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Christina Pawlowitsch

How much ambiguity can persist?
A complete characterization of neutrally stable states for an evolutionary proto-language game

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How much ambiguity can persist? A complete characterization of neutrally stable states for an evolutionary proto–language game

Christina Pawlowitsch*

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Abstract

In an evolutionary sender–receiver game that describes how signals become associated with objects (Hurford, 1989; Nowak and Krakauer, 1999), the set of evolutionarily stable states coincides with the set of strict Nash strategies—and a language is a strict Nash strategy if and only if it links each possible referent exclusively to 1 signal and vice versa (Trapa and Nowak, 2000). As a consequence, a language that displays homonymy (or synonymy)—the property that one signal is linked to more than one referent (or one referent to more than one signal)—cannot be an evolutionarily stable state. This seems to conflict with the results of the computer simulation reported in Nowak and Krakauer (1999) that lend support to the conjecture that a language in which the same signal is used for more than one object can be evolutionarily stable. This paper provides necessary and sufficient conditions for a *neutrally stable state* of this game—and, importantly, these conditions directly characterize a single strategy—showing that a language displaying homonymy or synonymy, even though it fails to be evolutionarily stable (in the strict sense), may still satisfy neutral stability, explaining why an evolutionary process does not necessarily lead away from it. *Journal of Economic Literature* Classification Number: C72

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1 Introduction

Bringing together theories about the evolution of language and *evolutionary game theory* recently has attracted some interest in both linguistics and applied mathematics.[1]

*Department of Economics, University of Vienna; Christina.Pawlowitsch@univie.ac.at

[1]See, for example, the work of J. Hurford and S. Kirby in linguistics; here a good reference is Christiansen and Kirby (2003). And the work initiated by M. Nowak and others in applied mathematics and theoretical biology; for a good review article of this strand of literature see Nowak, Komarowa and Niyogi (2003).

This paper takes up the issue of a specific sender–receiver game, first introduced in Hurford (1989) and further analyzed in Nowak and Krakauer (1999) and Trapa and Nowak (2000), that allows to study the evolution of so called *proto–languages* from an essentially prelinguistic environment. Here, the term proto–language is simply understood as a collection of correspondences between a finite number of *events* that are possibly communicated and a finite number of *signals* that are available to indicate these events.

Evolution of the Saussurean sign

Hurford (1989) is mainly interested in the evolution of the so–called *Saussurean sign*, a *bidirectional* mapping between a phonological form and some representation of a concept. He starts from the observation that most linguistic theories take lexical entries that incorporate the idea of the Saussurean sign as an integral part of the human Language Acquisition Device—that is, as a starting point. On purely logical grounds, however, successful communication does not necessarily require bidirectionality. He therefore hypothesizes that there must have been some evolutionary advantage of this design feature of human language over other possible communication strategies. The aim of his work is to provide a formal argument for this claim. In order to allow for the possibility of non–bidirectionality, in his model set–up, he first has to disentangle the lexical matrices individuals use for transmission and reception purposes. This leads to a specific sender–receiver game, which he approaches in a population based setting by computer experiments. His main argument finally consists in showing that individuals who adjust their receiver matrices to their sender matrices according to some consistency requirements fare considerably better in terms of their overall potential of successful communication than individuals with other behavioral rules.

Evolution of a common proto–language

Nowak and Krakauer (1999) take the same sender–receiver game as a starting point for their exploration of the evolution of language. They also study it via computer simulations, but they mainly see it as a framework to explain how meaning of signals can come into being by an evolutionary process that does not presuppose any ex–ante internalization of concepts, or, for that matter, rationality. Their simulation, therefore, is much more in the spirit of a replicator dynamics.

Starting from randomly drawn sender and receiver matrices, in every period, every individual communicates once with every other individual in the population. Payoffs are calculated, and in the following period each individual strategy, that is, a pair of a sender and a receiver matrix, replicates itself according to its payoff relative to the accumulated payoff in the overall population. After a certain number of rounds, indeed, specific signals start to correspond with specific events, and finally the population profile seems to converge to a common proto–language.

This type of replicator–imitation dynamics can be interpreted as biological as well as cultural transmission of the information captured by the sender and receiver matrices. Individuals that communicate more successfully leave more off spring, and parents transmit their sender and receiver matrices to their kids. Or, what is another interpretation, younger individuals are more likely to imitate the more successful individuals of the older generation, and so the more successful sender and receiver matrices spread at the cost of the less successful ones.

An interesting feature of the common proto–language for which the computer simulation lends support is that, even though there are as many signals as there are events, there are some events that share the use of one signal; or, put differently, there are some signals that are linked to more than one event. In linguistics, this phenomenon is called *homonymy*, as opposed to *synonymy* that refers to a situation where one event is linked to more than one signal.

Clearly, as it is generally true for imitation and replication dynamics, the language to which the simulation converges crucially depends on the initial conditions. What is surprising, indeed, is that their results also seems to be stable against the introduction of mutant strategies. From this the authors draw the conclusion that evolution does not always lead to a proto–language with maximal communicative potential—that is, where each event is exclusively linked to one signal and vice versa—but that certain suboptimum solutions where some signals are linked to more than one event can be evolutionarily stable.

Is homonymy evolutionarily stable?

Trapa and Nowak (2000) analyze the Nash equilibria and evolutionarily stable states of this game. They show that a proto–language, that is, a pair of a sender and receiver matrix, is an evolutionarily stable state if and only if it is a strict Nash strategy; and that a language is a strict Nash strategy if and only if there are as many signals as there are events, the sender matrix attributes to each event exactly one signal and vice versa—in mathematical terms, if it is a permutation matrix, and the receiver matrix is the transpose of the sender matrix. As a consequence, a language with the property of homonymy (or synonymy) *cannot* be a strict Nash strategy, and thus it also *cannot* be an evolutionarily stable.

Nevertheless, under specific conditions, homonymy or synonymy can well prevail in a Nash, though not in a strict Nash equilibrium. Being a Nash equilibrium is a necessary condition for an evolutionarily stable state. The language to which the computer simulation reported in Nowak and Krakauer (1999) converges seems to be compatible with these equilibrium conditions. Therefore, an interesting question to ask is whether a Nash, but not strict Nash language that displays homonymy (or synonymy), even though it fails to be evolutionarily stable, may still satisfy a weaker criterion for stability in an evolutionary sense that allows to understand why an evolutionary process does not lead away from it.

In their discussion of Nash strategies, Trapa and Nowak present examples of both, Nash, but not strict Nash strategies that are, and that are not *weak evolutionarily stable state*. They conjecture that, in principle, it should be possible to classify weak evolutionarily stable states for this game.

In the literature, the concept of weak evolutionary stability is also known as *neutral stability*[2]. Neutral stability means that there can be room for drift. There can be mutant strategies that can invade—there can be coexistence—but that do not have to invade necessarily. But if a state fails to be neutrally stable, then this means that there is at least one mutant strategy that, if introduced into the current strategy profile—even in only small amounts—eventually will take over. In this sense, there is some evolutionary pressure that leads away from a state that does not satisfy neutral stability, which is not necessarily true for a state that does satisfy neutral stability.

This paper gives necessary and sufficient conditions for a Nash strategy of this game to be a neutrally stable state, showing that Proto–languages with multi–valued correspondences between signals and events (homomymy) or events and signals (synonymy), even though they fail to be evolutionarily stable in the “strict” sense, may still satisfy neutrally stability as long as the degree of ambiguity is not too high: An event can be linked to more than one signal, but if this is the case, these signals cannot be used to refer to any other event. A signal can refer to more than one event, but if this is the case, these events cannot be linked to any other signal.

Methodologically, the emphasis is on the double symmetry of this game the best–response properties of sender and receiver matrices, thereby also providing an alternative discussion of Nash strategies.

Section 2 reviews the model and some definition. Section 3 restates them in a convenient way for the class of double symmetric games to which this specific sender–receiver game belongs. Section 4 explores the best–response properties of sender and receiver matrices. Section 5 shows how they can be used to characterize the Nash strategies of this game. Section 6, finally, develops the results on neutrally stable states.

2 The basic model

There is a large number of individuals among whom communication potentially takes place. There are n events that possibly become the object of communication, and there are m signals that can be used to communicate these events.

[2]See the discussion of neutrally stable states in Hofbauer and Sigmund (1998), and Samuelson (1997)

The $n \times m$–matrix

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{i1} & \cdots & p_{ij} & \cdots & p_{im} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nj} & \cdots & p_{nm} \end{pmatrix}$$

denotes an individuals *sender matrix*, where

$$P \in \Delta_w^{n\times m} = \{P \in \mathbb{R}_+^{n\times m} : \sum_{j=1}^{m} p_{ij} \leq 1, \forall i\},$$

the set of all weak stochastic matrices of dimension $n \times m$, and p_{ij} indicates the probability with which signal j will be transmitted if event i is to be communicated. On the other hand, the $m \times n$–matrix

$$Q = \begin{pmatrix} q_{11} & \cdots & q_{1i} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ q_{j1} & \cdots & q_{ji} & \cdots & q_{jn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ q_{m1} & \cdots & q_{mi} & \cdots & q_{mn} \end{pmatrix}$$

denotes an individual's *receiver matrix*,

$$Q \in \Delta_w^{m\times n} = \{Q \in \mathbb{R}_+^{m\times n} : \sum_{i=1}^{n} q_{ji} \leq 1, \forall i\},$$

and q_{ji} gives the probability with which event i will be inferred if signal j is received. If $\sum_{j=1}^{m} p_{ij} < 1$, the interpretation is that there is a residual probability, $1 - \sum_{j=1}^{m} p_{ij}$, that the possible event i does not induce the remittance of any signal at all. In this sense, the individual has "no name" for i; and if $\sum_{i=1}^{n} q_{ji} < 1$, then with probability $1 - \sum_{i=1}^{n} q_{ji}$ signal j is not associated with any meaning at all. A pair $L = (P, Q)$ is a language.

If an individual with receiver matrix P observes event i and wants to communicate this to an individual with language Q', then the probability that she successfully is doing so, is

$$\sum_{j=1}^{m} p_{ij} q'_{ji}.$$

Assuming that the objects of communication occur with equal frequencies,

$$f(P, Q') = \sum_{i} \frac{\sum_{j=1}^{m} p_{ij} q'_{ji}}{n}$$

can be taken as a measure for the *potential of successful communication* of an individual with the sender matrix P relative to an individual with the receiver matrix Q'.

The stage game

This set–up can be rephrased as a *two–player symmetric game*. The players of this game are the individuals. They all face the same strategy sets and payoff functions. A *strategy* of the game is a language,

$$L = (P, Q) \in \mathcal{L}^{n,m}, \tag{1}$$

where

$$\mathcal{L}^{n,m} = \Delta_w^{n\times m} \times \Delta_w^{m\times n} \tag{2}$$

is the *strategy set*. Assuming that communication is mutually beneficial and that the events of potential communication occur with equal frequencies—or, are equally important—a suitable form of the *payoff function* is given by

$$\begin{aligned} F(L, L') &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}q'_{ji} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} p'_{ij}q_{ji} \\ &= \frac{1}{2}\mathrm{tr}(PQ') + \frac{1}{2}\mathrm{tr}(P'Q). \end{aligned} \tag{3}$$

Note that this payoff function is symmetric,

$$F(L, L') = \mathrm{tr}(PQ') + \mathrm{tr}(P'Q) = \mathrm{tr}(P'Q) + \mathrm{tr}(PQ') = F(L', L),$$

giving rise to a *double symmetric game*.

Definitions

In the terminology of evolutionary game theory, for a symmetric game, a strategy is called "Nash" if it is a best reply to itself—that is, if it is the strategy played in a symmetric Nash equilibrium; and it is called "strict Nash" if it is the unique best reply to itself.

Definition 1. *Let* $L = (P, Q) \in \mathcal{L}^{n,m}$.

(a) L *is a* Nash strategy *if*

$$F(L, L) \geq F(L', L) \quad \textit{for all } L' \in \mathcal{L}^{n,m};$$

(b) L *is a* strict Nash strategy *if*

$$F(L, L) > F(L', L) \quad \textit{for all } L' \in \mathcal{L}^{n,m} \textit{ with } L' \neq L.$$

The most central stability concept in evolutionary game theory is *evolutionary stability*. It captures the idea that the population profile has reached a state that cannot be invaded by any mutant strategy that occurs within sufficiently small boundaries, whereas in a *neutrally stable state* there is room for drift. Equivalent criteria are given by the following definitions.

Definition 2. *Let* $L = (P, Q) \in \mathcal{L}^{n,m}$.

(a) L *is an* evolutionarily stable state *if and only if*

(i) L *is a Nash strategy, and*

(ii) $F(L, L) = F(L', L) \Rightarrow F(L', L') < F(L, L')$ *for all* $L' \in \mathcal{L}^{n,m}$*;*

(b) L *is a* neutrally stable state *if and only if*

(i) L *is a Nash strategy, and*

(ii) $F(L, L) = F(L', L) \Rightarrow F(L', L') \leq F(L, L')$ *for all* $L' \in \mathcal{L}^{n,m}$*.*

From these definitions one directly sees that:

strict Nash $\Rightarrow$ evolutionarily stable state $\Rightarrow$ neutrally stable state $\Rightarrow$ Nash.

3 Symmetrized games

The underlying stage game we are dealing with is best understood as the symmetrized version of an asymmetric game according to Selten (1980). In the asymmetric game there are two types of players, senders and receivers. Let senders be of type 1 and receivers of type 2. Then the strategy sets and payoff functions are

$$\begin{aligned} S_1 &= \Delta_{\mathrm{w}}^{n \times m}, \\ S_2 &= \Delta_{\mathrm{w}}^{n \times m}; \end{aligned}$$

and

$$\begin{aligned} f_1(s_1, s_2) &= \mathrm{tr}(s_1 s_2), \\ f_2(s_1, s_2) &= \mathrm{tr}(s_1 s_2), \end{aligned}$$

with $(s_1, s_2) \in S_1 \times S_2$ for players 1 and 2, respectively. Note that $f_1(s_1, s_2) = f_2(s_1, s_2)$, which means that even though interacting players are of different types, they get the same payoff out of their interaction.

Assuming that individuals find themselves in the roles of sender and receivers with equal probabilities, the payoff function of the symmetrized game is given by

$$F[(s_1, s_2), (s_1', s_2')] = \frac{1}{2} f_1(s_1, s_2') + \frac{1}{2} f_2(s_1', s_2),$$

where $(s_1, s_2) \in S_1 \times S_2$ is a strategy of the symmetrized game.

It is *not* generally true that the payoff function of a symmetrized game is symmetric! This is only the case if $f_1(s_1, s_2) = f_2(s_1, s_2)$. Then,

$$\begin{aligned} F[(s_1, s_2), (s_1', s_2')] &= \frac{1}{2} f_1(s_1, s_2') + \frac{1}{2} f_2(s_1', s_2) \\ &= \frac{1}{2} f_2(s_1, s_2') + \frac{1}{2} f_1(s_1', s_2) = F[(s_1', s_2'), (s_1, s_2)]. \end{aligned}$$

Let $P \in \Delta_w^{n\times m}$ and $Q \in \Delta_w^{m\times n}$. Then according to the usual definition,

$$B(P) \quad = \quad \{Q \in \Delta_w^{m\times n} \ : \ \mathrm{tr}(PQ) \geq \mathrm{tr}(PQ') \,\forall\, Q' \in \Delta_w^{m\times n}\}$$

denotes the *set of best responses to* P; and

$$B(Q) \quad = \quad \{P \in \Delta_w^{n\times m} \ : \ \mathrm{tr}(PQ) \geq \mathrm{tr}(P'Q) \,\forall\, P' \in \Delta_w^{n\times m}\}$$

denotes the *set of best responses to* Q. Clearly, for fixed P, the continuous function $\mathrm{tr}(PQ)$ attains a maximum on the compact set $\Delta_w^{m\times n}$; and, for fixed Q, $\mathrm{tr}(PQ)$ attains a maximum on $\Delta_w^{m\times n}$. So, $B(P)$ and $B(Q)$ are both non–empty.

It is generally true that (s_1, s_2) is a Nash strategy of the symmetrized game if and only if s_1 is a best response to s_2, and s_1 is a best response to s_2—and (s_1, s_2) is a strict Nash strategy if and only if s_1 is the unique best response to s_2, and s_1 is the unique best response to s_2.

To see why this is so, suppose that $(s_1, s_2) \in S_1 \times S_2$ is a Nash strategy, but that $s_1 \notin B(s_2)$. So, there is some $s_1' \in S_1$, $s_1' \neq s_1$ such that $f_1(s_1', s_2) > f_1(s_1, s_2)$. Consider (s_1', s_2) as an alternative strategy in the symmetrized game, then

$$\begin{aligned} f_1(s_1, s_2) + f_2(s_1, s_2) \quad &< \quad f_1(s_1', s_2) + f_2(s_1, s_2) \\ \Longleftrightarrow\ F[(s_1, s_2), (s_1, s_2)] \quad &< \quad F[(s_1', s_2), (s_1, s_2)] \end{aligned}$$

but this cannot be true if (s_1, s_2) is a Nash strategy.

The strict part works by an analogous argument. Suppose that (s_1, s_2) is a strict Nash strategy, but that there also exists some $s_1' \in B(s_2)$ with $s_1' \neq s_1$. Then,

$$\begin{aligned} f_1(s_1, s_2) + f_2(s_1, s_2) \quad &= \quad f_1(s_1', s_2) + f_2(s_1, s_2) \\ \Longleftrightarrow\ F[(s_1, s_2), (s_1, s_2)] \quad &= \quad F[(s_1', s_2), (s_1, s_2)], \end{aligned}$$

which cannot be true if (s_1, s_2) is a *strict* Nash strategy.

Note that this argument *does not* require that $f_1(s_1, s_2) = f_2(s_1, s_2)$. Sufficiency follows directly from the definition of best–response sets. [3]

For the sender–receiver game under consideration this implies the following:

[3] Selten (1980) shows that for the symmetrized version of an asymmetric game it is generally true that (s_1, s_2) is an evolutionarily stable state if an only if it is a strict symmetric Nash equilibrium. The proof of this is quite intuitive. Let $[(s_1, s_2)]$ a Nash but not a strict Nash strategy. Suppose that in addition to s_1 there is some $s_1' \neq s_1$ with $s_1' \in B(S_2)$, and consider the pair (s_1', s_2) as an invading strategy. Then, $f_1(s_1', s_2) + f_2(s_1, s_2) = f_1(s_1, s_2) + f_2(s_1, s_2)$, which means that $F[(s_1', s_2), (s_1, s_2)] = F[(s_1, s_2), (s_1, s_2)]$. But, as a consequence, $f_1(s_1, s_2) + f_2(s_1', s_2) = f_1(s_1', s_2) + f_2(s_1', s_2)$, which means that $F[(s_1, s_2), (s_1', s_2)] = F[(s_1', s_2), (s_1', s_2)]$; and so (s_1, s_2) cannot be evolutionarily stable. The crucial point here is that, since $f_1(s_1', s_2) = f_1(s_1, s_2)$, the strategy (s_1, s_2) has no chance to attain a higher payoff against (s_1', s_2), then (s_1', s_2) attains against itself. Note, that this argument does not require that $f_1(s_1, s_2) = f_2(s_1, s_2)$. The coincidence of evolutionarily stable states and strict Nash strategies for the sender–receiver game under consideration can be seen as a specific incidence of this more general result.

Lemma 1. *Suppose $L = (P, Q) \in \mathcal{L}^{n,m}$. Then*

(a) L is a Nash strategy if and only if $P \in B(Q)$ and $Q \in B(P)$;

(b) L is a strict Nash strategy if and only if $B(Q) = \{P\}$ and $B(P) = \{Q\}$, that is, P is the unique element in $B(Q)$ and Q is the unique element in $B(P)$.

Corollary 1. *$L = (P, Q) \in \mathcal{L}^{n,m}$ is a Nash but not strict Nash strategy if and only if*

- *$P \in B(Q)$ and $Q \in B(P)$, and*
- *there exists some $P' \in B(Q)$ with $P' \neq P$, or some $Q' \in B(P)$ with $Q' \neq Q$.*

This follows directly from Lemma 1. Just note that, since (P, Q') and (P', Q) are both different from (P, Q), it is indeed sufficient that either P is not unique in $B(Q)$ or that Q is not unique in $B(P)$.

Remark 1. For the specific payoff function of this game, if $Q \in B(P)$ and $Q' \in B(P)$, as well as $P \in B(Q)$ and $P' \in B(Q)$ then

$$\mathrm{tr}(PQ') = \mathrm{tr}(PQ) = \mathrm{tr}(P'Q).$$

Double symmetric games

For a symmetric payoff function—irrespective of whether this comes from the symmetrization of an asymmetric game or not—the stability notions of Definition 2 can be stated in a more convenient way.

Lemma 2. *If $F(L, L') = F(L', L)$, then Definition 2 can be rewritten as:*
Let $L = (P, Q) \in \mathcal{L}^{n,m}$.

(a) L is an evolutionarily stable state *if and only if*

  (i) L is a Nash strategy, and

  (ii) $F(L, L) = F(L', L) \Rightarrow F(L', L') < F(L, L)$ for all $L' \in \mathcal{L}^{n,m}$;

(b) L is a neutrally stable state *if and only if*

  (i) L is a Nash strategy, and

  (ii) $F(L, L) = F(L', L) \Rightarrow F(L', L') \leq F(L, L)$ for all $L' \in \mathcal{L}^{n,m}$.

Remark 2. For the specific form of the payoff function that we use, Definition 2 can be restated as follows:
Let $L = (P, Q) \in \Delta^{n \times m} \times \Delta^{m \times n}$.

(a) L is an evolutionarily stable state if and only if

(i) $tr(PQ') \leq tr(PQ)$ *for all* $Q' \in \Delta^{m\times n}$ *and*
$tr(P'Q) \leq tr(PQ)$ *for all* $P' \in \Delta^{n\times m}$*; and*

(ii) $tr(PQ') = tr(PQ) = tr(P'Q) \implies tr(P'Q') < tr(PQ)$.

(b) L *is a neutrally stable state if and only if*

(i) $tr(PQ') \leq tr(PQ)$ *for all* $Q' \in \Delta^{m\times n}$ *and*
$tr(P'Q) \leq tr(PQ)$ *for all* $P' \in \Delta^{n\times m}$*; and*

(ii) $tr(PQ') = tr(PQ) = tr(P'Q) \implies tr(P'Q') \leq tr(PQ)$.

4 Best–response properties of sender and receiver matrices

Suppose now that we are given a specific P and we want to find all the receiver matrices Q that maximize $\mathrm{tr}(PQ)$. Since the elements in Q are row–wise bounded to add up to 1, for fixed P, it is convenient to understand the operator $\mathrm{tr}(PQ)$ as multiplying the j-th column of P with the j-th row of Q, and then summing over all j:

$$\begin{aligned}
\mathrm{tr}(PQ) &= \sum_i \sum_j p_{ij} q_{ji} = \sum_j \sum_i p_{ij} q_{ji} \\
&= p_{11}q_{11} + p_{21}q_{12} \cdots + p_{n1}q_{1n} \\
&+ p_{12}q_{21} + p_{22}q_{22} \cdots + p_{n2}q_{2n} \\
&\vdots \\
&+ p_{1m}q_{m1} + p_{2m}q_{m2} \cdots + p_{nm}q_{mn}.
\end{aligned}$$

Finding a Q that maximizes $\mathrm{tr}(PQ)$ then amounts to choosing optimal "weights" q_{ji} to their corresponding elements p_{ij} such that $\sum_i p_{ij} q_{ji}$ is maximal for every j.

Fix, for example, the $j^\star$-th column of P and suppose that it contains a unique maximal element, say $p_{i^\star j^\star}$. Then in order to maximize $\sum_i p_{ij^\star} q_{j^\star i}$ it is clearly the optimal choice to put "full weight" to $p_{i^\star j^\star}$—that is, to set $q_{j^\star i^\star}$ equal to 1, and all the other elements in the $j^\star$-th row equal to zero. If, on the other hand, the $j^\star$-th column of P contains more than one maximal element, then there is more than one optimal appointment of the elements in the $j^\star$-th row of Q. All the corner solutions, where full weight is put to any of the maximal elements in the $j^\star$-th column of P, as well as any of their convex combinations fulfill the task of maximizing $\sum_i p_{ij^\star} q_{j^\star i}$. But no matter *how* the total mass of 1 is attached to the elements in the $j^\star$-th column of P, there is no way of doing better than to "extract" from the $j^\star$-th column of P the value of its maximum.

If Q is fixed and the entries in P are to be chosen optimally so that $\mathrm{tr}(PQ)$ is maximized, exactly the same logic applies only with the roles of P and Q reversed. Note that, in this case, one proceeds by columns of Q and rows of P.

The next two lemmas summarize these best–response properties of P and Q. Some extra notation helps exposition. We define

$$A(p_{\cdot j}) = \text{argmax}_i(p_{ij})$$

as the *index set of maximal row elements of the j-th column of P*, and analogously,

$$A(q_{\cdot i}) = \text{argmax}_j(q_{ji})$$

as the *index set of maximal row elements of the i-th column of Q.*

Lemma 3 (Best–response properties of P and Q). *Let $P \in \Delta_w^{n\times m}$ and $Q \in \Delta_w^{m\times n}$.*

(1) Suppose $Q \in B(P)$.

(a) If $p_{i^\star j^\star} \neq \max_i(p_{ij^\star})$ then $q_{j^\star i^\star} = 0$.

(b) If $\max_i(p_{ij^\star}) \neq 0$ then

$$\sum_{i\in A(p_{\cdot j^\star})} q_{j^\star i} = 1 \quad \textit{and} \quad q_{j^\star i} = 0 \quad \forall\, i \notin A(p_{\cdot j^\star}).$$

(2) Suppose $P \in B(Q)$.

(a) If $q_{j^\star i^\star} \neq \max_j(q_{ji^\star})$ then $p_{i^\star j^\star} = 0$.

(b) If $\max_j(q_{ji^\star}) \neq 0$ then

$$\sum_{j\in A(q_{\cdot i^\star})} p_{i^\star j} = 1 \quad \textit{and} \quad p_{i^\star j} = 0 \quad \forall\, j \notin A(q_{\cdot i^\star}).$$

Corollary 2. *Let $P \in \Delta^{n\times m}$ and $Q \in \Delta^{m\times n}$.*

(1) Suppose $Q \in B(P)$. If $A(p_{\cdot j^\star}) = \{i^\star\}$ then

$$q_{j^\star i^\star} = 1 \quad \textit{and} \quad q_{j^\star i} = 0 \quad \forall\, i \neq i^\star.$$

(2) Suppose $P \in B(Q)$. If $A(q_{\cdot i^\star}) = \{j^\star\}$ then

$$p_{i^\star j^\star} = 1 \quad \textit{and} \quad p_{i^\star j} = 0 \quad \forall\, j \neq j^\star.$$

Lemma 4. *Let $P \in \Delta^{n\times m}$ and $Q \in \Delta^{m\times n}$.*

(1) For fixed P,

$$\max_{q_{ji}} \left(\sum_j \sum_i p_{ij} q_{ji} \right) = \sum_j \max_i \left(p_{ij}\right).$$

(2) For fixed Q,

$$\max_{p_{ij}} \left(\sum_i \sum_j p_{ij} q_{ji} \right) = \sum_i \max_j (q_{ji}) .$$

How to find the set of all Q–s that are a best response to a given P or vice versa is best understood by looking at specific examples.

Example 1.

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1-x & x \end{pmatrix}_{x\in(0,1)}, \quad B(P_1) = \left\{ \begin{pmatrix} 1-y & y & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} : y \in [0,1] \right\}.$$

Note that $B(P_1)$ also includes the two "corner solutions"

$$Q_{1,y=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } Q_{1,y=1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that

$$\mathrm{tr}(P_1 Q_1) = \sum_j \max_i p_{ij} = 1 + (1-x) + x = 2.$$

for all $Q_1 \in B(P_1)$.

Note that for all $Q_1 \in B(P_1)$ it is also true that P_1 is a best response to Q_1. So, for this special example, all pairs $L_1 = (P_1, Q_1)$ with $Q \in B(P_1)$ are Nash strategies of this game. Note that this also includes the two pairs involving the two corner solutions

$$(P_1, Q_{1,y=0}) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1-x & x \end{pmatrix}_{x\in(0,1)}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right],$$

and

$$(P_1, Q_{1,y=1}) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1-x & x \end{pmatrix}_{x\in(0,1)}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right].$$

Clearly, non of these pairs is a strict Nash strategy, which is immediate from the fact that $B(P_1)$ contains more than one element. As languages all these pairs $L_1 = (P_1, Q)$ with $Q_1 \in B(P_1)$ display homonymy as well as synonymy.

Information revealed by best response

On the other hand, if we are given a specific Q and we know that $Q \in B(P)$, is there anything we can learn about P? To illustrate this, consider one of the elements Q_1 in $B(P_1)$ from the previous example, say

$$Q_{1,y=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now we must try to derive conclusions from rows in $Q_{1,y=0}$ to corresponding columns in P_1. The unique 1 in the third row of $Q_{1,y=0}$ came from the fact that $p_{33} = x$ was the unique maximal entry in the third column of P_1. The question is, can we learn this, just by observing $q_{33} = 1$? We definitely understand that p_{33} must have been *a* maximal element of the third column of P_1—otherwise putting some weight to this element could not have been optimal, but we cannot conclude that p_{33} was the *unique* maximal element of this column. There could have been some other maximal elements in this column. As we have seen above, all the corner solutions that attribute full weight to any of the maximal elements of a specific column are optimal solutions. Analogously, from $q_{23} = 1$ we learn that p_{32} is *a* maximal element of the second column of P_1, as well as that we learn from $q_{11} = 1$ that p_{11} is *a* maximal element of the first column of P_1. So, all we possibly know about P_1 from the fact that $Q_{1,y=0} \in B(P_1)$ is that

$$P_1 = \begin{pmatrix} \max_i(p_{i1}) & - & - \\ - & - & - \\ - & \max_i(p_{i2}) & \max_i(p_{i3}) \end{pmatrix}.$$

Note that, since this specific $Q_{1,y=0} \in B(P_1)$ puts no weight to the second maximal element of the first column of P_1, we have no way to understand that there was any other maximal element in this column.

As another example, consider the matrix

$$Q_{1,0<y<1} = \begin{pmatrix} 1-y & y & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{y\in(0,1)},$$

which, as we have seen above, also is an element of $B(P_1)$. By the same logic as above, we now learn that

$$P_1 = \begin{pmatrix} \max_i(p_{i1}) & - & - \\ \max_i(p_{i1}) & - & - \\ - & \max_i(p_{i2}) & \max_i(p_{i3}) \end{pmatrix}.$$

Nevertheless, we still cannot exclude the possibility that there were other maximal elements in *any* of the columns of P_1. But, the interesting thing in this case is that just by observing *any of the two non–zero elements in the first row of* $Q_{1,0<y<1}$ *in isolation* we could have learned that the first column of P_1 does not have a *unique* maximal element. Otherwise setting some element in the first row of $Q_{1,0<y<1}$ equal to some positive value that is not equal to 1 could not have been optimal for maximizing $\text{tr}(P_1 Q_1)$. Note that this does not exclude the possibility that the first column of P_1 is a zero column, but if we can further assume that the first column of P_1 does not consist entirely of zeros, then we also know that there must be some other non–zero elements in the first *row* of $Q_{1,0<y<1}$ (at least one) whose corresponding elements in P_1 are maximal elements of the first column of P_1 and which together with the originally observed non–zero element in the first row of $Q_{1,0<y<1}$ add up to 1. Otherwise the first row of $Q_{1,0<y<1}$ would not fully extract the maximum value of the first column of P_1. The next lemma generalizes these observations.

Lemma 5 (Properties of P and Q revealed by best response). *Let $P \in \Delta_w^{n\times m}$ and $Q \in \Delta_w^{m\times n}$.*

(1) Suppose $Q \in B(P)$.

- *(a) If $q_{j^\star i^\star} \neq 0$ then $p_{i^\star j^\star} = \max_i(p_{ij^\star})$.*
- *(b) If $0 < q_{j^\star i^\star} < 1$ then in addition there is some other $p_{i'j^\star} = \max_i(p_{ij^\star})$ with $i' \neq i^\star$. And if it furthermore can be assumed that the $j^\star$-th column in P does not consist entirely of zeros, then it also must be true that $\sum_{i\in A(p_{\cdot j^\star})} q_{j^\star i} = 1$.*

(2) Suppose $P \in B(Q)$.

- *(a) If $p_{i^\star j^\star} \neq 0$ then $q_{j^\star i^\star} = \max_j(q_{ji^\star})$.*
- *(b) If $0 < p_{i^\star j^\star} < 1$ then in addition there is some $q_{j'i^\star} = \max_j(q_{ji^\star})$ with $j' \neq j^\star$. And, if it furthermore can be assumed that the $i^\star$-th column in Q does not consist entirely of zeros, then it also is true that $\sum_{j\in A(q_{\cdot i^\star})} p_{i^\star j} = 1$.*

Note that (1.a) and (2.a) of Lemma 5 are just the contrapositives of the corresponding statements in Lemma 3.

5 Nash strategies

Via the bridge of Lemma 1—the notion of a Nash strategy that P has to be a best response to Q, and Q has to be a best response to P—Lemmas 3 and 5 together characterize the Nash strategies of this game.

Coming back to the question about the bidirectionality of event–signal correspondences raised by Hurford (1989) this can be stated as some minimal consistency requirements that must prevail in a symmetric Nash equilibrium:

Proposition 1. *Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a Nash strategy.*

(1) If $p_{i^\star j^\star} \neq 0$, then $q_{j^\star i^\star} = 0$ if and only if

(a) $q_{ji^\star} = 0$ *for all* j*, that is, event* $i^\star$ *is never successfully communicated; and*

(b) there are some other (at least one) events that use signal $j^\star$ *with at least the same probability as* $i^\star$*, and which cumulatively are inferred by signal* $j^\star$ *with full probability.*

(2) If $q_{j^\star i^\star} \neq 0$*, then* $p_{i^\star j^\star} = 0$ *if and only if*

(a) $p_{ij^\star} = 0$ *for all* i*, which means that signal* $j^\star$ *remains idle; and*

(b) there are some other (at least one) signals that induce event $i^\star$ *with at least the same probability as* $j^\star$*, and which cumulatively communicate event* $i^\star$ *with full probability.*

Proof. Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a Nash strategy.

If $p_{i^\star j^\star} \neq 0$, then by Lemma 5, $q_{j^\star i^\star} = \max_j(q_{\cdot i^\star})$. That is, if event $i^\star$ uses signal $j^\star$ with some probability to get communicated, then no other signal induces event $i^\star$ with a higher probability than $j^\star$. This does not necessarily imply that $q_{j^\star i^\star} \neq 0$. But $q_{j^\star i^\star} = 0$ can only be the case if $q_{ji^\star} = 0$ for all j, which means that event $i^\star$ is never successfully communicated. This proves part (a) of the proposition.

In addition one of the two cases must be met: (i) Either $p_{i^\star j^\star} \neq \max_i(p_{ij^\star})$, that is, there is at least some other event that uses $j^\star$ with a higher probability—indeed, this only is possible if $q_{ji^\star} = 0$ for all j, or (ii) $p_{i^\star j^\star} = \max_i(p_{ij^\star})$. But in both cases, $\max_i(p_{ij^\star}) \neq 0$, which by Lemma 3 implies that $\sum_{i^\star \neq i \in A(q_{\cdot i^\star})} q_{j^\star i} = 1$, which completes the proof (b).

Part (2) works by an analogous argument. □

There is an uncountable infinity of Nash strategies in this game. Note also that the Ps and Qs of a Nash strategy may have columns that consist entirely of zeros. In particular, a pair of zero–matrices with appropriate dimensions is a Nash strategy of the corresponding sender–receiver game. [4]

[4] The discussion of Nash strategies presented here can be seen as a complementary approach to the one taken in Trapa and Nowak (2000). They first characterize the specific class of Nash strategies, that is given by the condition that neither P nor Q contains columns that consist entirely of zeros; and then they show how by deleting and adding zero–columns and corresponding rows, all other Nash strategies can be reduced to and constructed from these particular Nash strategies, respectively. If neither P nor Q contains a zero–column, they find that $L = (P, Q) \in \mathcal{L}^{n,m}$ is a Nash strategy if and only if there exist real numbers $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$ such that for each j, the j-th column of P has its entries drawn from $\{0, p_j\}$—and $p_{ij} = p_j$ if and only if $q_{ji} = q_i$; and such that for each i, the i-th column of Q has its entries drawn from $\{0, q_i\}$. As a matter of consistency, $q_{ji} = q_i$ if and only if $p_{ij} = p_j$. Lemmas 1, 3 and 5 together with the no–zero–columns condition can be used to prove this result. The advantage of the approach taken here is that it also can be used to encompasses other conditions on P or Q, which proves to be particularly useful when it comes to characterize neutrally stable states.

6 Neutrally stable states

Consider any of the pairs $L_1 = (P_1, Q_1)$ with $Q_1 \in B(P_1)$ from Example 1. It is easily verified that L_1 is not only a Nash strategy but also a neutrally stable state. Whenever $\mathrm{tr}(P_1 Q_1') = \mathrm{tr}(P_1 Q_1)$ for some $Q_1' \in \Delta_w^{3\times 3}$ or $\mathrm{tr}(P_1' Q_1) = \mathrm{tr}(P_1 Q_1)$ for some $P_1' \in \Delta_w^{3\times 3}$, then $\mathrm{tr}(P_1' Q_1') = \mathrm{tr}(P_1 Q_1)$. By Remark 2 this is sufficient for L_1 to be a neutrally stable state. Consider instead the following language.

Example 2.

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$L_2 = (P_2, Q_2)$ definitely is a Nash strategy. Nevertheless, it does not fulfill the condition for a neutrally stable state. To see why this is so, consider $L_2' = (P_2', Q_2')$ with

$$P_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

as an invading language. L_2' is doing as well against L_2, as L_2 is doing against itself,

$$F(L_2, L_2') = 2 = F(L_2, L_2).$$

But L_2' against itself yields a strictly higher payoff than L_2 against L_2',

$$F(L_2', L_2') = 3 > 2 = F(L_2', L_2),$$

and so L_2 cannot be a neutrally stable state.

An obvious characteristic of L_2 as opposed to L_1 is that its sender *as well as* its receiver matrix contains a column that consists entirely of zeros. It can be shown that in general this is sufficient to destroy neutral stability.

Lemma 6. *Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a Nash strategy. If each of the two matrices, P and Q, contains at least one column that consists entirely of zeros, then L cannot be a neutrally stable state.*

The proof is given in the appendix. The intuition is straight forward. If there are events that are never possibly successfully communicated (a zero column in Q), then it cannot be evolutionarily stable—not even in the neutral sense—that there are any idle signals (a zero column in P). Analogously, in the presence of signals that are never used it cannot even be a neutrally evolutionarily stable state if there are messages that are never understood. An invading language that changes nothing about the existing linkages between events and signals but that links in addition the idle signal to the event that is never understood clearly is not doing worse against the resident language but can do better against itself.

Nevertheless, a zero column in both matrices, P and Q, is not the only thing that can happen to prevent a state from being neurally stable, nor is it the case that the condition that neither P nor Q contains a zero column is sufficient to guarantee neutral stability. The followig example indicates this.

Example 3.

$$P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\alpha & \alpha \\ 0 & 1-\alpha & \alpha \end{pmatrix}_{\alpha\in(0,1)} \quad \text{and} \quad Q_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\beta & \beta \\ 0 & 1-\beta & \beta \end{pmatrix}_{\beta\in(0,1)}.$$

$L_3 = (P_3, Q_3)$ is a Nash strategy. But, if everybody is playing L_3, and $L'_3 = (P'_3, Q'_3)$ with

$$P'_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } Q'_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is introduced into the population, L'_3 eventually will take over. $F(L_3, L'_3) = F(L_3, L_3)$, but $F(L'_3, L'_3) > F(L'_3, L_3)$, and so $L_3 = (P_3, Q_3)$ cannot even be neutrally stable. Note that this is true even though neither P_3 nor Q_3 contains a column that consists entirely of zeros. What destroys neutral stability in this case is the fact that in both matrices, P and Q, there are columns with multiple maximal elements that are positive but not equal to 1.

Some caution is in order: Contrary to what the above example suggests, it is *not* always true that in a symmetric Nash equilibrium where some events share the use of one particular signal, each of them also must share the use of all the other signals that one of them uses in parallel. The following example clarifies this remark.[5]

Example 4.

$$P_4 = \begin{pmatrix} 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \\ 0 & 0,5 & 0,5 \end{pmatrix} \text{ and } Q_4 = \begin{pmatrix} 0,5 & 0,5 & 0 \\ 0 & 0,5 & 0,5 \\ 0,5 & 0 & 0,5 \end{pmatrix}.$$

It is easily checked that $P_4 \in B(Q_4)$ and that Q_4 in $B(P_4)$, and so L_4 is a Nash strategy. To see that L_4 cannot be neutrally stable, it is sufficient to consider L'_3 from the previous example as the invading strategy.

Other instances of Nash strategies that are not weakly evolutionarily stable are cases where one of the two matrices—P or Q—contains a column with multiple maximal elements strictly between 0 and 1, and the other matrix a zero column.

Example 5.

$$P_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\beta & \beta \\ 0 & 0 & \beta \end{pmatrix}_{\beta\in(0,1)}.$$

[5] Trapa and Nowak (2000) wrongly conclude this from an example with a P matrix of dimensions 2×3. Example 4 is a specific 3×3 continuation of their example.

Taking again L_3' as the competing language, it can be checked that L_5 also fails to be a neutrally evolutionarily stable state.

The crucial thing in the case where one of the two matrices, P *or* Q, contains a column with multiple maximal elements that are strictly between 0 and 1, is that by the Nash property of $L = (P, Q)$ alone the other matrix is bound to contain a zero column *or* a column with multiple maximal elements that are strictly between 0 and 1 as well.

Lemma 7. *Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a Nash strategy. If P [Q] contains at least one column that has non-zero multiple maximal elements that are not equal to 1, then Q [P] contains*

(i) at least two columns that have non-zero multiple maximal elements that are not equal to 1, or

(ii) a zero column,

and L cannot be a neutrally stable state.

Here the intuition is that if the ambiguity created by instances of homonymy or synonymy works into both directions, then this leaves enough degrees of freedom for rearranging the existing linkages between events and signals in such a way that the ambiguity is resolved—and thereby increasing the total of correctly communicated messages—without loosing anything against the resident language. The proof, again, is given in the appendix.

Necessary and sufficient conditions

We finally show that a Nash language is an evolutionarily stable state if and only if P *or* Q satisfies the condition that if it has a column with multiple maximal elements, then they are equal to 1. Note that this implies that at least P *or* Q has no zero column.

Combining the Lemmas 6 and 7 we have that a Nash strategy *cannot* be a neutrally stable state if

(i) P and Q contain a zero column, or if

(ii) P or Q contains a column with multiple maximal elements that are strictly between 0 and 1.

The contrapositive of this statement yields a necessary condition for neutral stability.

Proposition 2. *Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a neutrally stable state. Provided that $L = (P, Q)$ is a Nash strategy, then*

(i) at least one of the two matrices P or Q has no zero column; and

(ii) neither P nor Q contains a column with multiple maximal elements that are strictly between 0 and 1.

Note that these conditions, of course, are also necessary conditions for an evolutionarily stable state. For a neutrally stable state they also prove to be sufficient.

Proposition 3. *Let* $L = (P, Q) \in \mathcal{L}^{n,m}$ *a Nash strategy. If* P *[*Q*] has no column with multiple maximal elements that are not equal to* 1*,*

(i) then Q *[*P*] has no column with non–zero multiple maximal elements that are not equal to* 1*; and*

(ii) $L = (P, Q)$ *is a neutrally stable state.*

The proof is given in the appendix. It basically relies on crosswise making use of the best–response properties that must hold true between P and Q, as well as between P and Q', and Q and P' of any invading language. Figure 1 offers a shortcut to its intuition.

Combining Proposition 2 and Proposition 3 finally yields a complete characterization of neutrally stable states for this game.

Theorem 1. *Suppose* $L = (P, Q) \in \mathcal{L}^{n,m}$ *is a Nash strategy.* $L = (P, Q)$ *is a neutrally stable state if and only if* P *or* Q *satisfies the following condition: If a column has multiple maximal elements, then they are equal to* 1*.*

Proof. Just note that if conditions (i) and (ii) of Proposition 2 are satisfied for P *or* Q, by Proposition 3 this already implies that condition (ii) of Proposition 2 is also satisfied for Q or P respectively. □

In general verifying neutral stability of a specific strategy involves taking into account all the other possible strategies of this game. What we gain by this statement is a criterion for neutral stability that allows us to tell whether a specific strategy is neutrally stable, or not, just by checking the properties of the single strategy. This provides a good tool for equilibrium selection.

Conclusions

An interpretation of this result is that in a neutrally stable state there can be some but not too much ambiguity: One signal can be linked to two or more events—honomymy; but if this is the case, then these events cannot make use of any other signal to get communicated. One event can be linked to two or more signals—synonymy; but if this is the case, then these signals cannot be used to communicate any other event. Furthermore, as long as there are events that are never possibly successfully communicated (a zero column in Q), there cannot be any idle signal (a zero column in P), and vice versa.

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7 Appendix

Lemma 6.

Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a Nash strategy. If each of the two matrices, P and Q, contains at least one column that consists entirely of zeros, then L cannot be a neutrally stable state.

Proof. Suppose that $L = (P, Q) \in \mathcal{L}^{n,m}$ is a Nash strategy, and assume that the $j^\star$–th column of P as well as the $i^\star$–th column of Q consist entirely of zero elements.

To show that $L = (P, Q)$ is not neutrally stable, we first have to show that there exists some $L' = (P', Q') \in \mathcal{L}^{n,m}$ with $L' \neq L$ such that $F(L, L') = F(L, L)$, and second that $F(L', L') > F(L', L)$. Following Remark 2, for the specific payoff function that we use, this means that we have to look for a $L' = (P', Q') \in \mathcal{L}^{n,m}$ such that $\mathrm{tr}(P'Q) = \mathrm{tr}(PQ) = \mathrm{tr}(PQ')$ and such that $\mathrm{tr}(P'Q') > \mathrm{tr}(PQ)$.

Now, take as a candidate P' the original P but with the entries in its $i^\star$–th *row* substituted by the vector

$$p'_{i^\star j} = \begin{cases} 1 & \text{for } j = j^\star \\ 0 & \text{otherwise} \end{cases} ;$$

and take as a candidate Q' the original Q but with the entries in its $j^\star$-th *row* substituted by the vector

$$q'_{j^\star i} = \begin{cases} 1 & \text{for } i = i^\star \\ 0 & \text{otherwise} \end{cases} .$$

We first check that, in deed, $\mathrm{tr}(P'Q) = \mathrm{tr}(PQ)$. Since the elements of the $i^\star$–th *column* of Q are all zero, the product of the $i^\star$–th *column* of Q with the $i^\star$–th *row* of any sender matrix will be zero. So, whatever the elements in the $i^\star$–th *row* of P might have been, we "loose" nothing by setting $p'_{i^\star j^\star}$ equal to 1. Since in constructing P' from P we did not change the elements of any other row, $\mathrm{tr}(P'Q) = \mathrm{tr}(PQ)$. The analogous argument also gives us that $\mathrm{tr}(PQ') = \mathrm{tr}(PQ)$.

What remains to be done, it to show that $\mathrm{tr}(P'Q') > \mathrm{tr}(PQ)$.

Note first that

$$p'_{ij} q'_{ji} = p_{ij} q_{ji} \quad \text{whenever} \quad i \neq i^\star \textit{ or } j \neq j^\star.$$

On the other hand,

$$p'_{i^\star j^\star} q'_{j^\star i^\star} = 1,$$

whereas

$$p_{i^\star j^\star} q_{j^\star i^\star} = 0.$$

So, summing over all i and j, we have that

$$\sum_i \sum_j p'_{ij} q'_{ji} = \sum_i \sum_j p_{ij} q_{ji} + 1,$$

and $L = (P, Q)$ cannot be a neutrally stable state. □

Lemma 7.

Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a Nash strategy. If P [Q] contains at least one column with multiple maximal elements strictly between 0 and 1, then Q [P] contains

(i) at least two columns with multiple maximal elements strictly between 0 and 1, or a zero column;

(ii) and L cannot be a neutrally stable state.

Proof. Suppose that $L = (P, Q) \in \mathcal{L}^{n,m}$ is a Nash strategy. As for Lemma 6, we have to show that there is some $L' = (P', Q') \in \mathcal{L}^{n,m}$ with $\mathrm{tr}(P'Q) = \mathrm{tr}(PQ) = \mathrm{tr}(PQ')$ and $\mathrm{tr}(P'Q') > \mathrm{tr}(PQ)$.

We give the proof for the case where the condition of the proposition applies to Q. Analogous conclusions hold true for the case where it applies to P.

Suppose that the $i^\star$–th column of Q contains more than one maximum element that is positive but not equal to 1. Since Q is a best response to P, by Lemma 5, for all $j \in A(q_{\cdot i^\star})$, $p_{i^\star j}$ is a maximal, but not the unique maximal element of its respective column in P: If $p_{i^\star j}$ with $j \in A(q_{\cdot i^\star})$ was not a maximal element of the j–th column of P, then $q_{ji^\star}$ could not be positive. If, on the other hand, $p_{i^\star j}$ with $j \in A(q_{\cdot i^\star})$ was the unique maximal element of the j–th column of P, then $q_{ji^\star}$ would have to be exactly equal to 1. Note that this does not exclude the possibility that for some $j \in A(q_{\cdot i^\star})$ the j–th column of P is a zero column.

On the other hand, P is a best response to Q. By Lemma 3, this implies that $\sum_{j \in A(q_{\cdot i^\star})} p_{i^\star j} = 1$, and that $p_{i^\star j} = 0$ whenever $j \notin A(q_{\cdot i^\star})$. This means that even though some of the $p_{i^\star j}$ with $j \in A(q_{\cdot i^\star})$ might be zero, not all of them can be zero. At least one of them has to be positive—and if it is really the only one, it has to be exactly equal to 1. Since, by assumption, $A(q_{\cdot i^\star})$ has at least two elements, this implies that P has at least (i) two columns with multiple maximal elements strictly between 0 and 1, or (ii) a zero column. This proves the first part of the proposition.

Suppose that for $j^{\star\star}$ with $j^{\star\star} \in A(q_{\cdot i^\star})$, $p_{i^\star j^{\star\star}} \neq 0$. Note that we do not rule out the possibility that $p_{i^\star j^{\star\star}}$ is equal to 1. Since Q is a best response to P, it must be true that $\sum_{i \in A(p_{\cdot j^{\star\star}})} q_{j^{\star\star} i} = 1$. Remember, we know from above that $i^\star \in A(p_{\cdot j^{\star\star}})$. But since $0 < q_{j^{\star\star} i^\star} < 1$ (and since the $j^{\star\star}$–th column of P is not a zero column) there must be some $i^{\star\star} \neq i^\star$ with $i^{\star\star} \in A(p_{\cdot j^{\star\star}})$ such that $q_{j^{\star\star} i^{\star\star}} \neq 0$. Of course, since P is a best response to Q, $q_{j^{\star\star} i^{\star\star}}$ is a maximal element of the $i^{\star\star}$–th column of Q. In the case where $\max_i(p_{ij^{\star\star}}) = 1$, $q_{j^{\star\star} i^{\star\star}}$ might well be the unique maximal element of this column.

For later use note that

$$\sum_{j \in A(q_{\cdot i^\star})} \sum_i p_{ij} q_{ji} = 1.$$

To see why this is so, consider the following argument: We know from above that $\sum_{j \in A(q_{\cdot i^\star})} p_{i^\star j} = 1$ and that all these elements in the $i^\star$–th row of P for which $j \in A(q_{\cdot i^\star})$ are maximal elements of their respective columns. So

$$\sum_{j \in A(q_{\cdot i^\star})} \max_i p_{ij} = 1,$$

and Lemma 4 tells us that any maximizing receiver matrix Q "extracts" from P exactly the sum of its column maxima, which gives the claim of the statement.

Now, we try to create an alternative Q' that is doing as well against P, as Q is doing against P; and an alternative P' that is doing as well against Q, as P is doing against Q.

Take as a candidate Q' the original Q but exchange the entries in its $j^{\star\star}$–th row by the vector

$$q'_{j^{\star\star} i} = \begin{cases} 1 & \text{for } i = i^{\star\star} \\ 0 & \text{otherwise} \end{cases},$$

and exchange its $j^\star$–th row by the vector

$$q'_{j^\star i} = \begin{cases} 1 & \text{for } i = i^\star \\ 0 & \text{otherwise} \end{cases},$$

where $j^\star$ is some $j \in A(q_{\cdot i^\star})$ with $j^\star \neq j^{\star\star}$. Note that, since $q_{j^\star i^\star} \neq 0$ and Q is a best response to P, by Lemma 5 this also implies that $p_{i^\star j^\star}$ is a maximal element of the $i^\star$–th column of P.

Since the original Q was a best response to P, and since in constructing Q' form Q we did not change any rows other than the $j^{\star\star}$–th and $j^\star$–th row of Q, all we have to do in order to show that $\mathrm{tr}(PQ') = \mathrm{tr}(PQ)$, is to show that the $j^{\star\star}$–th and the $j^\star$–th row of Q' successfully extract the maximum value of $j^{\star\star}$–th and the $j^\star$–th column of P, respectively. We know from above that $p_{i^{\star\star} j^{\star\star}}$ is a maximal element of the $j^{\star\star}$–th column of P, and that $p_{i^\star j^\star}$ is a maximal element of the $j^\star$–th column of P. So setting their corresponding elements in Q', $q_{j^{\star\star} i^{\star\star}}$ and $q_{j^\star i^\star}$, equal to 1, clearly is optimal in order to maximize $\mathrm{tr}(PQ)$, and so $\mathrm{tr}(PQ') = \mathrm{tr}(PQ)$.

As an alternative P' take the original P but exchange the entries in its $i^{\star\star}$–th row by the vector

$$p'_{i^{\star\star} j} = \begin{cases} 1 & \text{for } j = j^{\star\star} \\ 0 & \text{otherwise} \end{cases},$$

and the entries in its $i^\star$–th row by the vector

$$p'_{i^\star j} = \begin{cases} 1 & \text{for } j = j^\star \\ 0 & \text{otherwise} \end{cases}$$

We also know from above that $q_{j^\star i^\star}$ is a maximal element of the $i^\star$–th column of Q (by assumption definitely not its unique maximal element) and that $q_{j^{\star\star} i^{\star\star}}$ is a maximal element of the $i^{\star\star}$–th column of Q. As pointed out above, it even might be the unique maximal element of this column. In any case, setting $p_{i^\star j^\star}$ and $p_{i^{\star\star} j^{\star\star}}$ equal to 1 clearly is an optimal choice in order to maximize $\mathrm{tr}(PQ)$, and so $\mathrm{tr}(P'Q) = \mathrm{tr}(PQ)$.

What remains to be done, is to compare $\mathrm{tr}(P'Q')$ to $\mathrm{tr}(PQ)$. Since $p'_{i^\star j^\star} q'_{j^\star i^\star} = 1$ and $p'_{i^{\star\star} j^{\star\star}} q'_{j^{\star\star} i^{\star\star}} = 1$, we have that

$$\sum_{j \in A(q_{\cdot i^\star})} \sum_i p'_{ij} q'_{ji} \geq 2, \tag{4}$$

whereas, as we have noted above,

$$\sum_{j \in A(q_{\cdot i^\star})} \sum_i p_{ij} q_{ji} = 1.$$

We distinguish two cases now: Suppose first that for the $j^{\star\star} \in A(q_{\cdot i^\star})$ that we have chosen above, $p_{i^\star j^{\star\star}} = 1$. Then, of course, also $p_{i^{\star\star} j^{\star\star}} = 1$. But this implies that $p_{i^{\star\star} j} = 0$ for all $j \notin A(q_{\cdot i^\star})$, and so

$$\sum_{j \notin A(q_{\cdot i^\star})} \sum_i p'_{ij} q'_{ji} = \sum_{j \notin A(q_{\cdot i^\star})} \sum_i p_{ij} q_{ji}. \tag{5}$$

Summing over all j, we have that

$$\sum_j \sum_i p'_{ij} q'_{ji} \geq \sum_j \sum_i p_{ij} q_{ji} + 1,$$

and we are done.

The case where $0 < p_{i^\star j^{\star\star}} < 1$ is a little bit more complicated. Equation (4) still holds, but equation (5) is no longer necessarily true. It might well be that there are some $j \notin A(q_{\cdot i^\star})$ for which $p_{i^{\star\star} j} \neq 0$. Hence, in constructing P' form P, when we replace the $i^{\star\star}$ row in P by the vector $(p'_{i^{\star\star} j})$ that is 1 for $j = j^{\star\star}$ and 0 otherwise, it might well be the case that we nullify some positive entries $p_{i^{\star\star} j}$ for which $j \notin A(q_{\cdot i^\star})$ *that might be attributed some positive weight to by* Q! So when we multiply the elements $p'_{i^{\star\star} j}$ for $j \notin A(q_{\cdot i^\star})$ with their corresponding elements in Q'—which definitely are unchanged for $j \notin A(q_{\cdot i^\star})$—it might well be that we "loose" something as compared to the same expressions in $\mathrm{tr}(PQ)$. Nevertheless, since $p_{i^\star j^{\star\star}} \neq 0$, what we "loose" this way cannot be greater than 1, and so overall we still have that

$$\sum_j \sum_i p'_{ij} q'_{ji} > \sum_j \sum_i p_{ij} q_{ji},$$

which completes the proof that $L = (P, Q)$ cannot be a neutrally stable state. □

Proposition 3.

Let $L = (P, Q) \in \mathcal{L}^{n,m}$ a Nash strategy. If P [Q] has no column with multiple maximal elements that are not equal to 1,

(i) then Q [P] has no column with non–zero multiple maximal elements that are not equal to 1; and

(ii) $L = (P, Q)$ is a neutrally stable state.

Proof. Suppose that $L = (P, Q)$ is a Nash strategy. We give the proof for the case where the condition of the proposition is given for P. Because of the symmetric roles of P and Q, analogous conclusions hold true for the case where it is satisfied for Q.

If P has no column with multiple maximal elements that are not equal to 1, then for every fixed column of P there are only three possible cases. Its maximum is either

(1) unique and equal to 1, or

(2) unique but not equal to 1, or

(3) not unique but equal to 1.

Note that, in particular, there is no zero column in P.

In order to show that $L = (P, Q)$ is a neutrally stable state, using the symmetry of the payoff function, we have to show that if there is a $L' = (P', Q') \in \mathcal{L}^{n,m}$ such that $F(L, L) = F(L, L')$, then it should be the case that $F(L', L') \leq F(L, L)$. We know already from Remark 2 that for the specific payoff function we use, this amounts to the condition that if

$$\mathrm{tr}(PQ') = \mathrm{tr}(PQ) = \mathrm{tr}(P'Q),$$

for some $L' = (P', Q') \in \mathcal{L}^{n,m}$, then

$$\mathrm{tr}(P'Q') \leq \mathrm{tr}(PQ).$$

Starting with the assumptions on the columns of P, for each of these three cases separately, we will first try to exploit all the information we can get about the corresponding rows in Q and the other columns in P that derive from the fact that P and Q are maximizers for each other. Second, we will consider the consequences for the corresponding rows of *all the* $Q' \in A(P)$ and the corresponding columns of *all the* $P' \in A(Q)$. Multiplying columns with their corresponding rows, we will see that, for each of these three cases, these column–times–row products for P' and Q' are always smaller than or equal to their corresponding expressions for the original P and Q. Summing over all these products finally yields the result.

Case 1. (One event exclusively linked to one signal.) Suppose that $p_{i^\star j^\star} = 1$ is the unique maximal element in the $j^\star$–th column of P.

Since Q is a best response to P, by Lemma 3, we have that $q_{j^\star i^\star} = 1$, and that $q_{j^\star i} = 0$ whenever $i \neq i^\star$. Note that, since there are no entries greater than 1, this immediately implies that $q_{j^\star i^\star}$ is *a* maximal element of the $i^\star$–th column of Q.

Since the elements in each row of P add up at most to 1, we also have that $p_{i^\star j} = 0$ whenever $j \neq j^\star$. Since by assumption there are no columns in P that consist entirely of zeros, all these elements in the $i^\star$-th *row* of P that are equal to 0, cannot be maximal elements of their respective *columns* $j \neq j^\star$. Since Q is a best response to P, by Lemma 3 this implies that $q_{ji^\star} = 0$ whenever $j \neq j^\star$. So,

$$q_{ji^\star} = \begin{cases} 1 & \text{for } j = j^\star \\ 0 & \text{otherwise} \end{cases} . \tag{6}$$

This means that $q_{j^\star i^\star} = 1$ is not only *a* but *the unique* maximal element in the $i^\star$–th *column* of Q.

Now, we turn to $Q' \in A(P)$ and $P' \in A(Q)$. Since by assumption, $p_{i^\star j^\star}$ is the unique maximal element in the $j^\star$–th column of P, by Lemma 3, we have that

$$q'_{j^\star i^\star} = 1 = q_{j^\star i^\star} \text{ and } q'_{j^\star i} = 0 = q_{j^\star i} \; \forall\, i \neq i^\star, \tag{7}$$

for all $Q' \in A(P)$.

Since in this case, we also have that $q_{j^\star i^\star} = 1$ is the unique maximal element in the $i^\star$–th column of Q, Lemma 3 also tells us that

$$p'_{i^\star j^\star} = 1 = p_{i^\star j^\star}, \tag{8}$$

for all $P' \in A(Q)$.

Taking (7) and (8) together, we have that

$$\sum_i p'_{ij^\star} q'_{j^\star i} = 1 = \sum_i p_{ij^\star} q_{j^\star i}, \tag{9}$$

for all $L' = (P', Q') \in \mathcal{L}^{n,m}$ such that $P' \in A(Q)$ and $Q' \in A(P)$.

Figure 1.1 illustrates this case.

Case 2. (Synonymy.) Suppose that $0 < p_{i^\star j^\star} < 1$ is the unique maximal element in the $j^\star$–th column of P.

As in the previous case, from Q being a best response to P, we have that $q_{j^\star i^\star} = 1$, and that $q_{j^\star i} = 0$ for all $i \neq i^\star$. Since there are no elements greater than 1, this again implies that $q_{j^\star i^\star}$ is *a* maximal element of the $i^\star$–th *column* of Q. But now, since $p_{i^\star j^\star} \neq 1$, by Lemma 5, the fact that P is a best response to Q, implies that $q_{j^\star i^\star}$ cannot be the *unique* maximal element in the $i^\star$–th column of Q, and, of course, since the elements in each row may not add up to something greater than 1, we have that *for* *all* $j \in A(q_{\cdot i^\star})$, $q_{ji} = 0$ whenever $i \neq i^\star$. But since the maximum of the $i^\star$–th column of Q is not equal to zero, by Lemma 5 we also have that $\sum_{j \in A(q_{\cdot i^\star})} p_{i^\star j} = 1$, and that $p_{i^\star j} = 0$ for all $j \notin A(q_{\cdot i^\star})$. Together with the assumption that $0 < p_{i^\star j^\star} < 1$, this in turn implies that for all $j \in A(q_{\cdot i^\star})$, $p_{ji^\star} \neq 1$. On the other hand, since for all $j \in A(q_{\cdot i^\star})$, $q_{ji^\star} = 1 \neq 0$ and Q is a best response to P, by Lemma 5, we also know that for all $j \in A(q_{\cdot i^\star})$, $p_{ji^\star}$ is *a* maximal element of its respective column in P. Together with the assumption that P has no zero column and no column with multiple maximal elements strictly between 0 and 1, this implies that *for all* $j \in A(q_{\cdot i^\star})$, $0 < p_{i^\star j} < 1$ is *the unique* maximal element of its respective column in P.

In perfect analogy to the previous case, the fact that $p_{i^\star j} = 0$ for all $j \notin A(q_{\cdot i^\star})$ together with the assumption that P does not contain any zero column implies that $q_{ji^\star} = 0$ whenever $j \notin A(q_{\cdot i^\star})$. So,

$$q_{ji^\star} = \begin{cases} 1 & \text{for } j \in A(q_{\cdot i^\star}) \\ 0 & \text{otherwise} \end{cases} . \tag{10}$$

We now turn again to Q' and P'. Since *for all* $j \in A(q_{\cdot i^\star})$ it is true that $0 < p_{i^\star j} < 1$ is *the unique* maximal element of its respective column, by Lemma 3 we have that that *for all* $j \in A(q_{\cdot i^\star})$

$$q'_{ji^\star} = 1 = q_{ji^\star}, \text{ and } q'_{ji} = 0 = q_{ji} \; \forall \, i \neq i^\star, \tag{11}$$

for all $Q' \in A(P)$. On the other hand, the fact that the $i^\star$–th column of Q has multiple maximal elements implies that

$$\sum_{j \in A(q_{\cdot i^\star})} p'_{i^\star j} = 1 = \sum_{j \in A(q_{\cdot i^\star})} p_{i^\star j}, \tag{12}$$

for all $P' \in A(Q)$.

Putting (11) and (12) together we have that

$$\sum_{j \in A(q_{\cdot i^\star})} \sum_i p'_{ij} q'_{ji} = \sum_{j \in A(q_{\cdot i^\star})} \sum_i p_{ij} q_{ji} = 1, \tag{13}$$

for all $L' = (P', Q') \in \mathcal{L}^{n,m}$ such that $P' \in A(Q)$ and $Q' \in A(P)$.

This case is illustrated by Figure 1.2.

Case 3. (Homonymy.) Suppose that $p_{i^\star j^\star}$ is equal to 1, but not the unique maximal element in the $j^\star$-th column of P. So, $i^\star \in A(p_{\cdot j^\star})$, but there is at least one $i^{\star\star} \neq i^\star$ such that $i \in A(p_{\cdot j^\star})$.

In this case, from Q being a best response to P, by Lemma 3 we only have that $\sum_{i \in A(p_{\cdot j^\star})} q_{j^\star i} = 1$, and that $q_{j^\star i} = 0$ whenever $i \notin A(p_{\cdot j^\star})$. In particular, this does not imply that $q_{j^\star i} \neq 0$ for all $i \in A(p_{\cdot j^\star})$.

On the other hand, from the constraint that the elements in each row cannot add up to something greater than 1, we also have that *for all* $i \in A(p_{\cdot j^\star})$, $p_{ij} = 0$ whenever $j \neq j^\star$. Since by assumption P does not contain any column that consist entirely of zeros, any zero element in P can never be a maximal element of its respective column, and since Q is a best response to P, we have that *for all* $i \in A(p_{\cdot j^\star})$, $q_{ji} = 0$ for all $j \neq j^\star$. This implies that all the $q_{j^\star i}$ such $i \in A(p_{\cdot j^\star})$ are *maximal* elements of their respective *columns*. But since not all them are necessarily non–zero, this means that they are not necessarily the unique maximal element of this column, and there might be a zero column in Q. But, if $q_{j^\star i}$ with $i \in A(p_{\cdot j^\star})$ is equal to some positive value, then it definitely will be *the unique* maximal element of its column in P. So,

$$q_{ji} = \begin{cases} \max_j(q_{ji}) & \text{for } j = j^\star \\ 0 & \text{otherwise} \end{cases} . \tag{14}$$

We now turn to Q' and P'. By Lemma 3 we have that

$$\sum_{i \in A(p_{\cdot j^\star})} q'_{j^\star i} = 1 = \sum_{i \in A(p_{\cdot j^\star})} q_{j^\star i}, \text{ and } q'_{j^\star i} = 0 = q_{j^\star i} \, \forall \, i \notin A(p_{\cdot j^\star}), \tag{15}$$

for all $Q' \in A(P)$.

The case of $P' \in A(Q)$ is a little bit more complicated. As we have seen above, whenever $q_{j^\star i} \neq 0$ for some $i \in A(p_{\cdot j^\star})$, then it definitely will be the unique maximal element of its respective column in Q. By Lemma 3, its corresponding element in P' then *has to be* equal to 1. But if for some $i \in A(p_{\cdot j^\star})$, $q_{j^\star i} = 0$, then $p'_{ij^\star}$ *does not have to be* equal to 1—even though it can be equal to 1 or to some other positive value. So, if for some $i \in A(p_{ij^\star})$,

$$p'_{ij^\star} \neq 0 \Rightarrow p_{ij^\star} = 1. \tag{16}$$

Taking (15) and (16) together, we have that

$$\sum_i p'_{ij^\star} q'_{j^\star i} \leq \sum_i p_{ij^\star} q_{j^\star i} = 1, \tag{17}$$

for all $L' = (P', Q') \in \mathcal{L}^{n,m}$ such that $P' \in A(Q)$ and $Q' \in A(P)$.

Figure 1.3 illustrates this case.

So, whatever cases out of these three possible ones might be captured by any sender matrix P that is part of a Nash strategy $L = (P, Q)$, we have that

(i) the maximum of each column in Q is either unique or equal to 1; and

(ii) summing *over all* j and over all i, we see that

$$\sum_j \sum_i p'_{ij} q'_{ji} = \operatorname{tr}(P'Q') \leq \operatorname{tr}(PQ) = \sum_j \sum_i p_{ij} q_{ji},$$

which means that $L = (P, Q)$ is a neutrally stable state.

(i) comes from (6), (14) and (10); whereas (ii) is an implication of (9), (17) and (13) together. □

$$P = \begin{pmatrix} - & - & <1 \\ - & - & <1 \\ 0 & 0 & 1 \end{pmatrix} \quad \underset{P\in B(P)}{\overset{Q\in B(P)}{\longleftrightarrow}} \quad Q = \begin{pmatrix} - & - & 0 \\ - & - & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q'\in A(P),\ P'\in A(Q)$$

$$P' = \begin{pmatrix} - & - & - \\ - & - & - \\ 0 & 0 & 1 \end{pmatrix} \qquad Q' = \begin{pmatrix} - & - & 0 \\ - & - & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 1.1. One event exclusively linked to one signal.

$$P = \begin{pmatrix} - & <1-\alpha & <\alpha \\ - & <1-\alpha & <\alpha \\ 0 & 1-\alpha & \alpha \end{pmatrix} \quad \underset{P\in B(P)}{\overset{Q\in B(P)}{\longleftrightarrow}} \quad Q = \begin{pmatrix} - & - & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q'\in A(P),\ P'\in A(Q)$$

$$P' = \begin{pmatrix} - & - & - \\ - & - & - \\ 0 & 1-\alpha' & \alpha' \end{pmatrix} \qquad Q' = \begin{pmatrix} - & - & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure 1.2. Synonymy.

$$P = \begin{pmatrix} - & - & <1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \underset{P\in B(P)}{\overset{Q\in B(P)}{\longleftrightarrow}} \quad Q = \begin{pmatrix} - & 0 & 0 \\ - & 0 & 0 \\ 0 & 1-\beta & \beta \end{pmatrix}$$

$$Q'\in A(P),\ P'\in A(Q)$$

$$P' = \begin{pmatrix} - & - & - \\ 0? & 0? & 1? \\ 0? & 0? & 1? \end{pmatrix} \qquad Q' = \begin{pmatrix} - & 0 & 0 \\ - & 0 & 0 \\ 0 & 1-\beta' & \beta' \end{pmatrix}$$

Figure 1.3. Homonymy.

Figure 1. Proof of Proposition 3. We basically always apply the same procedure. First we exploit the fact that P and Q are reciprocally maximizers for each other. Second, we consider the properties of *all the* possible P' and Q' that are also best responses to Q and P, respectively. Finally, we show that taking together any P' with any Q' cannot yield something that is doing better than P together with Q.