquad (25)$$

It follows that, due to the assumptions $u''<0$ and $f''\leq 0$,

$$\Psi_1^s\left(\omega,\omega_s^{(n)}\right)>0 \quad\text{and}\quad \Psi_2^s\left(\omega,\omega_s^{(n)}\right)<0\ ,$$

$$\text{for all } x>0,\ \omega\in\left(0,\frac{1}{N}\right),\omega_s^{(n)}\in\left(0,\frac{1}{N}\right]\ .$$

Therefore, if $\omega_s^{(n+1)}$, the solution to (24) exists, and lies in the open interval $(0,1/N)$, after applying the implicit function theorem to (24),

$$\frac{d\omega_s^{(n+1)}}{d\omega_s^{(n)}}=-\frac{\Psi_2^s\left(\omega_s^{(n+1)},\omega_s^{(n)}\right)}{\Psi_1^s\left(\omega_s^{(n+1)},\omega_s^{(n)}\right)}>0\ ,\quad \text{for all } x>0,\ \omega_s^{(n)}\in\left(0,\frac{1}{N}\right]\ . \qquad (26)$$

The same remarks hold for the deterministic case, where the recursive mapping on $\omega_d^{(n)}$ is the solution to

$$\Psi^d\left(\omega_d^{(n+1)},\omega_d^{(n)}\right)=0\ , \qquad (27)$$

where

$$\Psi^d\left(\omega,\omega_d^{(n)}\right)=$$

$$=-u'(\omega x)+\delta\left[1-(N-1)\,\omega_d^{(n)}\right]f'((1-N\omega)\,x)\,\bar{\theta}u'\left(\omega_d^{(n)}\bar{\theta}f((1-N\omega)\,x)\right)\ . \qquad (28)$$

The two conditions, (24) and (27) define two recursive mappings.

Definition 3 The mapping $M_s:[0,1/N]\rightarrow[0,1/N]$ is given by $\Psi^s(M_s(\omega),\omega)=0$. The mapping $M_d:[0,1/N]\rightarrow[0,1/N]$ is given by $\Psi^d(M_d(\omega),\omega)=0$.

The following results provide a characterization for these two mappings. In particular, Lemma 1 shows the importance of imposing an Inada condition on the utility function for obtaining interior solutions (this Inada condition is a sufficient condition).

Lemma 1 If $\lim_{c \to 0} u'(c) = \infty$, for all $\omega_s^{(1)} \in (0,\ 1/N]$ and all $\omega_d^{(1)} \in (0,\ 1/N]$, the sequences $\left\{\omega_s^{(n)}\right\}_{n=2}^{\infty}$ generated by the mapping M_s, and $\left\{\omega_d^{(n)}\right\}_{n=2}^{\infty}$ generated by M_d, are such that $\omega_s^{(n)}$ and $\omega_d^{(n)}$ are unique with $\omega_s^{(n)}, \omega_d^{(n)} \in (0,\ 1/N)$, $n = 2, 3, ...$

Proof

See Appendix A1.□

Lemma 1 leads to a proof of the following result.

Theorem 3 If $\lim_{c \to 0} u'(c) = \infty$, then for $\omega_s^{(1)} = \omega_d^{(1)} = 1/N$, M_s and M_d are convergent, i.e., $\lim_{n \to \infty} \omega_s^{(n)} = \omega_s$ and $\lim_{n \to \infty} \omega_d^{(n)} = \omega_d$.

Proof

See Appendix A1.□

Theorem 3 shows that the procedure that Levhari and Mirman (1980) suggested and implemented in their example, leads to recursive computability of the infinite-horizon strategies for a more general class of games, i.e., the class of games $\langle u, f, \Theta \rangle$ with $\lim_{c \to 0} u'(c) = \infty$, where $\mathbb{S}_s$, $\mathbb{S}_d$, $\mathbb{S}_s^{(n)}$ and $\mathbb{S}_d^{(n)}$, $n = 1, 2, ...$, are all singletons. While this result is interesting in its own right (as we have identified a reliable calculation procedure), some properties of the mappings M_s and M_d are useful in the analysis of uncertainty. Lemma 2 states these properties of M_s and M_d.

Lemma 2 If $\lim_{c \to 0} u'(c) = \infty$, then for any interval $\mho_s = [\check{\omega}_s, \hat{\omega}_s] \subseteq (0,\ 1/N]$,

$$M_s(\check{\omega}_s) \geq \check{\omega}_s \text{ and } M_s(\hat{\omega}_s) \leq \hat{\omega}_s \Rightarrow \omega_s \in \mho_s\ ,$$

$$M_s(\breve{\omega}_s) > \breve{\omega}_s \text{ and } M_s(\hat{\omega}_s) < \hat{\omega}_s \Rightarrow \omega_s \in (\breve{\omega}_s, \hat{\omega}_s) \ ,$$

and

$$\lim_{n\to\infty} \omega_s^{(n)} = \omega_s \text{ for all } \omega_s^{(1)} \in \mho_s \ .$$

Moreover, for any interval $\mho_d = [\breve{\omega}_d, \hat{\omega}_d] \subseteq (0,\ 1/N]$,

$$M_d(\breve{\omega}_d) \geq \breve{\omega}_d \text{ and } M_d(\hat{\omega}_d) \leq \hat{\omega}_d \Rightarrow \omega_d \in \mho_d \ ,$$

$$M_d(\breve{\omega}_d) > \breve{\omega}_d \text{ and } M_d(\hat{\omega}_d) < \hat{\omega}_d \Rightarrow \omega_d \in (\breve{\omega}_d, \hat{\omega}_d) \ ,$$

and

$$\lim_{n\to\infty} \omega_d^{(n)} = \omega_d \text{ for all } \omega_s^{(1)} \in \mho_d \ .$$

Proof

See Appendix A1.□

Lemma 2 is crucial for the comparisons that follow. Specifically, in comparing two distinct models (e.g. the stochastic and the deterministic, or two models with different stochastic structures), we can view the solution to one model as a starting point for calculating the solution to the other model. If this starting point drives the necessary condition of the second model to be positive or negative, then we can identify the direction in which the strategy must be updated. The results, stated by Lemma 2, yield a method for identifying where the fixed point (infinite-horizon equilibrium) of the second model lies.

3.2.3 Uncertainty and strategic decisions

With Lemma 2, we are able to identify the primitive features of the model that are responsible for the impact of uncertainty on strategies. For games $\langle u, f, \Theta \rangle$ with linear symmetric Markov-Nash strategies, we show that the result of Theorem 2 hinges on features of the utility function, u, alone, and not on f. Of course, f plays an implicit role, since features of both f and u interact for the model to have linear strategies. Nevertheless, our results identify simple conditions on u that lead to specific effects of uncertainty on strategies.

Apart from analyzing the comparison between the stochastic and the deterministic game, (i) we employ the concept of second-order stochastic dominance of the distribution of the shock in order to 'increase risk', and, (ii) we alter the stochastic structure by using first-order stochastically dominated shocks. We find that, in games $\langle u, f, \Theta \rangle$ with linear-symmetric strategies, only the structure of the utility function is needed to explain the impact of changing risk on strategies. In particular, in the case of first-order stochastic dominance, it is the coefficient of relative risk aversion that is responsible for the effect of changes in risk on strategies.

Before proceeding, for a given pair u and f, we declare the set of distribution functions of the shock, Θ, which guarantee that the optimization problem of all players is well-defined and that linear symmetric Markovian strategies exist.[20]

Definition 4 For any pair u and f, with $\lim_{c \to 0} u'(c) = \infty$,

$$\eth(u, f) = \left\{ \Theta \;\middle|\; \text{the game } \langle u, f, \Theta \rangle \text{ is such that } \mathbb{S}_s, \mathbb{S}_d, \mathbb{S}_s^{(n)} \text{ and } \mathbb{S}_d^{(n)}, \; n = 1, 2, ..., \text{ are all non-empty.} \right\}.$$

Comparison of a stochastic game with its deterministic analogue

Theorem 4 Given a pair u and f, $\lim_{c \to 0} u'(c) = \infty$, for all $\Theta \in \eth(u, f)$, with $\mathbb{S}_s, \mathbb{S}_d, \mathbb{S}_s^{(n)}$ and $\mathbb{S}_d^{(n)}$, $n = 1, 2, ...,$ of the game $\langle u, f, \Theta \rangle$ being all singletons, (i)

[20] Whether a player's optimization problem is well-defined in a stochastic Markovian game can depend on the nature of the shock. For example, if the support of the shock is unbounded, conditions must be placed on the distribution of the shock in order to guarantee that value functions of players exist. In the context of the general single-controller stochastic growth model (which is the same as the model of Brock and Mirman (1972)), Stachurski (2002) identifies a simple condition on the mean of some monotonic transformation of the random shock that is sufficient to guarantee a well-behaved optimization problem and a well-defined long-run stationary distribution. Unlike Stachurski (2002), we do not provide such a condition for the general game, but we do so in the context of our more specific analysis in Section 5.

$\omega_s < \omega_d$, if and only if $\lambda(z)$ is strictly convex, (ii) $\omega_s > \omega_d$, if and only if $\lambda(z)$ is strictly concave, and (iii) $\omega_s = \omega_d$, if and only if $\lambda(z)$ is affine, where

$$\lambda(z) = zu'(z) \ . \tag{29}$$

Proof

See Appendix A2.□

Theorem 4 implies that, for the class of games with linear symmetric strategies, the model's characteristics behind the result of Theorem 2 are given solely by a condition that pertains to the utility function, u. This does not mean that f does not play any role in the linkup between uncertainty and strategic behavior. Together with u, the function f is crucial for placing a game $\langle u, f, \Theta \rangle$ in the class of games with linear strategies. The implication of Theorem 4 is that, in this class of games, the effect of uncertainty on strategic behavior is determined by a condition on the utility function. Moreover, notice that Theorem 4 does not require that $u(\cdot)$ be thrice continuously differentiable.[21]

We provide two additional characterizations based on comparisons of games using the notions of second- and first-order stochastic dominance.

Comparison of two stochastic games with the same u and f, where one shock second-order stochastically dominates the other We examine two games, $\langle u, f, \Theta \rangle$ and $\left\langle u, f, \tilde{\Theta} \right\rangle$, that have linear strategies, ω_s and $\tilde{\omega}_s$, with shocks denoted by $\theta \sim \Theta(\theta)$ and $\tilde{\theta} \sim \tilde{\Theta}\left(\tilde{\theta}\right)$, and such that one shock is riskier than the other. Changes in riskiness of shocks are captured by the concept of second-order stochastic dominance that is given by Definition 5.

[21]For a discussion of this point see Mirman (1971, p. 182) in his analysis of a two-period problem.

Definition 5 *Let two random variables,* $\tilde{X}$ *and* X*,defined on a common probability space, with both supports being subsets of* $Z \subseteq \mathbb{R}_+$*. Then* X *second-order stochastically dominates* $\tilde{X}$*, or* $\tilde{X} \preceq_{SSD} X$*, if*

$$E[h(X)] \geq E\left[h\left(\tilde{X}\right)\right] \quad \text{for all concave functions } h \ .$$

Theorem 5 provides conditions that dictate the effect of increasing risk on strategic decisions.

Theorem 5 Given a pair u and f, $\lim_{c\to 0} u'(c) = \infty$, for all $\Theta, \tilde{\Theta} \in \eth(u, f)$, $\Theta \neq \tilde{\Theta}$, such that $\tilde{\theta} \preceq_{SSD} \theta$, and $\mathbb{S}_s, \mathbb{S}_d, \mathbb{S}_s^{(n)}$ and $\mathbb{S}_d^{(n)}$, $n = 1, 2, ...$, are all singletons for both games $\langle u, f, \Theta \rangle$ and $\left\langle u, f, \tilde{\Theta} \right\rangle$, (i) if $\lambda(z)$ is strictly convex, then $\tilde{\omega}_s < \omega_s$, (ii) if $\lambda(z)$ is strictly concave, then $\tilde{\omega}_s > \omega_s$, and, (iii) if $\lambda(z)$ is affine, then $\tilde{\omega}_s = \omega_s$.

Proof

See Appendix A2.□

Comparison of two stochastic games with the same u and f, where one shock first-order stochastically dominates the other

Definition 6 *Let two random variables,* $\tilde{X}$ *and* X*, in a common probability space, with both supports being subsets of* $Z \subseteq \mathbb{R}_+$*. Then* X *first-order stochastically dominates* $\tilde{X}$*, or* $\tilde{X} \preceq_{FSD} X$*, if*

$$E[h(X)] \geq E\left[h\left(\tilde{X}\right)\right] \quad \text{for all non-decreasing functions } h \ .$$

Unlike the case of second-order stochastic dominance, first-order stochastic dominance, encompasses cases where the two shocks have different means.[22]

Theorem 6 Given a pair u and f, $\lim_{c \to 0} u'(c) = \infty$, for all $\Theta, \tilde{\Theta} \in \eth(u, f)$, $\Theta \neq \tilde{\Theta}$, such that $\tilde{\theta} \preceq_{FSD} \theta$, and $\mathbb{S}_s, \mathbb{S}_d, \mathbb{S}_s^{(n)}$ and $\mathbb{S}_d^{(n)}$, $n = 1, 2, ...$, are all singletons for both $\langle u, f, \Theta \rangle$ and $\left\langle u, f, \tilde{\Theta} \right\rangle$,

$$\lambda'(z) \gtreqqless 0 \quad \text{for all } z > 0 \Rightarrow \tilde{\omega}_s \gtreqqless \omega_s \ . \tag{30}$$

Proof

See Appendix A3.□

Remark 1

$$\lambda'(z) \gtreqqless 0 \quad \text{for all } z > 0 \Leftrightarrow -\frac{zu''(z)}{u'(z)} \lesseqqgtr 1 \text{ for all } z > 0 \ ,$$

i.e. the sufficient condition (30) is tightly linked with the value of the coefficient of relative risk aversion.

We proceed to examining specific examples. Through specific examples we are able to demonstrate the usefulness of our general results about games with unique linear symmetric Markov-Nash strategies.

[22] Moreover, Definition 6 is equivalent to the following condition (see Lippman and McCall (1981, pp. 215-6, Theorem 1)).

> *Let two random variables, $\tilde{X}$ and X, in a common probability space, with both supports being subsets of $Z \subseteq \mathbb{R}$, with distribution functions $\tilde{F}$ and F. We say that X first-order stochastically dominates $\tilde{X}$, or $\tilde{X} \preceq_{FSD} X$, if*
>
> $$\tilde{F}(z) \geq F(z) \quad \text{for all } z \in Z \ .$$

4. The stochastic Levhari-Mirman (1980) model

Using the Levhari-Mirman (1980) functions, namely $u(c) = \ln(c)$ and $f(x) = x^{\alpha}$, the value function of each player in the stochastic model is of the form,

$$V_s(x) = \frac{\alpha}{1-\alpha\delta}\ln(x) + b_s \ , \tag{31}$$

whereas the value function in the deterministic case is,

$$V_d(x) = \frac{\alpha}{1-\alpha\delta}\ln(x) + b_d \ , \tag{32}$$

where b_s and b_d are constants. So,

$$\Lambda_s(z) = \Lambda_d(z) = \frac{\alpha}{1-\alpha\delta} \quad \text{for all } z > 0 \ .$$

Theorem 2 implies that $\omega_s = \omega_d$. In fact,

$$\omega_s = \omega_d = \frac{1-\alpha\delta}{N(1-\alpha\delta)+\alpha\delta} \ . \tag{33}$$

In brief, the presence of uncertainty does not alter the rate of consumption in the Levhari-Mirman (1980) model. The difference between the stochastic and the deterministic model is that in the stochastic case the state variable evolves randomly and approaches a long-run stationary distribution. Consequently, the payoffs of players are also random in each period.

With the aid of Theorem 2 we show that the Levhari-Mirman model is a knife-edge case. We do this by extending the Levhari-Mirman model to a class of models for which uncertainty affects strategies and then showing that it is only in the Levhari-Mirman model that strategies are not affected by the presence of uncertainty, i.e., the Levhari-Mirman model is a knife edge. This extension to a more general "Great Fish War" model allow us to study the effect of uncertainty on the strategies in a game with uncertainty. This task is undertaken below.

5. An extended example

Consider the case where,

$$u(c) = \frac{c^{1-\frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}} \ , \tag{34}$$

and

$$f(x) = \left[\alpha x^{1-\frac{1}{\eta}} + (1-\alpha)\,\phi^{1-\frac{1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \ , \tag{35}$$

with $\eta > 0$, $\phi \geq 0$ and $\alpha \in (0,1]$. Notice that for $\eta = 1$, $u(c) = \ln(c)$ and $f(x) = \phi^{1-\alpha}x^{\alpha}$, i.e. the Levhari-Mirman model. A similar example, applied to problems of Cournot oligopoly, has been presented in Koulovatianos and Mirman (2007).[23]

5.1 Uniqueness of interior infinite-horizon linear-symmetric strategies

As in Stachurski (2002), we identify a single sufficient condition on the mean of the distribution of the transformed random variable, $\theta_t^{1-\frac{1}{\eta}}$, in order that the stochastic equilibrium be well-defined.

Proposition 1 *If* u and f are given by (34) and (35), and Θ is such that,

$$E\left(\theta^{1-\frac{1}{\eta}}\right) \equiv \zeta < \frac{1}{\alpha\delta} \ , \quad \text{and} \quad [E(\theta)]^{1-\frac{1}{\eta}} \equiv \bar{\zeta} < \frac{1}{\alpha\delta} \ , \tag{36}$$

[23]For a game with linear symmetric strategies, take (34) and consider a general CES production function

$$f(x) = \left[\alpha x^{1-\frac{1}{\gamma}} + (1-\alpha)\,\phi^{1-\frac{1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}} \ .$$

From the general first-order conditions of a game with linear strategies,

$$u'(\omega x) = \delta\,[1-(N-1)\,\omega]\, f'((1-N\omega)\,x)\, E\left[\theta u'(\omega f((1-N\omega)\,x)\,\theta)\right]$$

(see (3) and (7)). Using (34) and the CES production function,

$$x^{-\frac{1}{\eta}} = \delta\,[1-(N-1)\,\omega]\,\alpha\left(\frac{y}{x}\right)^{\frac{1}{\gamma}}(1-N\omega)^{-\frac{1}{\gamma}}\, y^{-\frac{1}{\eta}} E\left(\theta^{1-\frac{1}{\eta}}\right) \ ,$$

with $y \equiv f((1-N\omega)\,x)$. Setting $\eta = \gamma$ is the only way to obtain linear strategies with these functions. The single-controller version of this example with uncertainty (i.e. $N = 1$), is similar to this presented by Benhabib and Rustichini (1994).

then ω_s satisfying,

$$\omega_s = \frac{1}{\alpha\delta\zeta}\left[(1-N\omega_s)^{\frac{1}{\eta}} - \alpha\delta\zeta\,(1-N\omega_s)\right] \ , \tag{37}$$

and ω_d satisfying,

$$\omega_d = \frac{1}{\alpha\delta\bar{\zeta}}\left[(1-N\omega_d)^{\frac{1}{\eta}} - \alpha\delta\zeta\,(1-N\omega_d)\right] \ , \tag{38}$$

are the unique linear-symmetric strategies of the infinite-horizon stochastic game and the deterministic game, respectively.

Proof

It is easy to verify that (37) is equivalent to $\Psi^s(\omega_s, \omega_s, x) = 0$ and that (38) is equivalent to $\Psi^d(\omega_d, \omega_d, x) = 0$. Apart from showing that ω_s and ω_d are unique, we also show that the optimization problem of each player is well-defined. From the two value functions,

$$V_s(x) = \frac{\alpha\omega_s^{1-\frac{1}{\eta}}}{1-\alpha\delta\zeta\,(1-N\omega_s)^{1-\frac{1}{\eta}}}\frac{x^{1-\frac{1}{\eta}}-1}{1-\frac{1}{\eta}} + b_s \ , \tag{39}$$

and,

$$V_d(x) = \frac{\alpha\omega_d^{1-\frac{1}{\eta}}}{1-\alpha\delta\bar{\zeta}\,(1-N\omega_d)^{1-\frac{1}{\eta}}}\frac{x^{1-\frac{1}{\eta}}-1}{1-\frac{1}{\eta}} + b_d \ , \tag{40}$$

where b_s and b_d are constants, it is transparent that if $N\omega_s = \Omega_s \in (0,1)$ and $N\omega_d = \Omega_d \in (0,1)$, then the problem of each player is well-defined. To show that $\Omega_s \in (0,1)$, we examine two cases.

Case 1: $\eta > 1$

Focusing on aggregate consumption rates, $\Omega \equiv N\omega$, we express (37) as,

$$N = \frac{\alpha\delta\zeta\Omega}{(1-\Omega)^{\frac{1}{\eta}} - \alpha\delta\zeta\,(1-\Omega)} \equiv H(\Omega) \ . \tag{41}$$

Here,

$$H(0) = 1 \quad \text{and} \quad H(1) = \infty \ ,$$

while

$$H'(\Omega) = \alpha\delta\zeta \frac{\frac{1-\left(1-\frac{1}{\eta}\right)\Omega}{(1-\Omega)^{1-\frac{1}{\eta}}} - \alpha\delta\zeta}{\left[(1-\Omega)^{\frac{1}{\eta}} - \alpha\delta\zeta\,(1-\Omega)\right]^2} > 0 \ , \quad \text{for all } \Omega \in (0,1) \ .$$

To see the last statement, notice that $1 - \left(1 - \frac{1}{\eta}\right)\Omega \geq (1-\Omega)^{1-\frac{1}{\eta}}$ for all $\Omega \in [0,1)$, with equality if and only if $\Omega = 0$. So, applying the intermediate-value theorem to (41) shows that $\Omega_s \in (0,1)$, a unique symmetric equilibrium in linear strategies. This case is depicted by Figure 1. For $\Omega_d \in (0,1)$ and uniqueness, replace ζ with $\bar{\zeta}$ in the above argument.

Case 2: $\eta < 1$

For this case it is useful to express (37) as,

$$\Gamma(\Omega) \equiv \frac{N}{\Omega} = \frac{\alpha\delta\zeta}{(1-\Omega)^{\frac{1}{\eta}} - \alpha\delta\zeta\,(1-\Omega)} \equiv \Xi(\Omega) \ . \tag{42}$$

Where $\Gamma(0) = \infty$ and $\Xi(0) = \alpha\delta\zeta / (1 - \alpha\delta\zeta)$, and

$$\Xi(\Omega) > 0 \ \text{ if } \ \Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right] \ .$$

Due to (36) this interval is nonempty. Moreover,

$$\Xi'(\Omega) = \alpha\delta\zeta \frac{\frac{1}{\eta}(1-\Omega)^{\frac{1}{\eta}-1} - \alpha\delta\zeta}{\left[(1-\Omega)^{\frac{1}{\eta}} - \alpha\delta\zeta\,(1-\Omega)\right]^2} > 0 \quad \text{for all } \ \Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right]$$

as $0 < (1-\Omega)^{\frac{1}{\eta}} - \alpha\delta\zeta\,(1-\Omega) < \left[\frac{1}{\eta}(1-\Omega)^{\frac{1}{\eta}-1} - \alpha\delta\zeta\right](1-\Omega)$ for all $\Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right]$. Finally,

$$\Xi\left(1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right) = \infty \ .$$

Given that $\Gamma\left(1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right) < \infty$ and $\Gamma'(\Omega) < 0$ for all $\Omega \in \left[0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right]$, it follows that $\Omega_s \in \left(0\,,\, 1 - (\alpha\delta\zeta)^{\frac{\eta}{1-\eta}}\right) \subset (0,1)$, and it is unique. This is also shown by Figure 2. For $\Omega_d \in (0,1)$ and the uniqueness of it replace ζ with $\bar{\zeta}$ in the above argument.

For the last case of $\eta = 1$, see Section 4 above, to verify that (33) satisfies both (37) and (38), and also that $\Omega_s = \Omega_d \in (0,1)$.□

Note that Proposition 1 shows that the Levhari-Mirman model is indeed a knife edge case for $\eta = 1$ in our extended example. According to our analysis in Section 4, in the case of $\eta = 1$ of the extended example, uncertainty plays no role.[24] However, for other values of η, uncertainty plays a major role .

5.2 Uniqueness of interior finite-horizon linear-symmetric strategies

Substituting (34) and (35) into the necessary conditions of the finite-horizon problem given by (24) and (27), we find,

$$M_s(\omega) = \frac{1}{N + (\alpha\delta\zeta)^{\eta} \frac{[1-(N-1)\omega]^{\eta}}{\omega}} , \tag{43}$$

and

$$M_d(\omega) = \frac{1}{N + \left(\alpha\delta\bar{\zeta}\right)^{\eta} \frac{[1-(N-1)\omega]^{\eta}}{\omega}} . \tag{44}$$

It is straightforward from (43) and (44) that $M_s(\omega), M_d(\omega) \in (0, 1/N)$ for all $\omega \in (0, 1/N]$, implying that all finite-horizon strategies are interior and unique. Therefore, we can apply Theorems 4, 5 and 6, to our model.

5.3 Impact of uncertainty on strategies in the example

5.3.1 Application of Theorem 2

Let $\rho_d = f((1 - N\omega_d)x)$, and recall the definitions of Λ_s and Λ_d given in Theorem 2, notice that

$$E[\Lambda_s(\theta\rho_d)] = \frac{\alpha\zeta\omega_s^{1-\frac{1}{\eta}}}{1 - \alpha\delta\zeta(1 - N\omega_s)^{1-\frac{1}{\eta}}} \rho_d^{1-\frac{1}{\eta}} . \tag{45}$$

while

$$\Lambda_d\left(\bar{\theta}\rho_d\right) = \frac{\alpha\bar{\zeta}\omega_d^{1-\frac{1}{\eta}}}{1 - \alpha\delta\bar{\zeta}(1 - N\omega_d)^{1-\frac{1}{\eta}}} \rho_d^{1-\frac{1}{\eta}} . \tag{46}$$

[24]Notice also that, for $\eta = 1$, (39) and (40) yield (31) and (32).

From (37),

$$\frac{\alpha\zeta\omega_s^{1-\frac{1}{\eta}}}{1-\alpha\delta\zeta\left(1-N\omega_s\right)^{1-\frac{1}{\eta}}}=\left(\frac{1}{\omega_s}-N\right)^{\frac{1}{\eta}},$$

so substituting this expression into (45),

$$E\left[\Lambda_s\left(\theta\rho_d\right)\right]=\left(\frac{1}{\omega_s}-N\right)^{\frac{1}{\eta}}\rho_d^{1-\frac{1}{\eta}}\ . \tag{47}$$

After using (38), as was done with (37), (46) becomes,

$$\Lambda_d\left(\bar{\theta}\rho_d\right)=\left(\frac{1}{\omega_d}-N\right)^{\frac{1}{\eta}}\rho_d^{1-\frac{1}{\eta}}\ . \tag{48}$$

Proposition 2 identifies the parameters that are behind the comparison given in Theorem 2.

Proposition 2 *If* u and f are given by (34) and (35), and Θ is such that (36) is met, then

$$\eta \lesseqgtr 1 \Leftrightarrow \omega_s \lesseqgtr \omega_d\ .$$

Proof By applying the implicit function theorem to (41) and (42) and with the aid of Figures 1 and 2, it follows that

$$\omega_s = Z\left(\zeta\right) \quad \text{and} \quad \omega_d = Z\left(\bar{\zeta}\right)\ ,$$

where $Z'\left(z\right)<0$ for all $z\in\left(0,1/\left(\alpha\delta\right)\right)$. So,

$$\omega_s \lesseqgtr \omega_d \Leftrightarrow E\left(\theta^{1-\frac{1}{\eta}}\right) \gtreqless \left[E\left(\theta\right)\right]^{1-\frac{1}{\eta}} \Leftrightarrow \eta \lesseqgtr 1\ , \tag{49}$$

as the relationship between η and 1 determines whether the function $\lambda\left(z\right)=z^{1-\frac{1}{\eta}}$ is concave, convex or affine (in the present setting, when $\eta=1$, $\lambda\left(z\right)$ is equal to unity for all $z>0$). The result follows from Jensen's inequality.□

5.3.2 Application of Theorems 4, 5 and 6

In our example,

$$\lambda(c) = u'(c)\,c = c^{1-\frac{1}{\eta}} \Rightarrow \begin{cases} \lambda'(c) \gtreqless 0 \Leftrightarrow \eta \gtreqless 1 \\ \lambda''(c) \lesseqgtr 0 \Leftrightarrow \eta \gtreqless 1 \end{cases} . \tag{50}$$

Since u is thrice differentiable, the concavity of λ can be examined through the sign of its second derivative. Notice that, (50) illustrates the connection between Theorems 4, 5 and 6 and Theorem 2, as well as, with the result of Proposition 2. Most importantly, it shows how the properties of u can affect the role of uncertainty on strategies.

5.4 Changes in risk and the intensity of the tragedy of the commons

In this section we investigate whether increasing risk has an impact on the rate at which aggregate exploitation rates increase with the number of players. This investigation requires a comparison of aggregate exploitation rates along two dimensions, the number of players and the degree of riskiness (in the sense of second-order stochastic dominance). In performing such a comparison, if we employ two random variables, θ and $\tilde{\theta}$, with a discrete difference in the degree of riskiness, the comparison becomes unclear, as the initial aggregate exploitation rates can also be discretely different before increasing the number of players. For this reason, we must employ a concept of changing risk that allows for marginal increases in riskiness. For example, consider a lognormally distributed shock,

$$\ln(\theta) \sim N\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right). \tag{51}$$

The expectation of θ is,

$$E(\theta) = e^{\mu} \ ,$$

whereas its variance is,

$$Var(\theta) = e^{2\mu}\left(e^{\sigma^2} - 1\right),$$

i.e., the parameter σ has an impact only on the variance of θ and not on its mean (but the parameter μ has an impact on both the mean and the variance of θ). So, any two distributions given by (51) with different values of parameter σ are linked through second-order stochastic dominance. In particular, if $\theta \sim \Theta(\theta)$ with parameters (μ, σ) and $\tilde{\theta} \sim \tilde{\Theta}\left(\tilde{\theta}\right)$ with parameters $(\mu, \tilde{\sigma})$, then $\tilde{\sigma} > \sigma$ implies that $\tilde{\theta} \preceq_{SSD} \theta$.

Proposition 3 examines the impact of increases in risk on the intensity of the tragedy of the commons.

Proposition 3 *If* u and f are given by (34) and (35), Θ obeys (51) and condition (36) is met, then, (i) the tragedy of the commons is amplified by an increase in riskiness if and only if $\eta < 1$, (ii) the tragedy of the commons is mitigated by an increase in riskiness if and only if $\eta > 1$, (iii) the tragedy of the commons is unaffected by an increase in riskiness if and only if $\eta = 1$.

Proof

See Appendix A4.□

Proposition 3 shows that, if, $\eta > 1$, the overexploitation tendency is mitigated as both riskiness and the number of players increase. Note, however, Theorem 5 states that when $\eta > 1$, for a fixed number of players, all players would tend to increase their consumption rates as uncertainty increases. These results indicate the complex, yet interesting, strategic behavior in Nash equilibrium outcomes.

6. Appendix A1 - Proofs of Lemma 1, Theorem 3, and Lemma 2

Proof of Lemma 1

Fix any $x > 0$ and any $\omega_s^{(n)} \in (0,\ 1/N]$. Then, from (25), $\Psi^s\left(\omega, \omega_s^{(n)}\right)$ can be expressed as,

$$\Psi^s\left(\omega, \omega_s^{(n)}\right) = g(\omega) + h(\omega) \ ,$$

where

$$g(\omega) = -u'(\omega x) \ ,$$

and

$$h(\omega) = \delta\left[1 - (N-1)\,\omega_s^{(n)}\right] f'\left((1-N\omega)\,x\right) E\left[\theta u'\left(\omega_s^{(n)}\theta f\left((1-N\omega)\,x\right)\right)\right] > 0 \ .$$

Notice that,

$$h(\omega) > 0 \ , \ \text{ for all } \omega \in (0, 1/N) \ , \quad \text{and} \quad \lim_{\omega \to 0} g(\omega) = -\infty \ ,$$

so $\Psi^s\left(\omega, \omega_s^{(n)}\right)$ intersects the zero axis within the interval $(0, 1/N)$ at least once. Yet, the fact that, for all $\omega \in (0, 1/N)$, $h'(\omega), g'(\omega) > 0$, implies that $\Psi^s\left(\omega, \omega_s^{(n)}\right) = 0$ has a unique solution, $\omega^{(n+1)} \in (0, 1/N)$. The same argument can be used for the deterministic analogue of the stochastic model.□

Proof of Theorem 3

Fix $x > 0$, for the 1-period stochastic game, the only symmetric Nash equilibrium is $\omega_s^{(1)} = 1/N$ for all players. Using the same argument as in the proof of Lemma 1, there exists a unique $\omega_s^{(2)} \in (0, 1/N)$, i.e. $\omega_s^{(2)} < \omega_s^{(1)}$. Moreover,

$$\omega_s < \omega_s^{(2)} < \omega_s^{(1)} = \frac{1}{N} \ . \tag{52}$$

To see (52), suppose that, instead, $\omega_s^{(2)} \leq \omega_s$. Since $\Psi_1^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) > 0$, $\Psi_2^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) < 0$, and $\Psi^s(\omega_s, \omega_s) = 0$, $0 = \Psi^s(\omega_s, \omega_s) > \Psi^s\left(\omega_s, \omega_s^{(1)}\right) \geq \Psi^s\left(\omega_s^{(2)}, \omega_s^{(1)}\right)$, which contradicts that $\Psi^s\left(\omega_s^{(2)}, \omega_s^{(1)}\right) = 0$. From Lemma 1, $\omega_s^{(1)} = 1/N$ gives rise to a unique sequence $\left\{\omega_s^{(n)}\right\}_{n=1}^{\infty}$ that is generated from (24). Given that $\omega_s \in (0, 1/N)$ is unique, and the mapping M_s is a continuous function with $M_s'\left(\omega_s^{(n)}\right) = -\Psi_2^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) / \Psi_1^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) > 0$ for all $x > 0$, (52) and the intermediate value theorem imply that $\omega_s < \omega_s^{(n+1)} < \omega_s^{(n)}$, $n = 1, 2, ...$ So, $\lim_{n\to\infty} \omega_s^{(n)} = \omega_s$. The same argument holds for the deterministic analogue of the stochastic model.□

Proof of Lemma 2

Fix any $x > 0$ and let $\mho_s = [\breve{\omega}_s, \hat{\omega}_s] \subseteq (0,\ 1/N]$ with $M_s(\breve{\omega}_s) \geq \breve{\omega}_s$ and $M_s(\hat{\omega}_s) \leq \hat{\omega}_s$. Since M_s is a continuous function with $M_s'\left(\omega_s^{(n)}\right) = -\Psi_2^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) / \Psi_1^s\left(\omega_s^{(n+1)}, \omega_s^{(n)}\right) > 0$ for all $x > 0$, the intermediate value theorem implies that $\omega_s \in \mho_s$, as ω_s is, by assumption, unique. As $M_s(\breve{\omega}_s) \geq \breve{\omega}_s$, strict equality holds only if $\breve{\omega}_s = \omega_s$. If $M_s(\breve{\omega}_s) > \breve{\omega}_s$, then $\breve{\omega}_s < \omega_s$. From Lemma 1, any $\omega_s^{(1)} \in (0, 1/N]$ gives rise to a unique sequence $\left\{\omega_s^{(n)}\right\}_{n=1}^{\infty}$, so $\omega_s^{(n)} < \omega_s^{(n+1)} < \omega_s$, $n = 1, 2, ...$, for all $\omega_s^{(1)} \in [\breve{\omega}_s, \omega_s)$. Thus, M_s is stable for all $\omega_s^{(1)} \in [\breve{\omega}_s, \omega_s)$. By a similar argument, if $M_s(\hat{\omega}_s) < \hat{\omega}_s$, $\omega_s^{(n)} > \omega_s^{(n+1)} > \omega_s$, $n = 1, 2, ...$, for all $\omega_s^{(1)} \in (\omega_s, \hat{\omega}_s]$, so M_s is stable for all $\omega_s^{(1)} \in (\omega_s, \hat{\omega}_s]$. Finally, for $M_s(\hat{\omega}_s) \leq \hat{\omega}_s$, equality holds only if $\hat{\omega}_s = \omega_s$, completing the proof. The same argument can be used for the deterministic analogue of the stochastic model.□

7. Appendix A2 - Proof of Theorems 4 and 5

Proof of Theorem 4

Fix any $x > 0$. $\Psi^s\left(\omega, \omega^{(n)}\right)$ and $\Psi^d\left(\omega, \omega^{(n)}\right)$ can be expressed as,

$$\Psi^s\left(\omega, \omega^{(n)}\right) = -u'(\omega x) + \delta \frac{\left[1 - (N-1)\,\omega^{(n)}\right]}{\omega^{(n)}} \frac{f'((1 - N\omega)\,x)}{f((1 - N\omega)\,x)} E\left[\lambda\left(\omega^{(n)}\theta f((1 - N\omega)\,x)\right)\right] \tag{53}$$

and

$$\Psi^d\left(\omega, \omega^{(n)}\right) = -u'(\omega x) + \delta \frac{\left[1 - (N-1)\,\omega^{(n)}\right]}{\omega^{(n)}} \frac{f'((1 - N\omega)\,x)}{f((1 - N\omega)\,x)} \lambda\left(\omega^{(n)}\bar{\theta} f((1 - N\omega)\,x)\right). \tag{54}$$

From (53), (54), and Jensen's inequality,

(a) $\Psi^s\left(\omega, \omega^{(n)}\right) > \Psi^d\left(\omega, \omega^{(n)}\right)$ for all $\omega, \omega^{(n)} \in (0, 1/N)$, and for all Θ, if and only if, $\lambda(\cdot)$ is strictly convex,

(b) $\Psi^s\left(\omega, \omega^{(n)}\right) < \Psi^d\left(\omega, \omega^{(n)}\right)$ for all $\omega, \omega^{(n)} \in (0, 1/N)$, and for all Θ, if and only if, $\lambda(\cdot)$ is strictly concave,

(c) $\Psi^s\left(\omega, \omega^{(n)}\right) = \Psi^d\left(\omega, \omega^{(n)}\right)$ for all $\omega, \omega^{(n)} \in (0, 1/N)$, and for all Θ, if and only if, $\lambda(\cdot)$ is affine.

So, in case (a), $\Psi^d(\omega_s, \omega_s) < \Psi^s(\omega_s, \omega_s) = 0$, so $M_d(\omega_s) > \omega_s$. In the proof of Theorem 3 it is shown that $M_d(1/N) < 1/N$. Then, by Lemma 2, $\omega_d \in (\omega_s, 1/N)$, which proves statement (i). In case (b), $\Psi^s(\omega_d, \omega_d) < \Psi^d(\omega_d, \omega_d) = 0$, so $M_s(\omega_d) > \omega_d$. Since $M_s(1/N) < 1/N$ (from the proof of Theorem 3), Lemma 2 implies that $\omega_s \in (\omega_d, 1/N)$, which proves statement (ii). Finally, statement (iii) is straightforward.□

Proof of Theorem 5

Fix any $x > 0$, the necessary conditions, of the two problems, $\Psi^s\left(\omega, \omega^{(n)}\right)$ and $\tilde{\Psi}^s\left(\omega, \omega^{(n)}\right)$, can be expressed as,

$$\Psi^s\left(\omega, \omega^{(n)}\right) = -u'(\omega x) + \delta \frac{\left[1-(N-1)\,\omega^{(n)}\right]}{\omega^{(n)}} \frac{f'\left((1-N\omega)\,x\right)}{f\left((1-N\omega)\,x\right)} E\left[\lambda\left(\omega^{(n)}\theta f\left((1-N\omega)\,x\right)\right)\right] \tag{55}$$

and

$$\tilde{\Psi}^s\left(\omega, \omega^{(n)}\right) = -u'(\omega x) + \delta \frac{\left[1-(N-1)\,\omega^{(n)}\right]}{\omega^{(n)}} \frac{f'\left((1-N\omega)\,x\right)}{f\left((1-N\omega)\,x\right)} E\left[\lambda\left(\omega^{(n)}\tilde{\theta} f\left((1-N\omega)\,x\right)\right)\right]. \tag{56}$$

Using the expressions (55) and (56), and Definition 5,

(a) if $\lambda(\cdot)$ is strictly convex, then $\tilde{\Psi}^s\left(\omega, \omega^{(n)}\right) > \Psi^s\left(\omega, \omega^{(n)}\right)$ for all $\omega, \omega^{(n)} \in (0, 1/N)$,

(b) if $\lambda(\cdot)$ is strictly concave, then $\tilde{\Psi}^s\left(\omega, \omega^{(n)}\right) < \Psi^s\left(\omega, \omega^{(n)}\right)$ for all $\omega, \omega^{(n)} \in (0, 1/N)$,

(c) if $\lambda(\cdot)$ is affine, then $\tilde{\Psi}^s\left(\omega, \omega^{(n)}\right) = \Psi^s\left(\omega, \omega^{(n)}\right)$ for all $\omega, \omega^{(n)} \in (0, 1/N)$.

So, in case (a), $\Psi^s(\tilde{\omega}_s, \tilde{\omega}_s) < \tilde{\Psi}^s(\tilde{\omega}_s, \tilde{\omega}_s) = 0$, so $M_s(\tilde{\omega}_s) > \tilde{\omega}_s$. In the proof of Theorem 3 it was shown that $M_s(1/N) < 1/N$. By Lemma 2, $\omega_s \in (\tilde{\omega}_s, 1/N)$, which proves statement (i) of the Theorem. In case (b), $\tilde{\Psi}^s(\omega_s, \omega_s) < \Psi^s(\omega_s, \omega_s) = 0$, so $\tilde{M}_s(\omega_s) > \omega_s$. Since $\tilde{M}_s(1/N) < 1/N$ (see the proof of Theorem 3), Lemma 2 implies that $\tilde{\omega}_s \in (\omega_s, 1/N)$, which proves statement (ii) of the Theorem. Finally, statement (iii) is straightforward.□

8. Appendix A3 - Proof of Theorem 6

Proof of Theorem 6

Integration by parts yields,

$$E\left[h\left(\theta\right)\right]-E\left[h\left(\tilde{\theta}\right)\right]=\int_{S_\theta}\left[\tilde{\Theta}\left(z\right)-\Theta\left(z\right)\right]h'\left(z\right)dz\ ,$$

for all differentiable functions h. So, setting $h\left(z\right)=\lambda\left(\omega^{(n)}zf\left(\left(1-N\omega\right)x\right)\right)$, for any $\omega^{(n)}\in(0,1/N]$ and any $\omega\in(0,1/N)$, the fact that $\tilde{\theta}\preceq_{FSD}\theta$ implies,

$$\lambda'\left(z\right)\gtreqqless 0\quad\text{for all }z>0\Rightarrow E\left[\lambda\left(\omega^{(n)}\tilde{\theta}f\left(\left(1-N\omega\right)x\right)\right)\right]\lesseqqgtr E\left[\lambda\left(\omega^{(n)}\theta f\left(\left(1-N\omega\right)x\right)\right)\right]\ . \tag{57}$$

Based on (57), (55), and (56),

(a) $\lambda'\left(z\right)<0$ for all $z>0\Rightarrow\Psi^s\left(\tilde{\omega}_s,\tilde{\omega}_s\right)<\tilde{\Psi}^s\left(\tilde{\omega}_s,\tilde{\omega}_s\right)=0$,

(b) $\lambda'\left(z\right)>0$ for all $z>0\Rightarrow\tilde{\Psi}^s\left(\omega_s,\omega_s\right)<\Psi^s\left(\omega_s,\omega_s\right)=0$,

(c) $\lambda'\left(z\right)=0$ for all $z>0\Rightarrow\tilde{\Psi}^s\left(\omega,\omega^{(n)}\right)=\Psi^s\left(\omega,\omega^{(n)}\right)=0$ for all $\omega,\omega^{(n)}\in(0,1/N)$.

The rest of the proof follows exactly as in Theorem 5.□

9. Appendix A4 - Proof of Proposition 3

Proof of Proposition 3

We denote the aggregate exploitation rate as $\Omega\left(\sigma\right)=N\omega\left(\sigma\right)$. Equation (13) implies (after some algebra) that,

$$\frac{\partial\Omega\left(\sigma\right)}{\partial N}=\frac{\eta\zeta\left(\sigma\right)\omega\left(\sigma\right)\xi\left(\sigma\right)^{\eta-1}}{N\left[1-\frac{N-1}{N}\eta\zeta\left(\sigma\right)\xi\left(\sigma\right)^{\eta-1}\right]}\ . \tag{58}$$

where,

$$\zeta(\sigma) \equiv \left[\alpha\delta E\left(\theta(\sigma)^{1-\frac{1}{\eta}}\right)\right]^{\eta} \ ,$$

and

$$\xi(\sigma) \equiv [1-(N-1)\,\omega(\sigma)] \ .$$

Equation (58) can be re-written as,

$$\frac{\frac{\partial\Omega(\sigma)}{\partial N}}{\Omega(\sigma)} = \frac{\partial \ln[\Omega(\sigma)]}{\partial N} = \frac{\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}}{N^2\left[1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}\right]} \ . \tag{59}$$

Taking the partial derivative of this last expression with respect to σ, yields,

$$\frac{\partial^2 \ln[\Omega(\sigma)]}{\partial N\partial\sigma} = \frac{\eta\zeta'(\sigma)\,\xi(\sigma)^{\eta-2}\left[\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)}\right]}{N^2\left[1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}\right]^2} \ . \tag{60}$$

Since,

$$\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{d\xi(\sigma)}{d\zeta(\sigma)} \ ,$$

we find $d\xi(\sigma)/d\zeta(\sigma)$, by applying the implicit function theorem to (37). In particular, (37) can be written as,

$$\frac{N-1}{N}\zeta(\sigma)\,\xi(\sigma)^{\eta} + \frac{1}{N} - \xi(\sigma) = 0 \ , \tag{61}$$

so,

$$\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{\frac{N-1}{N}\xi(\sigma)^{\eta}}{1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}} \ . \tag{62}$$

After substituting (62), the expression $\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)}$ on the RHS of (60), becomes,

$$\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{\xi(\sigma)-\frac{N-1}{N}\zeta(\sigma)\,\xi(\sigma)^{\eta}}{1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}} \ .$$

Yet, (61) implies that, $\xi(\sigma)-\frac{N-1}{N}\zeta(\sigma)\,\xi(\sigma)^{\eta} = \frac{1}{N}$, so,

$$\xi(\sigma)+(\eta-1)\,\zeta(\sigma)\,\frac{\xi'(\sigma)}{\zeta'(\sigma)} = \frac{1}{N\left[1-\frac{N-1}{N}\eta\zeta(\sigma)\,\xi(\sigma)^{\eta-1}\right]} \ ,$$

and (60) gives,

$$\frac{\partial^2 \ln [\Omega(\sigma)]}{\partial N \partial \sigma} = \frac{\eta \zeta'(\sigma) \xi(\sigma)^{\eta-2}}{N^3 \left[1 - \frac{N-1}{N} \eta \zeta(\sigma) \xi(\sigma)^{\eta-1}\right]^3} . \tag{63}$$

As we have found in Theorem 1,

$$\frac{\partial \Omega(\sigma)}{\partial N} = \frac{\eta \zeta(\sigma) \omega(\sigma) \xi(\sigma)^{\eta-1}}{N \left[1 - \frac{N-1}{N} \eta \zeta(\sigma) \xi(\sigma)^{\eta-1}\right]} > 0 \Rightarrow 1 - \frac{N-1}{N} \eta \zeta(\sigma) \xi(\sigma)^{\eta-1} > 0 ,$$

so, (63) implies that,

$$\frac{\partial^2 \ln [\Omega(\sigma)]}{\partial N \partial \sigma} \gtreqless 0 \Leftrightarrow \zeta'(\sigma) \gtreqless 0 \Leftrightarrow \eta \lesseqgtr 1 ,$$

which proves the Proposition.□

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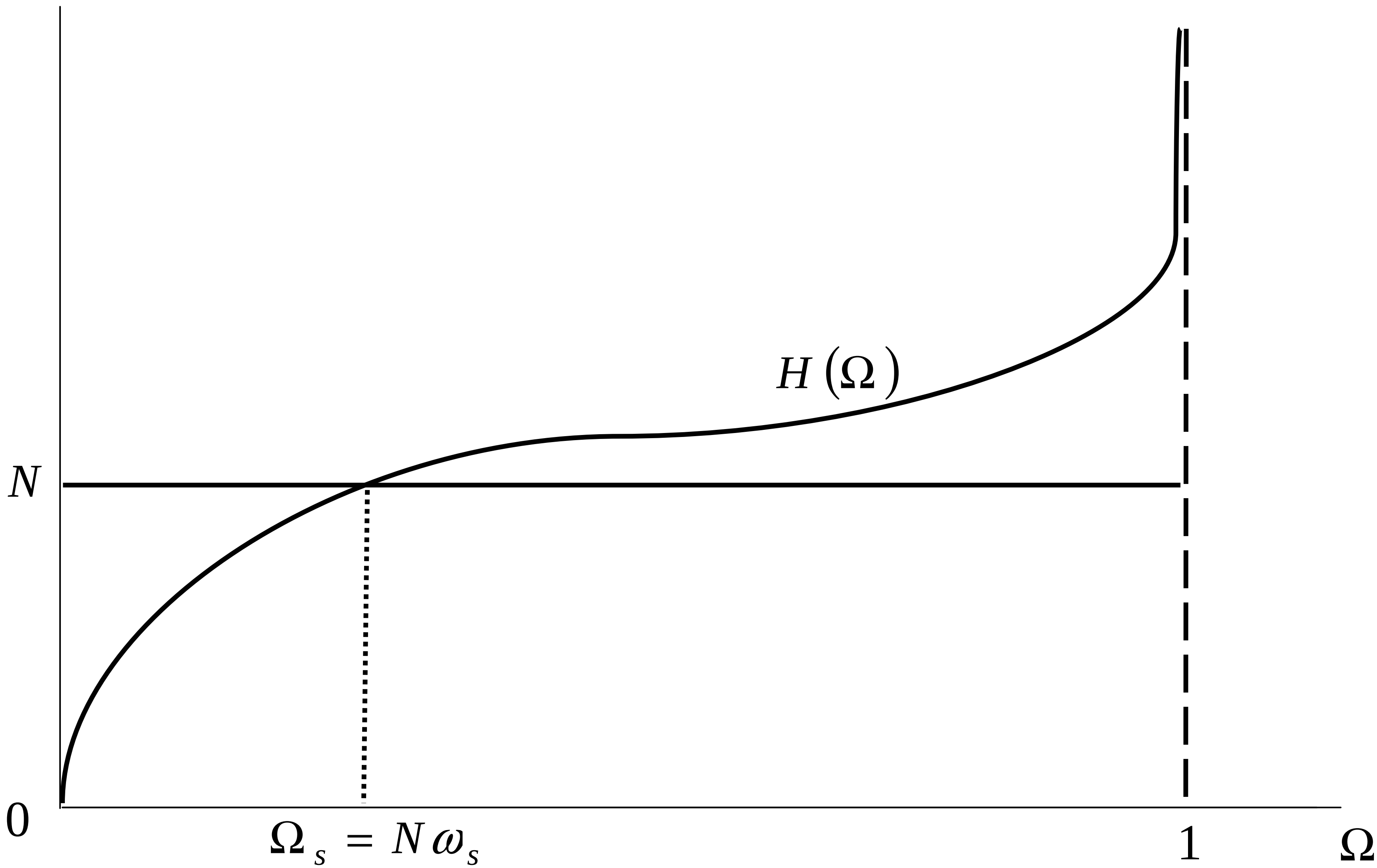

Figure 1 Equilibrium strategies when $\eta > 1$

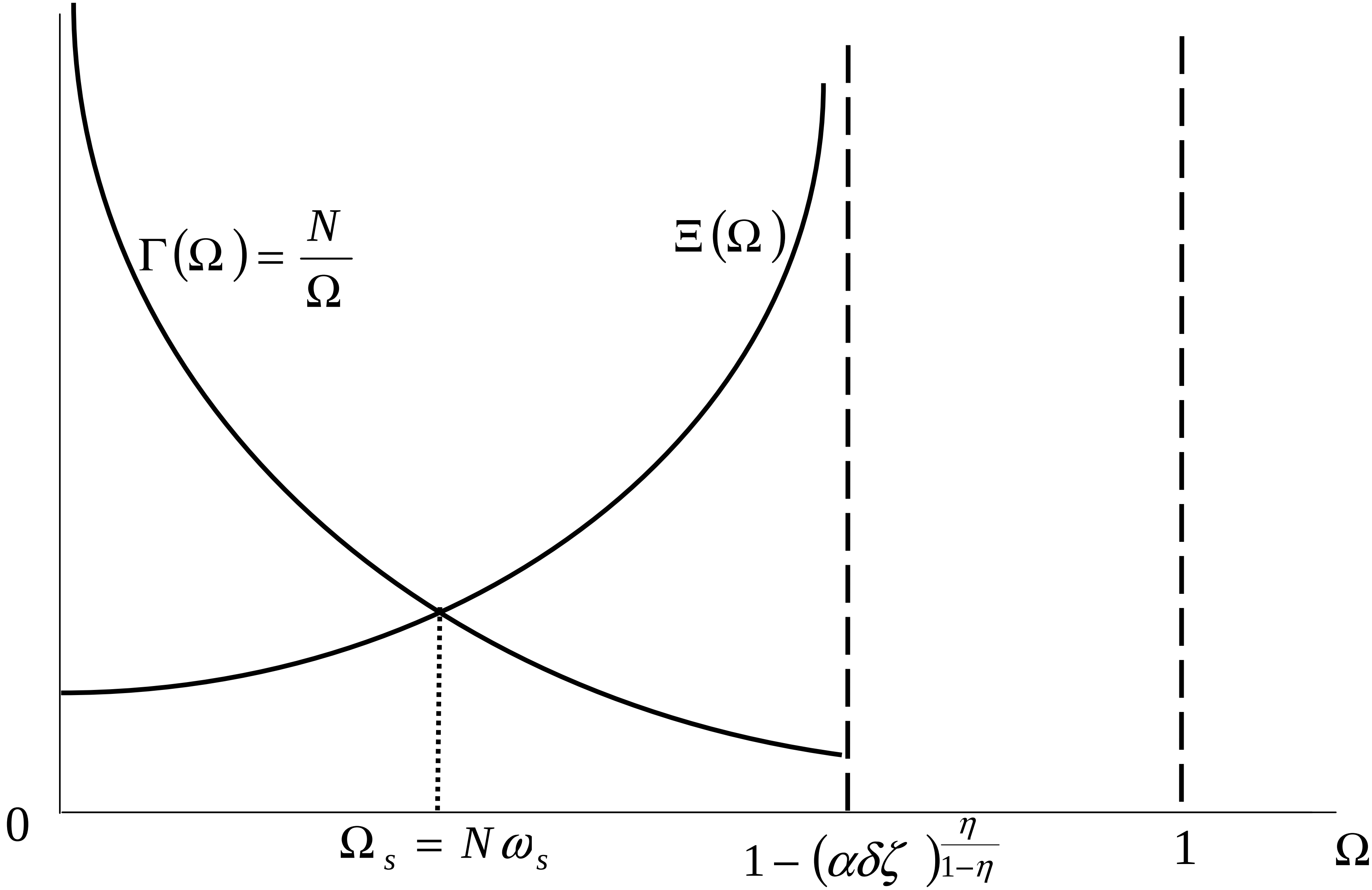

Figure 2 Equilibrium strategies when $\eta < 1$