WORKING PAPERS

Paul Pichler Gerhard Sorger

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October 2006

Working Paper No: 0610

DEPARTMENT OF ECONOMICS

UNIVERSITY OF VIENNA

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Paul Pichler and Gerhard Sorger

This Version: October 2006

Abstract

We study the Ramsey (1928) model under the assumption that households act strategically. We compute the Markov perfect equilibrium for this model and compare it to the original, competitive equilibrium and to a strategic open-loop equilibrium proposed by Sorger (2002, 2005b). We show that, if households are identical, strategic behavior has no influence on the long run evolution of the economy. If households are heterogeneous, however, the Markov perfect equilibrium has properties that differ from those of the competitive and the open-loop equilibrium.

Keywords: Ramsey model, strategic saving, Markov perfect equilibrium, projection methods

Journal of Economic Literature **classification codes**: C63, C73, D91, E21, O41

*The research reported in this paper forms part of the project 'Economic Growth with Strategic Savings Decisions' supported by the Austrian Science Fund (FWF) under project number P17886.

1 Introduction

The model of optimal capital accumulation motivated by Ramsey (1928) has arguably become the most important workhorse in modern dynamic macroeconomics. It forms the core of many models of economic growth, business cycles, monetary economics, international trade, and development economics, among many other fields. Models built on this framework are also used extensively to analyze policy issues.

In most applications, the Ramsey model is set up in the following way. The economy is populated by infinitely many rational households and a representative firm, both of which live forever. Households have time additive utility functions, and are usually identical with respect to their time preference rates. The firm produces a single output good on a perfectly competitive market, using a linear homogeneous production technology. Households own the production factors capital and labor, which they rent to the firm to earn factor income. As there are infinitely many households, each single household acts as a price taker on all markets. The output good is bought by the households and used for consumption, which is the only source of utility, or set aside to form capital for the next period.

This paper takes a different approach. We study a version of the Ramsey model under the assumption that households act strategically. The departure from the competitive framework of price taking household behavior is motivated by the following observations. First, the common assumption that there are infinitely many households in the economy is obviously unjustified. If we assume that the number of households is finite, rationality requires that households realize their market power. Obviously, this is uncontroversial as long as the number of households is small. This requirement is often met in models of international trade, in which households are interpreted as countries, or in models of development economics, in which households are interpreted as powerful groups, such as ethnic or regional communities. However, even if the number of households is large, a serious treatment of rationality requires that households are strategically interacting agents. Although their influence on prices may be very small, it is not considered negligible to a rational household.

The second reason for departing from the competitive framework stems from the Ramsey model itself. If time preferences are heterogeneous and households are price takers, then only the group of households who share the smallest time preference rate will hold wealth in the long run. This is a well known result, which has been conjectured already by Ramsey (1928) and has been formally proved by Becker (1980). A severe shortcoming of this result has been pointed out by Sorger (2002): if the number of households who share the smallest time preference rate is finite, then the result is conceptually inconsistent with one of the assumptions under which it is derived, namely the price taking behavior of the households. Sorger (2002) argues that, if all capital would be owned by only a finite number of rational households, then these households would realize that they have market power on the capital market and thus they would not take the return to capital as exogenously given. He proposes a version of the Ramsey model, in which households choose sequences of decision variables, knowing the inverse factor demand functions of the representative firm and sequences of decision variables for all other households in the economy. The households play an open-loop Nash equilibrium in the factor markets. Sorger (2002) shows by means of examples that in this model there exist stationary equilibria in which all households own a positive amount of capital, that is, the Ramsey conjecture does no longer hold. Becker (2004), Becker and Foias (2005), and Sorger (2005b) derive further, more general results for this model. A shortcoming of the open-loop concept used by Sorger, Becker, and Foias is that the resulting equilibrium is not sub-game perfect. We address this shortcoming by studying a recursive version of the Ramsey model. Within our framework, households choose optimal policy functions, given the inverse factor demand functions of the firm and the optimal policy functions of all other households in the economy. We solve for a stationary Markov perfect equilibrium of this model. We believe that this equilibrium concept is the most adequate in applications where the number of households is finite.

The remainder of the paper is organized as follows. Section 2 introduces the Ramsey model and describes the competitive equilibrium and the open-loop equilibrium of this model. Section 3 defines the stationary Markov perfect equilibrium of the Ramsey model. Section 4 analyzes stationary Markov perfect equilibria assuming that households are identical with respect to their utility functions, time preference rates, and initial capital endowments. Section 5 considers the more general case where households are heterogeneous. In this case, we are no longer able to compute the stationary Markov perfect equilibrium analytically, and thus we use computational methods to derive our results. Section 6 analyzes stability and dynamics of the stationary Markov perfect equilibrium. Section 7 summarizes our main results and concludes. Finally, we present a detailed description of our numerical algorithms in an appendix.

2 Model Formulation and Equilibrium Concepts

The Ramsey (1928) model economy consists of a representative firm and H infinitely-lived households, who own the production factors capital and labor. Time is measured in discrete periods $t \in \{0, 1, 2, \dots\}$. In every period t, the firm hires capital K_t and labor L_t from the households to produce a single output good using a linear homogeneous production function F. Since the firm takes market prices as given and maximizes its profit, factors are paid their marginal products. Together with the assumption of constant returns to scale, this implies that the firm earns zero profit in equilibrium. The output good is bought by the households and is either used for consumption or set aside to form capital for the next period. Households seek to maximize their lifetime utility derived from consumption of the single good. Formally, each household $h = 1, 2, \dots, H$ chooses consumption and labor supply to solve

$$\max_{c_t^h, l_t^h} \sum_{t=0}^{+\infty} (\beta^h)^t u^h(c_t^h) \tag{1}$$

subject to

$$\begin{aligned} k_{t+1}^h = R_t k_t^h + W_t l_t^h - c_t^h, \qquad & t = 0, 1, 2, \dots, \\ c_t^h \geq 0, \quad k_{t+1}^h \geq 0, \quad 0 \leq l_t^h \leq 1, \qquad & t = 0, 1, 2, \dots. \end{aligned}$$

The function $u^h : \mathbb{R}_+ \mapsto \mathbb{R}$ denotes household h's utility function. The remaining variables and parameters have the following interpretation: $\beta^h \in (0, 1)$ denotes a time discount factor, c_t^h and l_t^h denote consumption and labor supply of household h in period t, and k_t^h denotes its capital stock at the beginning of period t. The initial capital stock of household h is a given constant k_0^h. By R_t and W_t we denote the gross return on capital and the wage rate, respectively, for period t.

2.1 Competitive Equilibrium and the Ramsey Conjecture

Assume for the time being that households do not realize that their labor and capital decisions influence prices. This assumption is adequate, for example, if the number of households supplying labor and capital is very large.[1] Then, a competitive equilibrium may be defined as follows:

[1]Strictly speaking, this assumption is only justified if there are infinitely many households in the economy. Whenever the number of households is finite, rationality requires that these households realize their market power.

Definition 1. A competitive equilibrium from initial state $(k_0^1, k_0^2, \dots, k_0^H)$ is a sequence $E = (W_t, R_t, L_t, K_t, \{(c_t^h, k_{t+1}^h, l_t^h) | h = 1, 2, \dots, H\})_{t=0}^{+\infty}$ such that the following conditions are satisfied:

1. For each household $h \in \{1, 2, \dots, H\}$ the sequence $(c_t^h, k_{t+1}^h, l_t^h)_{t=0}^{+\infty}$ solves household h's maximization problem (1), given the sequences of prices $(W_t, R_t)_{t=0}^{+\infty}$ and the initial capital endowment k_0^h.

2. The firm behaves optimally, that is, $R_t = 1 + F_K(K_t, L_t) - \delta$ and $W_t = F_L(K_t, L_t)$ hold for all $t \in \{0, 1, \dots\}$, where δ denotes the depreciation rate of capital.

3. The factor markets clear, that is, $K_t = \sum_{h=1}^H k_t^h$ and $L_t = \sum_{h=1}^H l_t^h$ hold for all $t \in \{0, 1, \dots\}$.

Definition 2. A sequence $E = (W_t, R_t, L_t, K_t, \{(c_t^h, k_{t+1}^h, l_t^h) | h = 1, 2, \dots, H\})_{t=0}^{+\infty}$ is called a competitive equilibrium if there exists an initial state $(k_0^1, k_0^2, \dots, k_0^H)$ such that E is a competitive equilibrium from $(k_0^1, k_0^2, \dots, k_0^H)$. A competitive equilibrium that is a constant sequence is called a steady state competitive equilibrium.

Remark 1. In a competitive equilibrium it must hold that $l_t^h = 1$ and $L_t = H$ for all $h \in \{1, 2, \dots, H\}$ and all $t \in \{0, 1, \dots\}$. This is the case because every household will always find it optimal to supply its entire labor endowment.

Ramsey (1928) conjectured that, in a stationary competitive equilibrium, only the most patient households would own capital whereas all other households would have zero wealth. A formal proof of this conjecture has been given by Becker (1980). A severe shortcoming of the result, however, has been pointed out by Sorger (2002): if the number of households in the economy who share the smallest time preference rate is small, then the result is conceptually inconsistent with one of the main assumptions under which it is derived, namely with the price taking behavior of the households. Sorger argues that, if the entire capital stock belonged to only a few households, then these households would realize that they have market power on the capital market and thus they would not take the return on capital as exogenously given. Sorger (2002) addresses this issue by studying a model in which the households take the inverse capital demand function as given (instead of the return to capital) and play a Nash equilibrium in the capital market. He shows that there exist stationary equilibria in which all households own a positive amount of capital. Sorger (2005b) considers a variant of this model in which the households not only realize their market power on the capital market but on all markets. The next section discusses Sorger's results in greater detail.

2.2 Open-Loop Equilibrium

We consider the model economy introduced in the previous section. The households, however, do no longer take the rental rates of capital and labor as given, but know the inverse aggregate factor demand functions. The problem of the households is to choose sequences of labor and consumption, given the inverse factor demand functions and the sequences for capital and labor provided by the other households in the economy. Formally, each household $h = 1, 2, \dots, H$ solves the problem

$$\max_{c_t^h, l_t^h} \sum_{t=0}^{+\infty} (\beta^h)^t u^h(c_t^h) \tag{2}$$

subject to

$$\begin{aligned} k_{t+1}^h = R(K_t, L_t)k_t^h + W(K_t, L_t)l_t^h - c_t^h, \qquad & t = 0, 1, 2, \dots, \\ K_t = \sum_{j=1}^{H} k_t^j, \quad L_t = \sum_{j=1}^{H} l_t^j, \qquad & t = 0, 1, 2, \dots, \\ c_t^h \geq 0, \quad k_{t+1}^h \geq 0, \quad 0 \leq l_t^h \leq 1, \qquad & t = 0, 1, 2, \dots. \end{aligned}$$

R and W denote the inverse demand functions for capital and labor, respectively. Sorger (2005b) defines an open-loop equilibrium as follows.

Definition 3. An open-loop equilibrium from initial state $(k_0^1, k_0^2, \dots, k_0^H)$ is a sequence $E = (W_t, R_t, L_t, K_t, \{(c_t^h, k_{t+1}^h, l_t^h) | h = 1, \dots, H\})_{t=0}^{+\infty}$ such that the following conditions are satisfied:

1. For each household $h = 1, 2, \dots, H$ the sequence $(c_t^h, k_{t+1}^h, l_t^h)_{t=0}^{+\infty}$ solves household h's maximization problem (2), given the functions R and W, the initial capital endowment k_0^h, and the $H - 1$ sequences of capital stocks and labor, $\{(k_t^j, l_t^j)_{t=0}^{+\infty} | j \neq h\}$.

2. The firm behaves optimally, that is, $R(K, L) = 1 + F_K(K, L) - \delta$ and $W(K, L) = F_L(K, L)$ hold for all $K \geq 0$ and $L \geq 0$.

Definition 4. A sequence $E = (W_t, R_t, L_t, K_t, \{(c_t^h, k_{t+1}^h, l_t^h) | h = 1, 2, \dots, H\})_{t=0}^{+\infty}$ is called an open-loop equilibrium if there exists an initial state $(k_0^1, k_0^2, \dots, k_0^H)$ such that E is an open-loop equilibrium from $(k_0^1, k_0^2, \dots, k_0^H)$. An open-loop equilibrium that is a constant sequence is called a steady state open-loop equilibrium.

Sorger (2002) considers an equilibrium definition that differs from the one given above in that the households take the sequence of wage rates $(W_t)_{t=0}^{+\infty}$ as given rather than the inverse factor demand function W. He shows by means of examples that there exist steady state open-loop equilibria in which all households own positive amounts of capital. Becker (2004) generalizes Sorger's examples to the case of a general Cobb-Douglas production function and derives a necessary and sufficient condition for the Ramsey conjecture to hold. Becker and Foias (2005) study the dynamics of the open-loop equilibrium. All of these papers deal with economies consisting of two households only.

Sorger (2005b) studies economies with H households and uses the equilibrium definition stated above.[2] He shows that, in an open-loop equilibrium, all households supply their entire labor endowment provided that the production function satisfies standard assumptions and that labor demand is elastic. He also investigates how changes in the time preference profile of the economy affect the distribution of wealth in the steady state open-loop equilibrium.

3 Stationary Markov Perfect Equilibrium

The results discussed in the previous section are derived using an equilibrium in sequence formulation: each household perfectly anticipates the sequences of factor prices (in the competitive equilibrium) or of labor supplies and capital stocks of all other households (in the open-loop equilibrium), and maximizes conditional on these sequences. Although open-loop equilibria are time-consistent, they fail to be sub-game perfect. We address this shortcoming by studying stationary Markov perfect equilibria, in which the households choose policy functions that determine their decision variables, knowing the optimal policy functions of all other households.

To study stationary Markov perfect equilibria, it is convenient to reformulate the household's problem in a recursive way. Let $X = \mathbb{R}_+^H$ be the space of capital stocks. Then household h's problem is to choose policy functions $\mathbf{c^h} : X \mapsto \mathbb{R}_+$ and $\mathbf{l^h} : X \mapsto [0,1]$ such that

$$\begin{aligned}(\mathbf{c^h}(k^1, k^2, \ldots, k^H), \mathbf{l^h}(k^1, k^2, \ldots, k^H)) \\ = \operatorname{argmax}_{c^h,\, l^h} \left\{ u^h\left(c^h\right) + \beta^h V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime}) \right\},\end{aligned} \tag{3}$$

[2]Sorger (2005b) assumes that the depreciation rate δ is equal to zero. The model presented here allows for positive capital depreciation.

where

$$
\begin{aligned}
&V^h(k^1, k^2, \ldots, k^H) = \max_{c^h,\ l^h} \left\{ u^h\left(c^h\right) + \beta^h V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime}) \right\},\\
&k^{h\prime} = R(K, L)k^h + W(K, L)l^h - c^h,\\
&k^{j\prime} = R(K, L)k^j + W(K, L)\mathbf{l^j}(k^1, k^2, \ldots, k^H) - \mathbf{c^j}(k^1, k^2, \ldots, k^H),\ j \neq h,\\
&K = \sum_{j=1}^{H} k^j,\\
&L = l^h + \sum_{j \neq h}^{H} \mathbf{l^j}(k^1, k^2, \ldots, k^H).
\end{aligned}
$$

The maximization is subject to non-negativity constraints $c^h \geq 0$, $l^h \in [0, 1]$, and $k^{h\prime} \geq 0$. The function $V^h : X \mapsto \mathbb{R}$ is household h's value function. Variables without time subscript correspond to the current period and primed variables correspond to the next period.
We observe that a solution to household h's problem is given by a pair of policy functions $\mathbf{c^h}$ and $\mathbf{l^h}$ such that the Bellman equation

$$
V^h(k^1, k^2, \ldots, k^H) = u^h\left(\mathbf{c^h}(k^1, k^2, \ldots, k^H)\right) + \beta^h V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime}) \quad (4)
$$

holds, where, for all $j \in \{1, 2, \ldots, H\}$,

$$
k^{j\prime} = R(K, L)k^j + W(K, L)\mathbf{l^j}(k^1, k^2, \ldots, k^H) - \mathbf{c^j}(k^1, k^2, \ldots, k^H).
$$

We proceed by providing a formal definition of a stationary Markov perfect equilibrium.

Definition 5. A stationary Markov perfect equilibrium is a set of value functions $\{V^h | h = 1, 2, \ldots, H\}$, a set of policy functions $\{\mathbf{c^h}, \mathbf{l^h} | h = 1, 2, \ldots, H\}$, and a pair of pricing functions R and W such that:

1. Given the value function V^h and the other households' policy functions $\{\mathbf{c^j}, \mathbf{l^j} | j \neq h\}$, the policy functions $\{\mathbf{c^h}, \mathbf{l^h}\}$ solve household h's maximization problem (3); this holds for all households $h = 1, 2, \ldots, H$.

2. Given the policy functions $\{\mathbf{c^j}, \mathbf{l^j} | j = 1, 2, \ldots, H\}$, the value function V^h satisfies the functional equation (4); this holds for all households $h = 1, 2, \ldots, H$.

3. The firm behaves optimally, that is, $R(K, L) = 1 + F_K(K, L) - \delta$ and $W(K, L) = F_L(K, L)$ hold for all $K \geq 0$ and $L \geq 0$.

We call these equilibria *stationary* Markov perfect equilibria because of the fact that the policy functions and value functions do not depend explicitly on the time variable t. The dynamics generated by such an equilibrium can obviously be non-stationary. For this reason, let us also define what we mean by an equilibrium path and a steady state generated by a stationary Markov perfect equilibrium.

Definition 6. Let $(\{(V^h, \mathbf{c^h}, \mathbf{l^h}) | h = 1, 2, \ldots, H\}, R, W)$ be a stationary Markov perfect equilibrium. An equilibrium path generated by this equilibrium is a sequence of vectors $(\{k_t^1, k_t^2, \ldots, k_t^H\})_{t=0}^{+\infty}$ satisfying the equilibrium dynamics

$$k_{t+1}^h = R(K_t, L_t)k_t^h + W(K_t, L_t)\mathbf{l^h}(k_t^1, k_t^2, \ldots, k_t^H) - \mathbf{c^h}(k_t^1, k_t^2, \ldots, k_t^H)$$

for all $h \in \{1, 2, \ldots, H\}$, where $K_t = \sum_{j=1}^H k_t^j$ and $L_t = \sum_{j=1}^H \mathbf{l^j}(k_t^1, k_t^2, \ldots, k_t^H)$. A steady state generated by a stationary Markov perfect equilibrium is a fixed point of this set of difference equations.

We proceed by stating a set of necessary conditions which must be satisfied by any stationary Markov perfect equilibrium. To this end, let us define Γ^h by

$$\begin{aligned} & \Gamma^h \\ = \; & \frac{\partial V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime})}{\partial k^{h\prime}} \\ & \times [R_L(K, L)k^h + W(K, L) + W_L(K, L)\mathbf{l^h}(k^1, k^2, \ldots, k^H)] \\ & + \sum_{j \neq h} \frac{\partial V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime})}{\partial k^{j\prime}} [R_L(K, L)k^j + W_L(K, L)\mathbf{l^j}(k^1, k^2, \ldots, k^H)], \end{aligned}$$

where, as before, $K = \sum_{j=1}^H k^j$ and $L = \sum_{j=1}^H \mathbf{l^j}(k^1, k^2, \ldots, k^H)$.

Lemma 1. *A stationary Markov perfect equilibrium satisfies the following* *conditions for all* $h \in \{1, 2, \ldots, H\}$*:*

$$\frac{\partial u^h(\mathbf{c^h})}{\partial c^h} - \beta^h \frac{\partial V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime})}{\partial k^{h\prime}} \left\{ \begin{array}{c} \geq \\ = \end{array} \right\} 0 \;\; \textit{if} \;\; k^{h\prime} \left\{ \begin{array}{c} = \\ > \end{array} \right\} 0, \tag{5}$$

$$\Gamma^h \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} 0 \;\; \textit{if} \;\; \mathbf{l^h}(k^1, k^2, \ldots, k^H) \left\{ \begin{array}{l} = 0, \\ \in (0,1), \\ = 1, \end{array} \right. \tag{6}$$

$$k^{h\prime} = R(K,L)k^h + W(K,L)\mathbf{l^h}(k^1, k^2, \ldots, k^H) - \mathbf{c^h}(k^1, k^2, \ldots, k^H), \tag{7}$$

$$R(K,L) = 1 + F_K(K,L) - \delta, \tag{8}$$

$$W(K,L) = F_L(K,L), \tag{9}$$

$$K = \sum_{j=1}^{H} k^j \tag{10}$$

$$L = \sum_{j=1}^{H} \mathbf{l^j}(k^1, \ldots, k^H) \tag{11}$$

Proof. The proof follows immediately from the definition of a stationary Markov perfect equilibrium. Conditions (5)-(6) are the first-order conditions resulting from the household h's optimization problem. Condition (7) is the transition law for the individual capital stock of household h. Conditions (8)-(9) define the inverse factor demand functions, and the last two conditions (10)-(11) are the aggregate constraints. □

Lemma 1 indicates that, in order to derive a stationary Markov perfect equilibrium, we need to solve a system of functional difference equations. It is well known that such systems hardly ever allow for analytical solutions. In the special case where households are identical, however, one can analytically compute the stable steady state generated by the stationary Markov perfect equilibrium. If we allow for heterogeneity, this is no longer the case. The next two sections discuss both cases in detail.

4 Homogeneous Households

We assume that all H households are identical, that is, they share the same time preference factor, the same utility function, and the same initial capital endowment:

$$\beta^h = \beta, \; u^h = u, \; k_0^h = k_0 \text{ for all } h = 1, 2, \ldots, H$$

In this special case, one can derive the steady states of the competitive equilibrium, the open-loop equilibrium and the stationary Markov perfect equilibrium analytically. Lemmas 2, 3, and 4 present intermediate results. Theorem 1 provides our final result.

Let us start by noting the following simple observation. Since the production function F is homogeneous of degree 1, its partial derivatives F_K and F_L are homogeneous of degree 0. It follows therefore from Euler's theorem that

$$F_{KK}(K,L)K + F_{KL}(K,L)L = F_{KL}(K,L)K + F_{LL}(K,L)L = 0 \tag{12}$$

holds for all $K > 0$ and $L > 0$.

Lemma 2. *If households are identical, then it holds in every symmetric (competitive, open-loop, or stationary Markov perfect) equilibrium that each household provides its entire labor endowment.*

Proof. Lemma 2 is obvious for the competitive equilibrium; see Remark 1. Let us therefore consider the two strategic equilibria. In both cases, the symmetry assumption implies

$$k_t^h/K_t = l_t^h/L_t = 1/H \tag{13}$$

for all $h \in \{1, 2, \ldots, H\}$ and all $t \in \{0, 1, 2, \ldots\}$.

For the open-loop equilibrium, we prove the lemma in the following way. Since u^h is an increasing function, it follows from (2) that each household h chooses its labor supply in period t to maximize income in that period. The latter is given by

$$I_t^h = R(K_t^{-h} + k_t^h, L_t^{-h} + l_t^h)k_t^h + W(K_t^{-h} + k_t^h, L_t^{-h} + l_t^h)l_t^h.$$

In the above expression we use $K_t^{-h} = K_t - k_t^h$ and $L_t^{-h} = L_t - l_t^h$. Inserting the inverse factor demand functions $R(K_t, L_t) = 1 + F_K(K_t, L_t) - \delta$ and $W(K_t, L_t) = F_L(K_t, L_t)$, we see that the derivative of household h's income with respect to its labor supply is given by

$$\frac{\partial I^h}{\partial l_t^h} = F_{KL}(K_t, L_t)k_t^h + F_{LL}(K_t, L_t)l_t^h + F_L(K_t, L_t).$$

Substituting (13) and (12) into this equation, we obtain

$$\frac{\partial I_t^h}{\partial l_t^h} = F_L(K_t, L_t) > 0.$$

Obviously, this inequality is only consistent with an optimally chosen labor supply if $l_t^h = 1$.
Let us now consider a symmetric path generated by a stationary Markov perfect equilibrium. Along such an equilibrium path, each household provides a positive amount of capital. The equilibrium condition (5) implies therefore that

$$\frac{\partial V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime})}{\partial k^{h\prime}} = \frac{1}{\beta} \frac{\partial u(\mathbf{c^h})}{\partial c^h} > 0. \tag{14}$$

Furthermore, from the equilibrium conditions (8)-(9) it follows that

$$\begin{aligned} & R_L(K, L)k^h + W_L(K, L)\mathbf{l^h}(k^1, k^2, \ldots, k^H) \\ = \; & F_{KL}(K, L)k^h + F_{LL}(K, L)\mathbf{l^h}(k^1, k^2, \ldots, k^H). \end{aligned}$$

Combining this with (13) and (12) it follows that the condition

$$R_L(K, L)k^h + W_L(K, L)\mathbf{l^h}(k^1, k^2, \ldots, k^H) = 0 \tag{15}$$

holds along every symmetric equilibrium path generated by a stationary Markov perfect equilibrium. Together with (14) this implies

$$\Gamma^h = \frac{1}{\beta} \frac{\partial u(\mathbf{c^h})}{\partial c^h} W(K, L) > 0.$$

It follows therefore from condition (6) that $\mathbf{l^h}(k^1, k^2, \ldots, k^H) = 1$. □

The intuition behind this result is as follows. In a strategic equilibrium (open-loop or stationary Markov perfect), a change of the labor supply, $\mathrm{d}l_t^h$, has obviously two effects: a direct effect, $W(K_t, L_t)\mathrm{d}l_t^h$, and an indirect effect, $[R_L(K_t, L_t)k_t^h + W_L(K_t, L_t)l_t^h]\mathrm{d}l_t^h$. The direct effect is always positive, since the wage rate is positive. Equation (15) shows that the indirect effect is equal to 0 along any symmetric equilibrium path. Therefore, an increase in the labor supply has a positive overall effect, and thus households will provide their entire labor endowment.

Lemma 3. *If households are identical, then the aggregate capital stocks in the steady state open-loop equilibrium and the steady state competitive equilibrium coincide.*

Proof. Lemma 2 shows that all households provide their entire labor endowment in equilibrium. Thus, the Euler equation for problem (2) is given by

$$\frac{\partial u^h}{\partial c_t^h} \geq \beta^h \frac{\partial u^h}{\partial c_{t+1}^h}[R(K_{t+1}, H) + R_K(K_{t+1}, H)k_{t+1}^h + W_K(K_{t+1}, H)] \tag{16}$$

with equality if $k_{t+1}^h > 0$. From $R(K, L) = 1 + F_K(K, L) - \delta$, $W(K, L) = F_L(K, L)$, (12), and (13) it follows that the Euler equation simplifies to

$$\frac{\partial u^h}{\partial c_t^h} = \beta \frac{\partial u^h}{\partial c_{t+1}^h} R(K_{t+1}, H).$$

In a steady state of a symmetric open-loop equilibrium with positive aggregate capital, it must therefore hold that

$$\frac{1}{\beta} = R(K, H). \tag{17}$$

It is well known that this is exactly the condition that pins down the competitive steady state aggregate capital stock. □

The interpretation of Lemma 3 is similar to that regarding the full employment of labor. In the open-loop equilibrium, a change of k_{t+1}^h has the direct effect $R(K_{t+1}, H)\mathrm{d}k_{t+1}^h$ and an indirect effect $[R_K(K_{t+1}, H)k_{t+1}^h + W_K(K_{t+1}, H)]\mathrm{d}k_{t+1}^h$. The indirect effect is equal to 0 if the equilibrium path is symmetric. Since the indirect effect is not present in the competitive model, the steady state aggregate capital stocks of the competitive equilibrium and the open-loop equilibrium must coincide under symmetry.

Lemma 4. *If households are identical, then the aggregate capital stock in any stable steady state generated by a symmetric stationary Markov perfect equilibrium coincides with the aggregate capital stock in the steady state competitive equilibrium.*

Proof. We prove Lemma 4 by showing that it is again equation (17) which pins down the aggregate capital stock in a stable steady state generated by a symmetric stationary Markov perfect equilibrium. To this end, we recall that a stationary Markov perfect equilibrium must satisfy the Bellman equation (4). Differentiating the Bellman equation yields

$$\begin{aligned}
\frac{\partial V^h(k^1, k^2, \ldots, k^H)}{\partial k^h} &= \frac{\partial u(\mathbf{c^h})}{\partial c^h}\frac{\partial \mathbf{c^h}}{\partial k^h} + \beta \frac{\partial V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime})}{\partial k^{h\prime}} \\
&\times \left[R(K, H) + R_K(K, H)k^h + W_K(K, H) - \frac{\partial \mathbf{c^h}}{\partial k^h}\right] \\
&+ \beta \sum_{j \neq h} \frac{\partial V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime})}{\partial k^{j\prime}} \\
&\times \left[R_K(K, H)k^j + W_K(K, H) - \frac{\partial \mathbf{c^j}}{\partial k^h}\right]
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\frac{\partial V^h(k^1,k^2,\ldots,k^H)}{\partial k^j} &= \frac{\partial u(\mathbf{c^h})}{\partial c^h}\frac{\partial \mathbf{c^h}}{\partial k^j} + \beta\frac{\partial V^h(k^{1\prime},k^{2\prime}\ldots,k^{H\prime})}{\partial k^{h\prime}} \\
&\times \left[R_K(K,H)k^h + W_K(K,H) - \frac{\partial \mathbf{c^h}}{\partial k^j}\right] \\
&+ \beta\frac{\partial V^h(k^{1\prime},k^{2\prime},\ldots,k^{H\prime})}{\partial k^{j\prime}} \\
&\times \left[R(K,H) + R_K(K,H)k^j + W_K(K,H) - \frac{\partial \mathbf{c^j}}{\partial k^j}\right] \\
&+ \beta\sum_{i\neq h, i\neq j}\frac{\partial V^h(k^{1\prime},k^{2\prime},\ldots,k^{H\prime})}{\partial k^{i\prime}} \\
&\times \left[R_K(K,H)k^i + W_K(K,H) - \frac{\partial \mathbf{c^i}}{\partial k^j}\right]. \qquad (19)
\end{aligned}$$

Now, in a completely analogous way as (15) has been derived, one can show that

$$R_K(K,L)k^i + W_K(K,L)\mathbf{l^i}(k^1,k^2,\ldots,k^H) = R_K(K,L)k^i + W_K(K,L) = 0 \qquad (20)$$

holds for all $i \in \{1,2,\ldots,H\}$. From (14) and (20) it follows that (18) and (19) become

$$\begin{aligned}
\frac{\partial V^h(k^1,k^2,\ldots,k^H)}{\partial k^h} &= \frac{\partial u(\mathbf{c^h})}{\partial c^h}R(K,H) \\
&-\beta\sum_{j\neq h}\frac{\partial V^h(k^{1\prime},k^{2\prime},\ldots,k^{H\prime})}{\partial k^{j\prime}}\frac{\partial \mathbf{c^j}}{\partial k^h} \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial V^h(k^1,k^2,\ldots,k^H)}{\partial k^j} &= \beta\frac{\partial V^h(k^{1\prime},k^{2\prime},\ldots,k^{H\prime})}{\partial k^{j\prime}}\left[R(K,H) - \frac{\partial \mathbf{c^j}}{\partial k^j}\right] \\
&-\beta\sum_{i\neq h, i\neq j}\frac{\partial V^h(k^{1\prime},k^{2\prime},\ldots,k^{H\prime})}{\partial k^{i\prime}}\frac{\partial \mathbf{c^i}}{\partial k^j}. \qquad (22)
\end{aligned}$$

In a symmetric equilibrium path generated by a stationary Markov perfect equilibrium, it must hold for every $j \neq h$ and every $i \neq h$ that

$$\frac{\partial V^h(k^1,k^2,\ldots,k^H)}{\partial k^j} = \frac{\partial V^h(k^1,k^2,\ldots,k^H)}{\partial k^i}. \qquad (23)$$

In the steady state of a stationary Markov perfect equilibrium it must furthermore hold for all $j \in \{1, 2, \ldots, H\}$ that

$$\frac{\partial V^h(k^{1\prime}, k^{2\prime}, \ldots, k^{H\prime})}{\partial k^{j\prime}} = \frac{\partial V^h(k^1, k^2, \ldots, k^H)}{\partial k^j}. \tag{24}$$

Using (23) and (24), equation (22) becomes

$$\begin{aligned} & \frac{1}{\beta} \frac{\partial V^h(k^1, k^2, \ldots, k^H)}{\partial k^j} \\ = \; & \frac{\partial V^h(k^1, k^2, \ldots, k^H)}{\partial k^j} \left[R(K, H) - \frac{\partial \mathbf{c^j}}{\partial k^j} - \sum_{i \neq h, i \neq j} \frac{\partial \mathbf{c^i}}{\partial k^j} \right]. \end{aligned} \tag{25}$$

Obviously, there are only two scenarios in which (25) holds. Either

$$\frac{\partial V^h(k^1, k^2, \ldots, k^H)}{\partial k^j} = 0 \tag{26}$$

or

$$\frac{1}{\beta} = R(K, H) - \frac{\partial \mathbf{c^j}}{\partial k^j} - \sum_{i \neq h, i \neq j} \frac{\partial \mathbf{c^i}}{\partial k^j}. \tag{27}$$

In a stable symmetric steady state generated by a stationary Markov perfect equilibrium, however, the condition (27) cannot hold. To show this, we consider the Jacobian matrix associated with the equilibrium, which is given by

$$J = \begin{pmatrix} R(K,H) - \frac{\partial \mathbf{c^1}}{\partial k^1} & -\frac{\partial \mathbf{c^1}}{\partial k^2} & \cdots & -\frac{\partial \mathbf{c^1}}{\partial k^H} \\ -\frac{\partial \mathbf{c^2}}{\partial k^1} & R(K,H) - \frac{\partial \mathbf{c^2}}{\partial k^2} & \cdots & -\frac{\partial \mathbf{c^2}}{\partial k^H} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial \mathbf{c^H}}{\partial k^1} & \cdots & -\frac{\partial \mathbf{c^H}}{\partial k^{H-1}} & R(K,H) - \frac{\partial \mathbf{c^H}}{\partial k^H} \end{pmatrix}$$

In a symmetric steady state, all entries in the diagonal are identical to each other, as are all entries outside the diagonal. Formally,

$$\begin{aligned} \frac{\partial \mathbf{c^i}}{\partial k^i} = \frac{\partial \mathbf{c^j}}{\partial k^j} = \frac{\partial \mathbf{c^1}}{\partial k^1} \quad & \text{for } i, j = 1, \ldots, H \\ \frac{\partial \mathbf{c^i}}{\partial k^j} = \frac{\partial \mathbf{c^j}}{\partial k^i} = \frac{\partial \mathbf{c^1}}{\partial k^2} \quad & \text{for } i, j = 1, \ldots, H \end{aligned}$$

This structure implies that the Jacobian has the eigenvalues

$$\lambda \;=\; R(K,H) - \frac{\partial \mathbf{c^1}}{\partial k^1} - (H-1)\frac{\partial \mathbf{c^1}}{\partial k^2}$$

and

$$\mu \quad = \quad R(K,H) - \frac{\partial \mathbf{c^1}}{\partial k^1} + \frac{\partial \mathbf{c^1}}{\partial k^2},$$

whereby λ has multiplicity 1 and μ has multiplicity $H-1$; see Lemma A.2 in Sorger (2005a). Under (27) it would follow that

$$\lambda = \frac{1}{\beta} - \frac{\partial \mathbf{c^1}}{\partial k^2}, \text{ and } \mu = \frac{1}{\beta} + (H-1)\frac{\partial \mathbf{c^1}}{\partial k^2}.$$

Since $\beta \in (0,1)$, this would imply that either $\lambda > 1$ or $\mu > 1$, which cannot be the case in a stable steady state. Thus, in the symmetric and stable steady state generated by a stationary Markov perfect equilibrium, equation (26) must hold. Using (14), (23), (24), and (26) we see that (21) is equivalent to (17). This completes the proof of the lemma. □

Lemmas 2, 3, and 4 allow us to formulate our main result for homogeneous households in the following theorem.

Theorem 1. *If households are identical, the symmetric and stable steady state generated by any stationary Markov perfect equilibrium coincides with the steady state competitive equilibrium and the steady state open-loop equilibrium.*

Proof. The proof follows directly from Lemmas 2, 3, and 4. We know from Lemma 2 that each houshold provides the same amount of labor in every steady state equilibrium. From Lemmas 3 and 4 we know that the aggregate capital stocks coincide in all symmetric steady states. Since households are identical, this carries over to the individual capital stocks.

□

5 Heterogeneous Households

When we consider heterogeneous households, we can no longer solve analytically for the steady state of the stationary Markov perfect equilibrium.[3] In this case, we therefore find ourselves constrained to using computational methods.

[3]The steady states of the competitive model and the open-loop equilibrium have been fully characterized by Becker (1980) and Sorger (2005b), respectively.

5.1 Numerical Method

We choose a Least Squares Projection method to determine the stationary Markov perfect equilibria numerically. We select this method since it allows us to implement high-order approximations easily, such that we are confident that our results are very accurate. We approximate each household's value function by a parametric function in all individual capital stocks, using a least squares projection method. We derive the coefficient vector of the approximate value functions such that the set of equilibrium conditions (5)-(11) and the Bellman equation (4) are satisfied for every household. Given approximate value functions, we can derive the corresponding policy functions in a straightforward manner. The latter can be used to compute the Markov perfect steady state as the fixed point of the system dynamics. A detailed discussion and outline of our method is given in the appendix.
Using numerical methods requires us furthermore to specify the technology and utility functions. In order to guarantee comparability with previous studies, we select a Cobb-Douglas production function,

$$F(K, L) = AK^{\alpha}L^{1-\alpha} \tag{28}$$

and a constant elasticity of substitution utility function,

$$u^h(c^h) = \begin{cases} \frac{{c^h}^{1-\sigma^h}}{1-\sigma^h} & \text{if } \sigma^h \neq 1 \\ \log c^h & \text{if } \sigma^h = 1 \end{cases} \tag{29}$$

This is a very convenient choice, such that we do not present any further discussion. Finally, we need to assert values to the structural parameters of the model. Table 1 summarizes these choices. For computational convenience,

Table 1: Benchmark Parameter Values

H	A	α	δ	σ^1	σ^2	β^1	β^2
2	1	0.36	0.05	1	1	0.94	0.91

we set the number of households equal to two. We believe this generates interesting results while keeping the computational burden at a moderate level. The total factor productivity A is normalized to 1. Our benchmark calibration sets the capital share to 0.36, the depreciation rate equal to 0.05, both utility function parameters σ^1 and σ^2 equal to 1, and the discount factors equal to $\beta^1 = 0.94$ and $\beta^2 = 0.91$, respectively. To experiment with the model, and to check the sensitivity of our results with regard to different parameter choices, we vary δ, σ^1, σ^2, β^1, and β^2 throughout our analysis.

5.2 Results

We begin the discussion of our results by making the following remark on the full provision of labor.

Remark 2. In all cases we consider, households find it optimal to supply their entire labor endowment in equilibrium.

This result may not be valid generally, but our results show that the observation goes through for a broad variety of calibrations. We find this not surprising. As was already demonstrated in the previous section for homogeneous households, an increase in the labor supply has a first-order positive effect on income, as well as a second-order effect which can be positive or negative, depending on individual labor supplies and capital holdings. This suggests that in many cases the overall effect will be positive, such that households supply their entire labor endowment.

Keeping this result in mind, we restrict our discussion to the steady state capital stocks. For our benchmark calibration, we observe the following result. In both strategic equilibria, the aggregate capital stock is smaller than

Table 2: Steady State Capital Stocks: Benchmark Calibration

Markov perfect			Open-loop			Competitive		
k^1	k^2	K	k^1	k^2	K	k^1	k^2	K
7.1581	2.5781	9.7362	6.8472	2.8161	9.6633	12.0880	0	12.0880

$$H = 2,\ A = 1,\ \alpha = 0.36,\ \delta = 0.05,\ \sigma^1 = \sigma^2 = 1,\ \beta^1 = 0.94,\ \beta^2 = 0.91$$

in the competitive equilibrium. The stationary Markov perfect equilibrium delivers a slightly higher steady state aggregate capital stock than the open-loop equilibrium. Furthermore, capital holdings are more dispersed in the stationary Markov-perfect equilibrium: the patient household owns a larger fraction of the aggregate stock in the Markov perfect steady state as compared to the open-loop steady state. However, in both strategic equilibria, the less patient agent owns a positive amount of capital such that, contrary to the competitive equilibrium, the Ramsey conjecture does not hold. We find that these main characteristics of the results for the benchmark calibration carry over to a broad range of parameter values. The remainder of this section presents characteristics of our results for parameter sets different from the benchmark calibration.

Observation 1. If households are heterogeneous, utility functions have an impact on the steady state Markov perfect equilibrium.

We find this observation particularly interesting, since the utility functions have no impact on the steady state of the competitive and the open-loop equilibrium of the Ramsey model. For the stationary Markov perfect equilibrium, however, they play a non-negligible role, and they affect both the slope and the level of the policy functions. Table 2 provides some numerical examples that emphasize this result. It lists individual capital stocks for different values of the elasticity of substitution parameters σ^1 and σ^2. We observe that for a given σ^j, the steady state capital stock of household

Table 3: Steady State Capital Stocks

		Markov perfect			Open-loop		Competitive	
σ^1	σ^2	k^1	k^2	K	k^1	k^2	k^1	k^2
1	1	7.1581	2.5781	9.7362	6.8472	2.8161	12.0880	0
1	3	6.9303	2.7145	9.6448	6.8472	2.8161	12.0880	0
1	5	6.8469	2.7677	9.6146	6.8472	2.8161	12.0880	0
3	1	7.3361	2.4706	9.8067	6.8472	2.8161	12.0880	0
3	3	7.0604	2.6395	9.6999	6.8472	2.8161	12.0880	0
3	5	6.9594	2.7022	9.6616	6.8472	2.8161	12.0880	0
5	1	7.4165	2.4205	9.8370	6.8472	2.8161	12.0880	0
5	3	7.1213	2.6053	9.7266	6.8472	2.8161	12.0880	0
5	5	7.0128	2.6731	9.6859	6.8472	2.8161	12.0880	0
					$K = 9.6633$		$K = 12.0880$	

$H = 2$, $A = 1$, $\alpha = 0.36$, $\delta = 0.05$, $\beta^1 = 0.94$, $\beta^2 = 0.91$

$h \neq j$ increases when σ^h increases, whereas the steady state capital stock of household j decreases. This holds for all numerical examples we consider.

Observation 2. In the steady state of the stationary Markov perfect equilibrium, the ordering of households according to their capital holdings coincides with the ordering according to their time preference rates. The more patient a household is, the more capital it holds in the steady state.

This result seems obvious as long as we consider equal utility functions among households. However, if utility functions are different, our first observation suggests that there may be cases, in which the ordering of households according to their time preference rates does not coincide with the ordering according to their steady state capital holdings. Observation 2, however, states that this is not the case.

Since we are again not able to prove this in general, we give examples that

Table 4: Markov Perfect Steady State Capital Stocks

	k^1	k^2
$\beta^1 = 0.92510$ $\beta^2 = 0.92490$	4.8625	4.8335
$\beta^1 = 0.92501$ $\beta^2 = 0.92499$	4.8492	4.8475

$H = 2,\ A = 1,\ \alpha = 0.36,\ \delta = 0.05,\ \sigma^1 = 1,\ \sigma^2 = 5$

underscore this observation. From our first observation we would expect that the less patient household may own a larger capital stock than the more patient one, if both households are almost equally patient, and the less patient household has a substantially higher elasticity of substitution parameter. The results in Table 4 show that the *coincidence of orderings* holds even in this case. We are thus confident that the result carries over to a broad range of parametric cases.

Observation 3. A mean preserving spread of time preference factors increases substantially the dispersion between individual Markov perfect steady state capital holdings, but has only minor implications for the aggregate Markov perfect steady state capital stock.

Table 5: Mean Preserving Spreads in Discount Factors

		Markov perfect			Open-loop			Competitive		
β^1	β^2	k^1	k^2	K	k^1	k^2	K	k^1	k^2	K
0.925	0.925	4.85	4.85	9.70	4.85	4.85	9.70	4.85	4.85	9.70
0.93	0.92	5.62	4.08	9.70	5.52	4.17	9.69	10.41	0	10.41
0.94	0.91	7.16	2.58	9.74	6.85	2.82	9.66	12.09	0	12.09
0.95	0.90	8.69	1.14	9.82	8.13	1.47	9.61	14.21	0	14.21

$H = 2,\ A = 1,\ \alpha = 0.36,\ \delta = 0.05,\ \sigma^1 = 1,\ \sigma^2 = 1$

We see from the results in Table 5 that a mean preserving spread increases the competitive steady state capital stock substantially.[4] For both the steady state open-loop equilibrium and the steady state Markov perfect equilibrium, we observe that the dispersion between individual capital holdings increases,

[4]To facilitate the reading of Table 5, we display the steady state capital stocks with only two digits after the comma.

whereas the aggregate capital stock is only moderately affected.
In our numerical examples, we observe a small decrease in aggregate capital holdings for the open-loop equilibrium. This observation relates to a finding in Sorger (2005b), who proves that mean preserving spreads in discount rates $(\rho^1, \rho^2, \ldots, \rho^H)$, defined by $\rho^h = 1/\beta^h - 1$, lead to higher aggregate steady state capital stocks for the open-loop equilibrium.[5] These results, however, do not carry over to mean preserving spreads in time preference rates $(\beta^1, \beta^2, \ldots, \beta^H)$, since mean preserving spreads in $(\beta^1, \beta^2, \ldots, \beta^H)$ do not induce mean preserving spreads in $(\rho^1, \rho^2, \ldots, \rho^H)$.
Finally, we observe in our examples that the aggregate capital stock in the steady state generated by the stationary Markov perfect equilibrium increases for mean preserving spreads in (β^1, β^2). However, we cannot provide a formal proof that this relationship always holds.

Observation 4. The stationary Markov perfect equilibrium need not lead to a higher steady state aggregate capital stock as compared to the open-loop equilibrium, and need not exhibit more dispersion between the individual steady state capital holdings.

Our results so far have shown that for many calibrations, in the steady state of the stationary Markov perfect equilibrium the aggregate capital stock is higher, and individual capital holdings are more dispersed, as compared to the open-loop equilibrium. However, this is not generally true. Table 3 has demonstrated three situations in which the stationary Markov perfect equilibrium leads to a smaller aggregate steady state capital stock. These are cases in which the less patient household has a higher elasticity of substitution parameter. Among these situations, we believe the case where $\sigma^1 = 1$ and $\sigma^2 = 5$ is particularly interesting, since both households save less in the Markov perfect steady state than in the open-loop steady state.
Moreover, our results so far indicate that the stationary Markov perfect equilibrium exhibits more dispersion between the individual steady state capital holdings, as compared to the open-loop equilibrium. Table 6 demonstrates that this need not be the case. If σ^1 is very low and σ^2 is very high, then we observe situations in which the less patient agent saves more, and the patient agent saves less in the Markov perfect steady state, as compared to the open-loop steady state. Table 6 illustrates such a scenario. In the case of equal utility functions, $\sigma^1 = \sigma^2 = 0.5$, the patient household saves more in the Markov perfect steady state, whereas the less patient household saves less.

[5] In particular, Sorger (2005b) shows that if the production function is Cobb-Douglas, mean preserving spreads in discount rates $(\rho^1, \rho^2, \ldots, \rho^H)$ do not affect the aggregate steady state capital stock of the open-loop equilibrium, as long as the number of households owning positive wealth is not altered.

As stated by our first observation, increasing the elasticity of substitution parameter of the less patient household causes a decrease in the steady state capital stock of the patient household, whereas it increases the steady state capital stock of the less patient household. Indeed, if $\sigma^2 = 7$, we observe that the patient household saves less in the Markov perfect steady state as compared to the open-loop equilibrium, whereas the less patient household saves more.

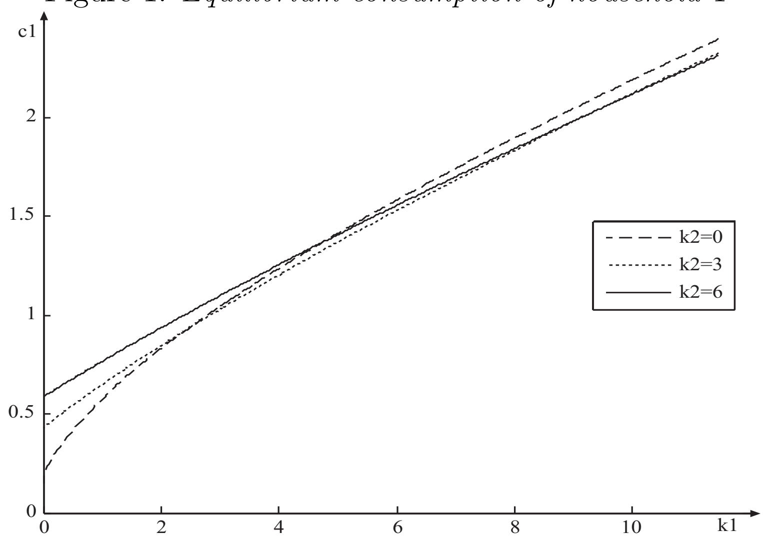
Table 6: Less Dispersed Capital Holdings

	Markov perfect		Open-loop		Competitive	
	k^1	k^2	k^1	k^2	k^1	k^2
$\sigma^1 = 0.5 \quad \sigma^2 = 0.5$	7.2001	2.5581	6.8472	2.8161	12.0880	0
$\sigma^1 = 0.5 \quad \sigma^2 = 7$	6.7431	2.8387	6.8472	2.8161	12.0880	0

$H = 2$, $A = 1$, $\alpha = 0.36$, $\delta = 0.05$, $\beta^1 = 0.94$, $\beta^2 = 0.91$

Observation 5. The equilibrium consumption function of household h, $\mathbf{c^h}$, is monotonically increasing in k^h, and may be increasing or decreasing in the other household's capital stock, $k^j, j \neq h$.

Figure 1: *Equilibrium consumption of household 1*



$H = 2$, $A = 1$, $\alpha = 0.36$, $\delta = 0.05$, $\sigma^1 = \sigma^2 = 1$, $\beta^1 = 0.94$, $\beta^2 = 0.91$

Figure 1 emphasizes this observation graphically. It displays optimal consumption of household 1 as a function of its own capital stock, holding the other household's capital stock fixed at $k^2 = 0$, $k^2 = 3$, and $k^2 = 6$, respectively.

We observe that for each k^2, consumption of household 1, $\mathbf{c^1}$, is increasing in its capital stock, k^1. Furthermore, we see that the role of the other household's capital stock, k^2, is ambiguous. For small values of k^1, $\mathbf{c^1}$ is increasing in k^2. For large values of k^1, the opposite is true. We find that these properties hold for a broad class of calibrations.

6 Dynamics and Stability

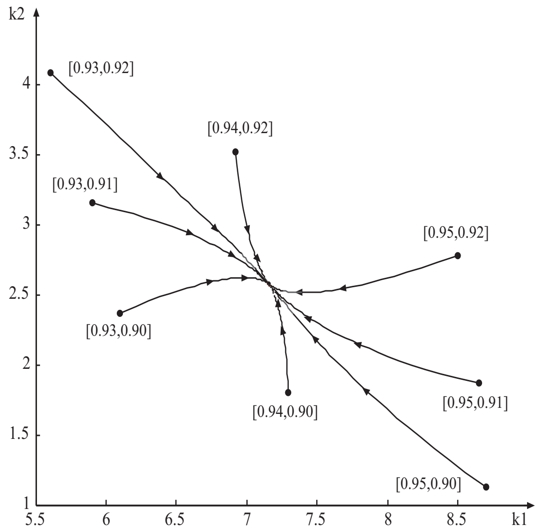
This section analyzes the stability and dynamics of Markov perfect equilibria in the Ramsey model. We find that for all parametric cases we consider, the steady state generated by the Markov perfect equilibrium is very stable. We observe furthermore that the structural parameters of the model, in particular the elasticity of substitution parameters σ^1 and σ^2, have a strong influence on the dynamics of the model around the steady state. This relates to one of our previous observations, that the elasticity of substitution parameters affect both the level and slope of the equilibrium policy functions. The following plots emphasize these observations.

We first consider the case where households are heterogeneous with respect to their time preference rates, but share the same elasticity of substitution parameters.[6] Figure 2 visualizes the convergence to the benchmark steady state, starting from steady states implied by different time preference profiles.[7] We see that the equilibrium exhibits standard convergence properties. Figure 2 does hardly change when we increase σ^1 and σ^2 simultaneously to three or five, respectively. As long as both households share the same elasticity of substitution parameters, an increase of σ^1 and σ^2 influences the speed of convergence to the steady state, but affects its direction only moderately. In the following we consider different elasticity of substitution parameters across households. We observe that the Markov perfect equilibrium dynamics are highly dependent on the choice of σ^1 and σ^2. Figure 3 visualizes one example that we find particularly interesting. One can see from Figure 3

[6] The case of identical households is not explicitly discussed, since the dynamics are very similar to the case of heterogeneous households.

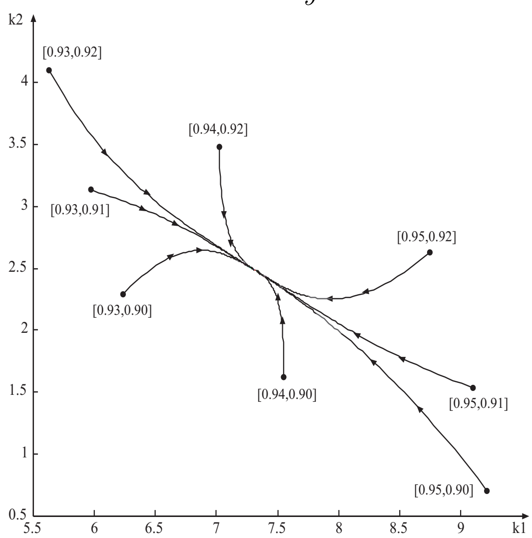
[7] The terms in brackets in the graphs quote the discount factors β^1 and β^2 corresponding to the respective steady states. For example, $[0.95, 0.90]$ labels the steady state corresponding to the parameter values $H = 2$, $A = 1$, $\alpha = 0.36$, $\delta = 0.05$, $\sigma^1 = \sigma^2 = 1$ and $\beta^1 = 0.95$, $\beta^2 = 0.90$, respectively.

Figure 2: *Transition to the benchmark steady state*



$H = 2,\ A = 1,\ \alpha = 0.36,\ \delta = 0.05,\ \sigma^1 = \sigma^2 = 1,\ \beta^1 = 0.94,\ \beta^2 = 0.91$

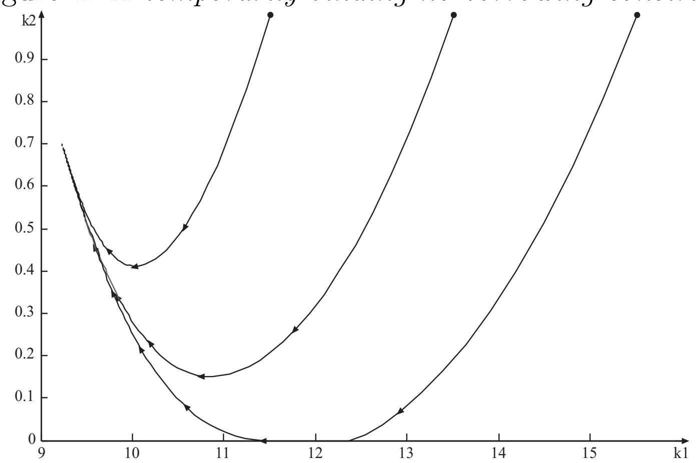
Figure 3: *Transition to a steady state with* $\sigma^1 = 3,\ \sigma^2 = 1$



$H = 2,\ A = 1,\ \alpha = 0.36,\ \delta = 0.05,\ \sigma^1 = 3,\ \sigma^2 = 1,\ \beta^1 = 0.94,\ \beta^2 = 0.91$

that during the transition from the steady state implied by $\beta^1 = 0.93$ and $\beta^2 = 0.90$ to the steady state implied by $\beta^1 = 0.94$ and $\beta^2 = 0.91$, the less patient household temporarily owns a bigger capital stock than in each of the two steady states. On the other hand, during transition from the steady state implied by $\beta^1 = 0.95$, $\beta^2 = 0.92$ to the steady state implied by $\beta^1 = 0.94$, $\beta^2 = 0.91$, it owns a smaller capital stock than in each of the two steady states. The latter observation suggests that there may exist situations, in which the less patient household temporarily holds no wealth during the transition to the steady state. Figure 4 displays such a scenario. It shows the convergence to the steady state implied by $\beta^1 = 0.95$ and $\beta^2 = 0.90$, starting from three different capital endowments. We see that if the initial

Figure 4: *A temporarily binding no-borrowing constraint*



$H = 2$, $A = 1$, $\alpha = 0.36$, $\delta = 0.05$, $\sigma^1 = 3$, $\sigma^2 = 1$, $\beta^1 = 0.95$, $\beta^2 = 0.90$

capital stocks are given by $k_0^1 = 15.5$ and $k_0^2 = 1$, the less patient household temporarily holds no capital during the transition to the steady state. The intuition behind this result is as follows: initially, both households own more capital than they want to in steady state, such that both households steadily reduce their capital stocks. After some time, the less patient household is no longer willing to hold any capital, whereas the more patient agent still reduces its capital holdings. Obviously, since capital supply is steadily decreasing, the rental rate of capital is steadily increasing. Once it has reached a certain level, the less patient household is again willing to save, since it receives a very high return on its savings. Whereas the more patient agent still reduces its capital holdings, the less patient agent accumulates capital until they reach the steady state.

7 Conclusion

We conclude by summarizing the main contributions of our paper, by discussing some possible extensions, and by pointing to possible applications. Our contributions are the following: we formulate Markov perfect equilibria for the Ramsey model economy; we prove that in the case of homogeneous households, strategic behavior does not affect the economy in the long run; we analyze steady state equilibria with heterogeneous households and study the implications of different equilibrium concepts; we study the stability and local dynamics of Markov perfect equilibria; finally, we provide a well-behaved algorithm implemented in the popular programming language MATLAB, which can easily be adapted to related problems.

We see three possible extensions of our analysis. First, we could study a model with more than two heterogeneous households. Although we are confident that our main characteristics of the Markov perfect equilibrium for two households carry over, there may obviously be additional insights gained by studying this more general case. A second possible extension relates to the role of firms. In our model, we make the crucial assumption that there is a single firm that operates on a competitive market. One could obviously extend the model by allowing for a finite number of strategically interacting firms. However, this is problematic, since we would need very strong assumptions on the firms' objectives throughout the analysis. A third possible extension is to incorporate uncertainty into the analysis, for example, by introducing shocks to households' preferences or to the firm's production technology. This, we believe, is the most interesting extension of our analysis, since valuable insights may be gained from studying the households' response to shocks in a strategic environment.

Finally, we see a number of possible applications, in which Markov perfect equilibria are the adequate model solution concept. For example, in trade theory, Markov perfect equilibria of small dynamic models could be used to investigate the distribution of capital across countries, or to study questions related to the balance of payments, among others. Within this field, one interesting exercise would be to set up a model with a small number of households, which are given the interpretation of countries. Countries differ by their time preferences, their utility functions, their initial capital endowments, and their production technologies. Each country produces the same output good, using labor and capital as inputs. Labor is immobile, whereas capital is mobile: countries may either use their capital for production in the domestic market, or may rent their capital to other countries to earn factor income. Within this model, we believe, interesting results can be obtained by studying strategic behavior of countries. For example, one could analyze the distribution of capital, or current accounts, in the Markov perfect equilibrium of this model.

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A Numerical Strategy

This appendix discusses the numerical strategy we use to derive the Markov perfect equilibrium of the Ramsey model. For notational convenience, we describe the algorithm for $H = 2$ households. Extensions to a larger number of households are straightforward.
Our algorithm is a variant of the least squares projection method introduced into economics by Judd (1992). We find that this method is very well

suited for our purpose, since it allows for the easy implementation of high-order approximations. By using families of orthogonal polynomials, such as Chebyshev polynomials, in the approximate value and policy functions, we are able to increase the accuracy of our results to practically any desired level by raising the order of approximation. Due to the orthogonality property of Chebyshev polynomials, we can avoid multicollinearity problems which are often encountered when applying high-order approximation methods.

The general idea underlying our algorithm is to find approximate value functions, such that the system of equilibrium conditions (5)-(11) and the Bellman equation (4) hold (approximately). To this end, we start by selecting parametric functional forms

$$V^1(k^1,k^2) \approx \tilde{V}(k^1,k^2;\theta^1) = \sum_{i,j=0,\dots,P;\, i+j\leq P} \theta^1_{ij} T_i(\xi(k^1))T_j(\xi(k^2)) \tag{30}$$

$$V^2(k^1,k^2) \approx \tilde{V}(k^1,k^2;\theta^2) = \sum_{i,j=0,\dots,P;\, i+j\leq P} \theta^2_{ij} T_i(\xi(k^1))T_j(\xi(k^2)) \tag{31}$$

where θ^1 and θ^2 denote the parameter vectors for households 1 and 2, respectively. P gives the order of approximation, and T_i denotes the Chebyshev polynomial of order i. Chebyshev polynomials, which are defined recursively by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_i(x) = 2xT_{i-1}(x) - T_{i-2}(x) \text{ for } i = 2,3,\dots$$

are well suited for approximation, since they constitute a family of orthogonal polynomials in the interval $[-1,1]$.[8] We derive upper and lower bounds of the state space, $\bar{k}$ and $\underline{k}$, by constructing a sufficiently large interval around the steady state generated by the open-loop equilibrium, and use

$$\xi(k) = 2\frac{k-\underline{k}}{\bar{k}-\underline{k}} - 1$$

to map the state space into the interval $[-1,1]$. We select $P = 14$ since we are confident that this guarantees that our result are sufficiently accurate.[9]

[8]For further information, see Judd (1998).

[9]Our choice of P is not arbitrary. We know that in the special case where households are identical, the steady states of the stationary Markov perfect equilibrium, the open-loop equilibrium and the competitive equilibrium coincide. Obviously, we can compute the latter two steady states analytically. This gives us the opportunity to fine tune our algorithm. We do that by determining the order of approximation for which we derive sufficiently accurate results for the steady state of the stationary Markov perfect equilibrium, when households are identical. In practice, we use the parametric case where $\alpha = 0.36$, $\delta = 0.05$, $\beta^1 = \beta^2 = 0.925$ and $\sigma^1 = \sigma^2 = 1$, and we raise the order of approximation until the relative error of aggregate capital stocks is smaller than 1e-5.

We continue by selecting n gridpoints for each capital stock as the zeros of the Chebyshev polynomial of order n, and by combining these points to create a bivariate grid of size $n^2 \times 2$. We denote this grid by $[k_m^1\, k_m^2]$, $m = 1, \ldots, M$, where we use $M = n^2$ for notational convenience. In our application, we set n equal to 15, since we find that a larger n has a negligible effect on the accuracy of the solution, while increasing the computational time substantially.[10]

For any given θ, it is straightforward to find approximate policy functions such that (5)-(11) hold. In a stationary Markov perfect equilibrium, however, also the Bellman equation (4) must be satisfied. We use an iterative procedure to find a parameter vector $\bar{\theta}$ for which this is true, that is, (5)-(11) and (4) hold.

Before discussing the steps involved in this procedure in detail, we introduce some new notation and place a remark on the full provision of labor.

We denote by $\tilde{\mathbf{c}}(k_m^1, k_m^2; \theta^1)$ and $\tilde{\mathbf{c}}(k_m^1, k_m^2; \theta^2)$ the approximate optimal consumption functions for households 1 and 2, respectively. These functions are uniquely implied by the approximate value functions $\tilde{V}(k^1, k^2; \theta^1)$ and $\tilde{V}(k^1, k^2; \theta^2)$. Finally, we define

$$\tilde{\mathbf{g}}(k^1, k^2; \theta^1) \quad = R(k^1 + k^2, 2)k^1 + W(k^1 + k^2, 2) - \tilde{\mathbf{c}}(k^1, k^2; \theta^1) \tag{32}$$

and

$$\tilde{\mathbf{g}}(k^1, k^2; \theta^2) \quad = R(k^1 + k^2, 2)k^2 + W(k^1 + k^2, 2) - \tilde{\mathbf{c}}(k^1, k^2; \theta^2) \tag{33}$$

Obviously, $\tilde{\mathbf{g}}(k^1, k^2; \theta^1)$ and $\tilde{\mathbf{g}}(k^1, k^2; \theta^2)$ are the households' saving functions when both provide one unit of labor. Since households derive no disutility of labor, we are confident that providing the entire labor endowment is indeed optimal in equilibrium. Throughout our algorithm we thus assume that $l^1 = l^2 = 1$ holds, such that we need not explicitly solve for optimal labor supply from the equilibrium conditions. Obviously, this implies that the next period capital stocks are given by ${k_m^1}' = \tilde{\mathbf{g}}(k^1, k^2; \theta^1)$ and ${k_m^2}' = \tilde{\mathbf{g}}(k^1, k^2; \theta^2)$, respectively. We are of course aware of the fact that there might be circumstances under which a household would want to reduce its labor supply to influence prices. Therefore, we check that the full labor supply assumption holds, after having derived optimal policy functions. Indeed, we find that the full provision of labor is optimal for all households in every parametric case we consider.[11]

[10]By choosing $n = 15$ our least squares projection method actually becomes very similar to a Collocation projection method, in which the residuals of the Bellman equation are set equal to zero for a finite number of points.

[11]If the full provision of labor was not optimal for any household, we would have to rewrite the algorithm to allow for flexible labor supply. Although such a more general

Our iterative procedure to compute the parameter vectors θ^1 and θ^2, which correspond to the stationary Markov perfect equilibrium, then involves the following steps.

1. First we derive initial estimates of θ^1 and θ^2. To this end, we compute the value of every grid point $[k^1_m\, k^2_m]$, $m = 1, \ldots, M$, for each household, assuming it lived only for one period and consumed all its wealth and income at once. Using the results of the one-period problem, we compute initial estimates, $\theta^* = [\theta^{1*}\, \theta^{2*}]$, by ordinary least squares.[12]

2. Given approximate value functions $\tilde{V}(k^1, k^2; \theta^{1*})$ and $\tilde{V}(k^1, k^2; \theta^{2*})$, we use the equilibrium condition (5) to express optimal current consumption of both households as functions of next period capital stocks. We use these expressions in the full labor supply saving functions (32) and (33), such that we obtain a system of two non-linear equations in two unknowns, ${k^1_m}'$ and ${k^2_m}'$, for each point in the grid. We derive ${k^1_m}'$ and ${k^2_m}'$, $m = 1, \ldots, M$, using a non-linear equations solver. Given next period capital stocks, we compute optimal consumption from (5), that is, we compute $c^1_m = \tilde{\mathbf{c}}(k^1_m, k^2_m; \theta^{1*})$ and $c^2_m = \tilde{\mathbf{c}}(k^1_m, k^2_m; \theta^{2*})$ for all m. We then derive the right hand-sides of the Bellman equations (4) as functions of current capital stocks, k^1_m and k^2_m, and of parameters, θ^{1*} and θ^{2*}. That is, we compute

$$\begin{aligned} rhs^1_m &= u^1(\tilde{\mathbf{c}}(k^1_m, k^2_m; \theta^{1*})) + \beta^1 \tilde{V}(\tilde{\mathbf{g}}(k^1_m, k^2_m; \theta^{1*}), \tilde{\mathbf{g}}(k^1_m, k^2_m; \theta^{2*}); \theta^{1*}) \\ rhs^2_m &= u^2(\tilde{\mathbf{c}}(k^1_m, k^2_m; \theta^{2*})) + \beta^2 \tilde{V}(\tilde{\mathbf{g}}(k^1_m, k^2_m; \theta^{1*}), \tilde{\mathbf{g}}(k^1_m, k^2_m; \theta^{2*}); \theta^{2*}) \end{aligned}$$

   for all gridpoints $m = 1, \ldots, M$. Then, we derive a new parameter estimate $\hat{\theta} = [\hat{\theta^1}\, \hat{\theta^2}]$ as

$$\hat{\theta} = \arg\min_{\theta} \sum_{m=1}^{M} \left((\tilde{V}(k^1_m, k^2_m; \theta^1) - rhs^1_m)^2 + (\tilde{V}(k^1_m, k^2_m; \theta^2) - rhs^2_m)^2 \right)$$

   Finally, we check whether $|\theta^* - \hat{\theta}| < \varepsilon$, where ε denotes the convergence criterion, which we set to $1e-6$. If converge has not been achieved, we set $\theta^* = \hat{\theta}$ and repeat the step. If $|\theta^* - \hat{\theta}| < \varepsilon$ holds, we let $\bar{\theta}$ denote the final parameter vector and proceed to the next step.

algorithm is relatively easy to implement, we think that our version is preferable. It is substantially faster and more reliable, since we do not have to account for occasionally binding constraints.

[12]In some cases it may be that the algorithm fails to converge, since the initial guess derived from the one-period problem is too far away from the true solution. In these cases, however, one can usually apply homotopy methods to derive a better initial guess.

3. We check whether the optimality assumption of full labor supply holds for $\bar{\theta}$. To this end, we inspect whether

$$\begin{aligned} 0 &< \tilde{V}'_{(1)}(\tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^1), \tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^2); \bar{\theta}^1) \\ &\quad \times \Big[R_L(K_m, 2)k_m^1 + W(K_m, 2) + W_L(K_m, 2) \Big] \\ &+ \tilde{V}'_{(2)}(\tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^1), \tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^2); \bar{\theta}^1) \Big[R_L(K_m, 2)k_m^2 + W_L(K_m, 2) \Big] \end{aligned}$$

and

$$\begin{aligned} 0 &< \tilde{V}'_{(2)}(\tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^1), \tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^2); \bar{\theta}^2) \\ &\quad \times \Big[R_L(K_m, 2)k_m^2 + W(K_m, 2) + W_L(K_m, 2) \Big] \\ &+ \tilde{V}'_{(1)}(\tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^1), \tilde{\mathbf{g}}(k_m^1, k_m^2; \bar{\theta}^2); \bar{\theta}^2) \Big[R_L(K_m, 2)k_m^1 + W_L(K_m, 2) \Big] \end{aligned}$$

hold for all points in the grid, $m = 1, \ldots, M$. Here, $K_m = k_m^1 + k_m^2$ and $\tilde{V}'_{(j)}(k_m^1, k_m^2; \bar{\theta}^h)$ denotes the derivative of household h's approximate value function with respect to the jth argument.

4. Finally, we use the policy functions to compute the fixed point

$$k^{1*} = \tilde{\mathbf{g}}(k^{1*}, k^{2*}; \bar{\theta}^1), \quad k^{2*} = \tilde{\mathbf{g}}(k^{1*}, k^{2*}; \bar{\theta}^2)$$

where k^{1*} and k^{2*} denote the steady state capital stocks generated by the stationary Markov perfect equilibrium.

For further technical details we refer the reader to the MATLAB codes, which are available from the authors upon request and will be available on the authors' websites by the time of publication.