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Trees and Extensive Forms

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Trees and Extensive Forms*

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Abstract. This paper addresses the question of what it takes to obtain a well-defined extensive form game. Without relying on simplifying finiteness or discreteness assumptions, we characterize the class of game trees for which (a) extensive forms can be defined and (b) all pure strategy combinations induce unique outcomes. The generality of the set-up covers "exotic" cases, like stochastic games or decision problems in continuous time (differential games). We find that the latter class fulfills the first, but not the second requirement.

1. Introduction

Non-cooperative game theory is the theory of games with complete rules. Why is this important? It is because the hallmark of non-cooperative game theory is a thought experiment that leaves all decisions exclusively to the players. Unlike cooperative game theory, where axioms like Pareto efficiency or symmetry restrict the solutions, or competitive equilibrium, where an auctioneer provides the agents with (equilibrium) price information, non-cooperative games provide an idealized 'world of its own,' where the effects of purely individual decisions can be studied without any external intervention.

When it comes to applications, games with complete rules are not any more realistic than the presumption of a perfect vacuum. But, as in physics the latter serves the study of the theory of gravity, the former serves to provide a benchmark of a purely individualistic "interactive decision theory" (Aumann (1987), p. 460).

How can it be verified that a game has complete rules? The formal device that accomplishes this is the *extensive form*, as introduced by von Neumann and Morgenstern (1944) and generalized by Kuhn (1953). The latter employs a *tree* to model the order of decisions by players—who can do what when—and how those generate an outcome. A tree is usually taken to be a directed connected graph without loops and with a distinguished node, the "root," where the game starts. To focus on conceptual issues, much of the theoretical literature on extensive forms tacitly assumes finite trees. (A notable early exception is Aumann (1964), who generalizes Kuhn's theorem to infinite extensive form games.)

Though a tree is an ingenious representation of sequential decision making, a graph is not a natural domain for received decision theory. In the classical theory of decisions under uncertainty a decision maker chooses between maps from states to consequences ("acts") once and for all. Thus, neither is the domain of preferences

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naturally a graph, nor is the traditional decision theory one about sequential decision making.

Expected utility theory, as introduced by von Neumann and Morgenstern (1944), offers, however, an attractive separability between probability assignments over states and a Bernoulli utility function on consequences, even without the "Archimedean" axiom (see Blume, Brandenburger, and Dekel (1991)). This separability lends itself quite naturally to a dynamic re-interpretation. Consecutive decision problems can be described by restricting probability assignments to the states that remain possible, without affecting the Bernoulli utility of consequences—a technique known as Bayes' rule. This has lead to the conundrum of "Bayesian decision theory."

The framework of expected utility—in particular, its independence axiom—is even more powerful, though. By replacing consequences with *sets* of consequences, genuinely dynamic decisions can be formalized. This approach has been used to characterize preference orderings allowing for a preference for flexibility (Kreps (1979)), temptation and self-control (Gül and Pesendorfer (2001) and Dekel, Lipman, and Rustichini (2004)), or unforeseen contingencies (Dekel, Lipman, and Rustichini (2001)).

In short, classical decision theory has proved capable of formalizing sequential decision making on a domain that considers maps from states, or sets of states, to sets of consequences. Contrary to how it may appear at first sight, this notion is perfectly compatible with the tree model of dynamic decision making. It has been shown elsewhere (Alós-Ferrer and Ritzberger (2005), henceforth AR) that every decision tree—in particular, Kuhn's (1953) graph—can be represented as a collection of sets (of plays, consequences, or outcomes) that is ordered by set inclusion, without loss of generality. The advantage of such a set representation is that nodes become sets of plays/states, thus "events" in the sense of decision theory.

Thus, trees do constitute an adequate representation of sequential decision making that is consistent also with the modern and more dynamic versions of decision theory. The concept of set-trees, moreover, is so general that it encompasses all examples that have been studied in the literature, including exotic cases like "differential games" (decision problems in continuous time), infinitely repeated games, stochastic games (Shapley (1953)), infinite bilateral bargaining (Rubinstein (1982)), and long cheap-talk games (Aumann and Hart (2003)), all of which have received considerable attention in the literature.[1]

The present paper takes advantage of this generality to address a more fundamental issue: What are the necessary structures (possibly hidden behind the veil of simplifying finiteness assumptions) for the formulation of extensive form games and the verification of the completeness of rules?

This paper employs a general definition of an extensive form that imposes no restrictions on neither the number of choices nor the length of plays. In particular, all the aforementioned "long" games are covered by this definition, including differential games, as are games with a continuum of players. This general framework allows us to turn to fundamental conceptual questions. Here we investigate one particular aspect: When do (pure) strategies generate outcomes, so that players can make up their minds as to which strategy to employ?

[1] An extended account of these classes of games is given in AR, Section 2.2.

It was shown in AR, for instance, that differential games (in continuous time) can be rigorously defined as extensive form games with the aid of the general concept of a set-tree. But are they such that players can indeed always decide on the basis of a well-defined outcome being associated with every strategy? This is an important question, because if a non-cooperative game is meant to leave all decisions to players, then players ought to be able to evaluate their strategies; otherwise, they may not be able to decide in the first place.

Non-cooperative solution concepts, like Nash equilibrium, are invariably defined in terms of strategy combinations.[2] For solutions to be meaningful, therefore, requires that they yield an object in the domain of the players' preferences—a play or an outcome. If there are strategy combinations that "evaporate" (do not induce any outcome at all) or yield multiple outcomes, they cannot be evaluated by the decision makers in the game. Hence, only extensive form games, where strategies do indeed always induce an outcome, can be "solved."

More precisely, we seek to characterize the class of extensive forms that satisfy the following three desiderata: (A1) For every outcome/play there is a strategy combination that selects this play. (A2) Every strategy combination induces some outcome/play. (A3) The outcome/play induced by a given strategy combination is unique.

All of these appear necessary for a game to be "solvable." That the rules of a game are complete does not necessarily imply that they are also consistent. Completeness of rules is a local criterion in the sense that at each "when" it is clear "who can do what." But consistency of the rules is a global criterion. And, as they are about strategies, the desiderata (A1)-(A3) indeed take such a global view. The differential game, for instance, has an extensive form representation and, thereby, complete rules. But, as shown in this paper, its rules are such that they allow for decisions that do not combine to a definite result. It is in this sense that the rules of the differential game are inconsistent.

Technically speaking, the three desiderata are also necessary and sufficient for an extensive form game to have a normal form representation. Indeed many prominent non-cooperative solution concepts are defined directly in the normal form. Since the extensive form represents the complete rules of the game, the translation into the normal form is a technical convenience that unifies the task of game theory.

> "The reduction of any specific game, except the simplest, to normal form is a task defying the patience of man; but, since the normal form of *all possible games* is comparatively simple, one may hope to carry out successfully a mathematical examination of all possible games in normal form." (Luce and Raiffa (1957), p. 53)

For this reduction to work, the three desiderata are necessary and sufficient. In this sense our results also characterize the class of extensive form games that possess a normal form and can, thus, be subjected to the examination alluded to in the above quotation.

[2]This is in contrast to cooperative solution concepts that are often defined in terms of utility allocations or "imputations."

The rest of the paper is organized as follows. First, in Section 2, the basic results on set-trees are briefly reviewed and three motivating examples are provided. Then, as an auxiliary step, nodes in a set-tree are classified in Section 3. The class of trees on which extensive decision problems can be defined is characterized in Section 4. After introducing strategies and formally defining the three desiderata (A1), (A2), and (A3) in Section 5, we study each of them in isolation. The first, (A1), is shown to be a consequence of the general definition of extensive decision problems employed here. As for the second, (A2), we characterize in Section 6 a slightly stronger criterion: that every strategy induces an outcome after every history. As histories initiate subgames, this forms the basis on which solution concepts can be built in the spirit of backwards induction. Section 7 provides a characterization of (A3). Finally, in Section 8 we characterize the class of trees on which extensive decision problems satisfy all three desiderata. Section 9 discusses the discrete trees commonly used in most parts of traditional game theory. Conclusions are in Section 10.

The proofs for all results, except for the main theorems and a few brief arguments, are relegated to an appendix.

2. Game Trees and Motivating Examples

The purpose of this section is to review key definitions and results from AR, and to provide motivating examples that can be referred to throughout the paper. The first part aims at introducing the concept of a game tree and demonstrating its generality.

2.1. Review on Game Trees. A (partially) *ordered set* (or a *poset*) is a pair $(N, \geq)$ consisting of a nonempty set N and a reflexive, transitive, and antisymmetric binary relation $\geq$ on N. In particular, a V*-poset* is an ordered set $(M, \supseteq)$ where M is a collection of nonempty subsets of a given set V and $\supseteq$ is set inclusion. A nonempty subset $h \in 2^N$ of a poset $(N, \geq)$ is a *chain* if for all $x, y \in h$ either $x \geq y$ or $y \geq x$ (or both), i.e., if the induced order on h is complete. An *order isomorphism* between two posets $(N_1, \geq_1)$ and $(N_2, \geq_2)$ is a bijection $\varphi : N_1 \to N_2$ such that

$$x \geq_1 y \text{ if and only if } \varphi(x) \geq_2 \varphi(y) \tag{1}$$

for all $x, y \in N_1$. This last property is referred to as "order embedding." It can be shown (AR, Proposition 1) that every poset $(N, \geq)$ *admits a set representation*, i.e., there is an order isomorphism between $(N, \geq)$ and some V-poset $(M, \supseteq)$.

Given a poset $(N, \geq)$ and an element $x \in N$ define the *up-set* (or *order filter*) $\uparrow x$ and the *down-set* (or *order ideal*) $\downarrow x$ by (see e.g. Davey and Priestley (2002))

$$\uparrow x = \{y \in N \,|\, y \geq x\} \text{ and } \downarrow x = \{y \in N \,|\, x \geq y\}. \tag{2}$$

A *tree* is an ordered set $(N, \geq)$ such that $\uparrow x$ is a chain for all $x \in N$. In a tree the elements of N are called *nodes*. For nodes $x, y \in N$ say that x is a *predecessor* (resp. successor) of y if $x \geq y$ (resp. $y \geq x$) and $x \neq y$. A tree is *rooted* if there is a node $x_o \in N$, called the root, such that $x_o \geq x$ for all $x \in N$.

Every tree $(N, \geq)$ has a set representation $(M, \supseteq)$ which satisfies that

$$\text{if } a \cap b \neq \emptyset \text{ then } a \subset b \text{ or } b \subseteq a \tag{3}$$

for all $a, b \in M$ (AR, Proposition 2(b)). This property is called "Trivial Intersection."

For a tree $(N, \geq)$ a *play* w is a maximal chain in N. Denote by W the set of all plays. For a node $x \in N$, let $W(x) = \{w \in W \,|x \in w\}$ be the set of all plays *passing through* x. If $(N, \geq)$ is a poset and h a chain in N, then there exists a maximal chain w in N such that $h \subseteq w$. This is the Hausdorff Maximality Principle, a version of the Axiom of Choice.[3] It guarantees that in a tree every chain is contained in a play.

A *decision tree* is a tree $(N, \geq)$ such that $W(x) = W(y)$ implies $x = y$, for all $x, y \in N$. This rules out trivial structures, where a node has only one successor. Every decision tree $(N, \geq)$ has a set representation $(M, \supseteq)$ which satisfies (3) and

$$\text{if } b \subset a, \text{ there is } c \in M \text{ such that } c \subseteq a \text{ and } b \cap c = \emptyset \tag{4}$$

for all $a, b \in M$ (AR, Proposition 3(b)). The latter property is called "Separability."

A particular "canonical" set representation is given by the collection of plays passing through nodes $(W(N), \supseteq)$, where $W(N) = \{W(x)\}_{x \in N}$. A tree $(N, \geq)$ has such a canonical set representation if and only if it is a decision tree (AR, Theorem 1).[4] This particular set representation always satisfies Trivial Intersection, (3), and Separability, (4).

Definition 1. *A V-**set tree** is a V-poset $(M, \supseteq)$ that satisfies Trivial Intersection, (3), and Separability, (4). It is **rooted** if $V \in M$.*

By the previous considerations, every decision tree is order isomorphic to a set tree. But, in general, the elements of the underlying set V of a V-set tree $(M, \supseteq)$ have no particular meaning. This changes once for all $v, v' \in V$

$$\text{if } v \neq v' \text{ then there are } a, b \in M \text{ such that } v \in a \setminus b \text{ and } v' \in b \setminus a \tag{5}$$

holds, in which case the V-poset is called *irreducible*. An irreducible V-poset that satisfies Trivial Intersection, (3), must satisfy Separability, (4) (AR, Lemma 9). By appropriately "cleaning" the underlying set V, any V-set tree can be turned irreducible (AR, Proposition 4 and Lemma 11). Thus, every decision tree can be represented by an irreducible set tree.

For irreducible V-set trees there is a one-to-one mapping between the elements of the underlying set V and the set of plays W (AR, Propositions 5 and 6). Thus, under irreducibility the elements $v \in V$ have meaning as representatives of plays.

But, not all plays of a V-set tree $(M, \supseteq)$ may be represented by elements $v \in V$. (That is, the mapping from V to W may not be onto.) This will only be the case if

$$\text{for all chains } h \in 2^M \text{ there is } v \in V \text{ such that } v \in a \text{ for all } a \in h \tag{6}$$

i.e., if the V-set tree is *bounded* (from below). Yet, once again, this poses no serious difficulty, because, by appropriately enlarging the underlying set, every irreducible

[3] This is also known as Kuratowski's Lemma; see e.g. Hewitt and Stromberg (1965) or Davey and Priestley (2002).

[4] In a related paper (Alós-Ferrer and Ritzberger (2003)) it is shown that an analogous characterization holds for trees that are not decision trees. This requires the use of certain non-maximal chains on top of plays.

V-set tree can be turned into a bounded irreducible set tree (AR, Proposition 8). Thus, every decision tree has a set representation that is a bounded irreducible set tree. Bounded irreducible set trees are called *game trees*. This can be alternatively stated as follows (AR, Lemma 14):

Definition 2. *A* ***game tree*** *is a V-poset $(M, \supseteq)$ that satisfies (5) and*

$$h \in 2^M \text{ is a chain if and only if } \exists\, v \in V : v \in a \text{ for all } a \in h \tag{7}$$

The "canonical" set representation by plays is such a game tree. Moreover, this particular set representation has the "fixed point" property of being its own set representation by plays. It turns out that this is equivalent to irreducibility and boundedness. A V-poset is a bounded irreducible V-set tree if and only if it is its own set representation by plays; and this is equivalent to the existence of a bijection between V and the set W of plays (AR, Theorem 3). Thus, every decision tree can be represented as a game tree and any such representation will essentially be the "canonical" representation by plays.

An additional advantage of a game (V-set) tree is that the singleton sets from the underlying set V can be added to the set of nodes—as "terminal nodes"—without changing any essential features of the tree (AR, Proposition 10).

Definition 3. *A* ***complete game tree*** *is a V-poset $(M, \supseteq)$ such that (7) holds and $\{v\} \in M$ for all $v \in V$.*

Condition (7) is equivalent to Trivial Intersection plus boundedness (AR, Lemma 14). Since irreducibility follows from the presence of the singletons (terminal nodes), complete game trees are precisely those set trees that are irreducible and where every play has a minimum (AR, Proposition 11).

In the sequel trees will be taken to be game trees. So, from now on W is considered indistinctly as a given underlying set (for a W-set tree) and as the set of all plays, and N is considered indistinctly as the set of nodes and as a collection of (nonempty) subsets of W. By Theorem 3 of AR the underlying set and the set of plays W can be identified, so that, for game trees, both *nodes* and *plays* can equivalently be taken as the primitives.

The elements of W will mostly be referred to as plays. It is the peculiar property of game trees that an element $w \in W$ can be viewed either as an outcome (element of some node) or as a play (maximal chain of nodes). Whenever a distinction is in order, we write w for the outcome and $\uparrow\{w\}$ for the play (chain of nodes), where

$$\uparrow\{w\} = \{x \in N \,|\, w \in x\}$$

is the play formed by all nodes containing w.

2.2. Leading Examples. Game trees (and extensive decision problems) as defined in AR encompass all classical examples of games, from finite games to repeated and stochastic games. They also allow for more exotic examples, like Aumann and Hart's (2003) "Long Cheap Talk" (where the length of the plays is $\omega + 1$, with ω being the first infinite ordinal). The following example, taken from AR (Section 2.2), shows that game trees also encompass decision problems in continuous time.

Example 1. ***(Differential Game)*** *Let W be the set of functions $f : \mathbb{R}_+ \to A$, where A is some fixed set of "actions," containing at least two elements, and let*

$$\begin{aligned} N &= \{x_t(g)\,|g \in W,\ t \in \mathbb{R}_+\}\,,\ \textit{where} \\ x_t(g) &= \{f \in W\,|\,f(\tau) = g(\tau),\, \forall \tau \in [0,t)\} \end{aligned}$$

for any $g \in W$ and $t \in \mathbb{R}_+$. Intuitively, at each point in time $t \in \mathbb{R}_+$ a decision $a_t \in A$ is taken. The "history" of all decisions taken in the past (up to, but exclusive of, time t) is a function $f : [0,t) \to A$, i.e. $f(\tau) = a_\tau$ for all $\tau \in [0,t)$. A node at "time" t is the set of all functions which coincide with f on $[0,t)$, all possibilities still open for their values thereafter.

It is shown in AR that $(N, \supseteq)$ (which is a W-poset) is a game tree. At each node $x_t(f)$, the decision that an agent has to take is merely her action at time t. Ultimately, a function $f \in W$ becomes a complete description of all decisions taken from the beginning to the end and, hence, the set W becomes naturally one-to-one with the set of plays.

The generality of game trees allows problems to surface that are associated with pushing the limits of extensive form games beyond the confines of finite games. These problems are often difficult to see in the differential game, but are easily identified in the following example.

Example 2. ***(Hole in the Middle)*** *Let $W = [0,1]$ and*

$$N = \left\{(x_t)_{t=1}^{\infty}, (y_t)_{t=1}^{\infty}, \left(y_t'\right)_{t=1}^{\infty}, (\{w\})_{w \in W}\right\},$$

where $x_t = [(t-1)\,/\,(4t)\,, (3t+1)\,/\,(4t)]$, $y_t = [1/4, (2t-1)\,/\,(4t)]$, and, finally, $y_t' = [(2t+1)\,/\,(4t)\,, 3/4]$ for all $t = 1, 2, ...$ (so, $x_1 = W$). It is not difficult to verify that this is a complete game tree.

Make this into a single-player extensive form game (without chance) by assigning all nodes except the root as choices to the personal player. Consider a strategy that assigns to each x_t the choice x_{t+1}, to each y_t the (singleton) choice $\{(2t-1)\,/\,(4t)\}$, and to each y_t' the (singleton) choice $\{(2t+1)\,/\,(4t)\}$, i.e., a strategy that "continues" at all x_t, but "stops" at all y_t and y_t', for all $t = 1, 2, ...$ This strategy will induce the outcome $1/2 \in W$. But now remove the element $1/2$ from the underlying set W and the singleton $\{1/2\}$ from N. It is intuitively clear that then this strategy will not induce an outcome at all.

There is also a strategy s that always "continues", i.e., it assigns to any move the non-singleton choice: $s(x_t) = x_{t+1}$, $s(y_{t+1}) = y_t$, and $s\left(y_{t+1}'\right) = y_t'$ for all t. Intuitively, this strategy selects the outcomes/plays $1/4, 1/2, 3/4 \in W$. The reason for this multiplicity is that there is no move, where a decision between $\cup_{t=1}^{\infty} y_t = [1/4, 1/2)$ and $\cup_{t=1}^{\infty} y_t' = (1/2, 3/4]$ is taken, because a move of the form $[1/4, 3/4]$ is missing from the tree.

Many problems that arise in the abstract can be illustrated with instances from a family of examples that encompasses the, in many respects, simplest trees of all.

Example 3. *(**Centipedes**) Let W be any completely ordered set (possibly infinite), i.e. there is an order relation $\geq$ defined on W such that, for any two elements $w, w' \in W$, either $w \geq w'$ or $w' \geq w$. Define $x_t = \{\tau \in W \,|\, \tau \geq t\}$ for all $t \in W$, and let $N = \left\{(\{t\})_{t \in W}, (x_t)_{t \in W}\right\}$.*

*Since $x_r \subseteq x_t \Leftrightarrow r \geq t$ and $\{r\} \subseteq x_t \Leftrightarrow t \leq r$, for all $t, r = 1, 2, ...$, Trivial Intersection, (3), holds. Moreover, if $r \neq t$, say, $t < r$, then $t \in \{t\}$, $r \in x_r$, and $\{t\} \cap x_r = \emptyset$, so Irreducibility, (5), also holds. Therefore, $(N, \supseteq)$ is an irreducible W-set tree. The set of plays for $(N, \supseteq)$ consists of sets of the form $v_t = \left\{\{x_\tau\}_{\tau \leq t}, \{t\}\right\}$ for all $t \in W$, plus the play $v_\infty = \{x_t\}_{t \in W}$. Since the last play might not have a lower bound (e.g. if W is taken to be the set of natural numbers), the tree might not be bounded. This is, however, a case in which the tree can be completed by the addition of an "infinite" element which does not affect the order-theoretic structure (see Proposition 8 of AR). Formally, if W has a maximum, $w_\infty \geq w$ for all $w \in W$, it follows that $t \in x$ for all $x \in v_t$, and $w_\infty \in x$ for all $x \in v_\infty$. Hence, $(N, \supseteq)$ is bounded, i.e. it is a game tree. Since all singleton sets are nodes, it is even a complete game tree. We refer to this tree as the W-*centipede.[5]

This family contains a large variety of qualitatively different game trees. If W is taken to be the set of natural numbers plus an infinity point, $W = \{1, 2, 3, ..., \infty\}$, an "infinite centipede" (see Figure 1) is obtained. If W is taken to be the real interval $[0, 1]$, a "continuous centipede" emerges. Other examples will be introduced below.

3. A Classification of Nodes in a Tree

This section introduces two classifications of the nodes in a game tree. The first is a traditional convention. The second will prove particularly useful in understanding how strategies relate to outcomes.

3.1. Moves and Terminal Nodes. Nodes of a tree $(N, \geq)$ that are properly followed by other nodes are called *moves*. That is,

$$X = \{x \in N \,|\, \downarrow x \setminus \{x\} \neq \emptyset\} \tag{8}$$

is the set of all moves. Nodes that are *not* properly followed by other nodes are called *terminal*, and $E = \{x \in N \,|\, \downarrow x = \{x\}\}$ denotes the set of terminal nodes.

Lemma 1. *For an irreducible W-set tree $(N, \supseteq)$, a node $x \in N$ is terminal if and only if there is $w \in W$ such that $x = \{w\}$.*

This implies that, whenever $(N, \supseteq)$ is an irreducible set tree, $X = N \setminus \{\{w\}\}_{w \in W}$ can be taken as an alternative definition of (the set of) moves. Yet, this result does not imply that $\{w\} \in N$ for all $w \in W$, unless $(N, \supseteq)$ is a complete game tree.

[5] Incidentally, this family of examples shows that there exist (complete) game trees with plays of arbitrary cardinality. It suffices to consider a set W' of the appropriate cardinality, endow it with a well order (hence a total order) by applying Zermelo's well-order theorem (see e.g. Hewitt and Stromberg (1965)), and adjoin a "top" (maximum) to it. If the resulting set is called W, the corresponding W-centipede proves the claim.

3.2. Finite, infinite, and strange nodes. If all chains in a W-poset $(N, \supseteq)$ are finite, then $(N, \supseteq)$ is a game tree *if and only if* it satisfies Trivial Intersection, (3), and all singleton sets are nodes (Proposition 12 of AR). In particular, this result holds always if W is a finite set. Yet, both W and the plays in N may be infinite and still certain nodes may have properties analogous to the finite case.

Definition 4. *Let $(N, \supseteq)$ be a W-poset and $x \in N \setminus \{W\}$. Say that x is* ***finite*** *if $\uparrow x \setminus \{x\}$ has a minimum,* ***infinite*** *if $x = \inf \uparrow x \setminus \{x\}$, and* ***strange*** *if $\uparrow x \setminus \{x\}$ has no infimum. Denote the sets of finite, infinite, and strange nodes of N by $F(N)$, $I(N)$, and $S(N)$, respectively.*

The three possibilities in this definition are exhaustive, i.e., all nodes (other than the root) are either finite, infinite, or strange. For, if $\uparrow x \setminus \{x\}$ has an infimum z, it is either a minimum (and then x is finite), or $z \notin \uparrow x \setminus \{x\}$. In the latter case, it follows by definition of an infimum that $z = x$.

The following characterization borrows from the standard notion of *finite elements* of an arbitrary complete partially ordered set (CPO).

Lemma 2. *For an irreducible W-set tree $(N, \supseteq)$ a node $x \in N \setminus \{W\}$ is*

(a) the infimum of a chain $h \in 2^N$ if and only if $x = \cap_{y \in h} y$;

(b) infinite if and only if $x = \cap_{y \in \uparrow x \setminus \{x\}} y$;

(c) strange if and only if $\uparrow x \setminus \{x\}$ has no minimum and $x \subset \cap_{y \in \uparrow x \setminus \{x\}} y$.

(d) If, moreover, $(N, \supseteq)$ is a game tree, then x is not infinite (that is, either finite or strange) if and only if, for every chain $h \in 2^N$,

$$\text{if } x \supseteq \cap_{y \in h} y \text{ then } x \supseteq y \text{ for some } y \in h.$$

Remark 1. *A (meet-)CPO is a poset with a top element (a maximum) such that all "directed" sets have an infimum. It is easy to show (see Alós-Ferrer and Ritzberger (2003)) that in a tree the only directed sets are the chains. In a CPO finite elements are defined as those elements such that if $x \supseteq \inf h$, then $x \supseteq y$ for some $y \in h$, whenever h is directed. By Lemma 2(a), though, for irreducible trees the infimum is actually the intersection, giving rise to the property in Lemma 2(d). In game trees the finite elements of CPOs translate into non-infinite nodes.*

In a sense, infinite nodes are those which can be generated from the other nodes in the tree (Lemma 2(b)). This implies that an infinite node $x \in N$ has infinitely many predecessors, justifying its name. For, if $\uparrow x \setminus \{x\}$ were finite, it would have a minimum $z \in \uparrow x \setminus \{x\}$, implying that x was a finite node.

By Lemma 2(b), for an irreducible W-set tree $(N, \supseteq)$ (thus, for any game tree) the set of infinite nodes can be alternatively defined by

$$I(N) = \left\{ x \in N \,\middle|\, x = \cap_{y \in \uparrow x \setminus \{x\}} y \right\}. \tag{9}$$

The following examples illustrate the classification of nodes.

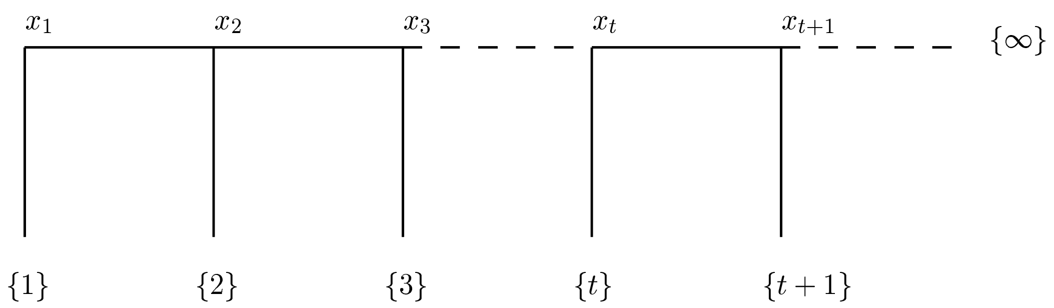

Figure 1: The infinite centipede. $W = \{1, 2, \ldots, \infty\}$, $x_t = \{t, t+1, \ldots, \infty\}$

Example 4. *(**Infinite Centipede**) Let $W = \{1, 2, ..., \infty\}$ be the set of natural numbers together with "infinity" ∞, and consider the corresponding W-centipede (see Example 3). Terminal nodes are singletons $\{t\}$ (including $\{\infty\}$). Moves are nodes of the form $x_t = \{t, t+1, ..., \infty\}$ (except for $x_\infty = \{\infty\}$ which is terminal). This example is illustrated in Figure 1. It constitutes an intuitive case, where all nodes, except the terminal node $\{\infty\}$, are finite. In particular, all moves are finite nodes. Only $\{\infty\}$ is infinite, because $\cap \{x_t \,|1 \le t < \infty\} = \{\infty\}$, yet $\{\infty\} \notin \{x_t \in N \,|t = 1, 2, ...\} = \uparrow\{\infty\} \setminus \{\{\infty\}\}$. This node represents a never-ending, infinite chain of decisions. Notice that, if $\{\infty\}$ were removed from the tree, there would still be an infinite play $\{x_t\}_{t=1}^{\infty}$ leading to the outcome $\infty \in W$. The resulting tree would be a game tree (not complete), that could be completed by adding $\{\infty\}$ back (see AR, Proposition 10). That is, whether the singleton $\{\infty\}$ is a node or not is purely a modelling decision that does not affect the structure of the tree. However, if a node $\{t\}$ with $t < \infty$ were suppressed, the chain $\{x_\tau\}_{\tau=1}^{t}$ would not be a play (since it is contained in the larger chain $\{x_\tau\}_{\tau=1}^{t'}$ for any $t' > t$), and hence the set of plays would change, affecting the structure of the tree. In fact, in such a case the resulting tree would cease to be a decision tree.*

Unfortunately, not all examples of game trees are so well behaved as the previous one. In particular, it may be the case that *moves* are infinite nodes. Intuitively, an infinite move represents a decision point that is never actually reached, because it follows an infinite sequence of other decisions. If, e.g., decisions are taken at each point in continuous time, there is neither a well defined "previous" decision to a given one, nor a "next" decision.

Example 5. *(**Continuous Centipede**) Let $W = [0, 1]$ be the unit interval. The corresponding W-centipede (see Example 3) is again a game tree, since $[0, 1]$ has a maximum. Nodes are either singletons $\{t\}$ or subsets of the form $x_t = [t, 1]$. All singletons $\{t\}$ are finite except the "last" node, $\{1\}$. To see this, note that, for any $t \le 1$, $\cap \{y \,|y \in\uparrow\{t\} \setminus \{\{t\}\}\} = \cap \{x_\tau \,|\tau < t\} = \cap_{\tau<t} [\tau, 1] = [t, 1] = x_t$. The latter is only identical to $\{t\}$ if $t = 1$. All moves, though, are infinite, except for the root $x_0 = [0, 1]$. For, $\cap \{y \,|y \in\uparrow x_t \setminus \{x_t\}\} = \cap \{x_\tau \,|\tau < t\} = [t, 1] = x_t$, provided $t > 0$ (otherwise the intersection is empty).*

This example is still intuitively clear. Moves represent decision points in continuous time, hence one cannot say at which point they are "chosen." Terminal nodes $\{t\}$ with $t < 1$, though, represent decisions that have been taken exactly at the decision point x_t. The terminal node $\{1\}$, on the other hand, represents the outcome of a never-ending chain of decisions and is, hence, analogous to the infinity node $\{\infty\}$ in the previous example.

The two previous examples do not contain strange nodes. They, therefore, satisfy the following definition.

Definition 5. *A game tree $(N, \supseteq)$ is* **regular** *if there are no strange nodes, i.e. if $S(N) = \emptyset$.*

Indeed, all centipedes are regular game trees. Examples with strange nodes, on the other hand, are the "Twins" Example 13 from AR and the "hole in the middle" Example 2 from Section 2.2.

Example 6. *Reconsider the "hole in the middle" example. In this tree the terminal node $\{1/2\} \in E$ is strange. The node $\{1/2\}$ is a lower bound for the chain $(x_t)_{t=1}^{\infty}$, as is any of the nodes y_t and y'_t. However, none of these nodes contains the element 1/2. It follows that $\uparrow \{1/2\} \setminus \{\{1/2\}\} = (x_t)_{t=1}^{\infty}$ has no infimum and hence $\{1/2\}$ is a strange node. Likewise, if the nodes $y = [1/4, 1/2)$ and $y' = (1/2, 3/4]$ were added to the tree, these new nodes would also be strange.*[6] *However, if a node $x = [1/4, 3/4]$ were added, then none of the previous ones would be strange, but $[1/4, 3/4]$ would be an infinite move. This later addition, though, qualitatively changes the tree by adding a further decision.*

3.3. Removing Singletons. This subsection presents an application of the classification of nodes to the following issue. To a game (W-set) tree the singletons $\{w\}$ with $w \in W$ can be added, as terminal nodes, without affecting the structure of the tree (see Section 2.1). This is, for instance, the case in the infinite centipede (Example 4) above, where, if the node $\{\infty\}$ were absent, it could be added without changing the set of plays. But can singletons also be *removed*?

It is clear that terminal nodes that are finite or strange cannot be removed without violating Irreducibility, (5). For, a node $x \in F(N) \cup S(N)$ satisfies $x \subset \bigcap_{y \in \uparrow x \setminus \{x\}} y$ (by definition if it is finite or by Lemma 2(c) if it is strange), so that there are $w \in x$ and $w' \in \left(\bigcap_{y \in \uparrow x \setminus \{x\}} y\right) \setminus x$. Therefore, if $x \in (S(N) \cup F(N)) \cap E$ were removed, then in the resulting tree $w \in y$ would imply $w' \in y$ (because $y \in \uparrow x \setminus \{x\}$ in the original tree) for all $y \in N \setminus \{x\}$ in contradiction to (5).[7] Hence, if at all, only infinite terminal nodes may be removed without harm.

Indeed, suppose that a singleton $\{w\} \subseteq W$, that is not already a node (i.e. $\{w\} \notin N$), can be added without affecting the set of plays (i.e. such that the set of plays of the resulting tree is bijective to the set of plays of the original tree; see Proposition 10

[6] If a node x is strange, then any maximal lower bound of $\uparrow x \setminus \{x\}$ is also strange.

[7] In order to recover Irreducibility, one would need to remove also the element w from the underlying set W of plays. But this, of course, changes the structure of the tree by changing the set of plays.

of AR). If $\{w\}$ were finite or strange in $(N \cup \{\{w\}\}, \supseteq)$, then $\{w\} \subset \cap_{x \in \uparrow \{w\} \setminus \{\{w\}\}} x$ by definition or by Lemma 2(c). Thus, there would be some element $w' \neq w$ in W such that $w' \in x$ for all $x \in \uparrow \{w\} \setminus \{\{w\}\}$. Since $(N, \supseteq)$ is a game tree, $\uparrow \{w\}$ and $\uparrow \{w'\}$ would be two distinct plays for $(N, \supseteq)$ (by Theorem 3(c) of AR). But it has just found that $\uparrow \{w\} \subseteq \uparrow \{w'\}$ - a contradiction. It follows that $\{w\}$ must be an infinite node in $(N \cup \{\{w\}\}, \supseteq)$. In other words, any singleton $\{w\} \notin N$ that can properly be added to a game tree without affecting its (set of) plays must be infinite in the resulting tree.

The following result states the converse that infinite terminal nodes can be *removed* from a (complete) game tree without affecting its structure. The resulting tree is order isomorphic to the original tree and has essentially the same plays. Furthermore, if the resulting set tree is completed by adding the now-missing singletons, the original game tree reemerges. For game trees infinite terminal nodes are *precisely* those, the presence or absence of which is immaterial to the structure of the tree.

Proposition 1. *If $(N, \supseteq)$ is a complete game tree with set of plays W, then for every $Y \subseteq E \cap I(N) \equiv E_I$ the poset $(N \setminus Y, \supseteq)$ is a game tree with set of plays W', the mapping $\Upsilon : W \to W'$ given by $\Upsilon(w) = w \setminus Y$ is bijective, and if $(N \setminus Y, \supseteq)$ is completed by adding all singletons (as in Proposition 10 of AR), the resulting poset is the complete game tree $(N, \supseteq)$.*

This is of some significance, because most textbook treatments of game theory specify the players' preferences over terminal nodes, rather than plays. This is justified if the underlying tree is a game tree, because for such trees infinite terminal nodes can be added (e.g., for repeated games) so as to provide a domain for preferences that is set-isomorphic to the set of plays. That is, in the class of game trees assuming completeness and specifying preferences on terminal nodes (rather than plays) is without loss of generality.

4. Extensive Decision Problems

In this section the general concept of an "extensive decision problem" is introduced, and the class of trees is characterized on which such problems can be defined.

4.1. Definition. To develop the notion of an extensive decision problem requires a few definitions. For a game tree $(N, \supseteq)$ with set W of plays and an arbitrary subset $a \subseteq W$ of plays (not necessarily a node), the *down-set* of a is given by $\downarrow a = \{x \in N \,|\, x \subseteq a\}$ and the *up-set* of a is given by $\uparrow a = \{x \in N \,|\, a \subseteq x\}$. Moreover, the set of *immediate predecessors* of $a \in 2^W$ is given by

$$P(a) = \{x \in N \,|\, \exists y \in \downarrow a : \uparrow x = \uparrow y \setminus \downarrow a\} \tag{10}$$

Since nodes in a game tree are sets of plays, they too may, but need not, have immediate predecessors. The following Lemma shows that the nodes having immediate predecessors are precisely the finite nodes.

Lemma 3. *Let $(N, \supseteq)$ be a game tree and $x \in N$ a node. Then, $P(x) \neq \emptyset$ if and only if x is finite. In this case, $P(x) = \{\min \uparrow x \setminus \{x\}\}$.*

The definition of an extensive decision problem from AR (Definition 7) follows.

Definition 6. *An* ***extensive decision problem*** *(EDP) with player set I is a pair (T, C), where $T = (N, \supseteq)$ is a game tree with set of plays W and $C = (C_i)_{i \in I}$ is a system consisting of collections C_i (the sets of players' choices) of nonempty unions of nodes (hence, sets of plays) for all $i \in I$, such that*

(EDP.i) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$, then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ for all $i \in I$;

(EDP.ii) $x \cap \left[\cap_{i \in J(x)} c_i\right] \neq \emptyset$ for all $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ and for all $x \in X$;

(EDP.iii) if $y, y' \in N$ with $y \cap y' = \emptyset$ then there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$;

(EDP.iv) if $x \supset y \in N$, then there is $c \in A_i(x)$ such that $y \subseteq c$ for all $i \in J(x)$, for all $x \in X$;

where $A_i(x) = \{c \in C_i \,|\, x \in P(c)\}$ are the choices available to $i \in I$ at $x \in X$, and $J(x) = \{i \in I \,|\, A_i(x) \neq \emptyset\}$ is the set of decision makers at x, which is required to be nonempty, for all $x \in X$.

Briefly, the interpretation of the conditions in Definition 6 is as follows (see AR, Section 5, for additional details). (EDP.i) stand in for information sets. In words, whenever two choices are simultaneously available at a common move, then they are disjoint and their immediate predecessors coincide; that is, whenever one of them is available, so is the other. (EDP.ii) requires that simultaneous decisions by different players at a common move do select some outcome. (EDP.iii) states that for any two disjoint nodes, there must be a player (possibly chance) who can eventually take a decision that selects among them. Finally, (EDP.iv) states that, if a player takes a decision at a given node, he must be able not to discard any given successor of the node. This requirement excludes absent-mindedness (see AR, Proposition 13).

The player set I for an extensive decision problem (T, C) may contain a distinguished player, called "chance" $i = 0$, whose behavior models what is not under the control of personal players. Property (EDP.i) is not necessary for such a player.

4.2. Perfect Information Choices. Whenever it is possible to define an EDP on a tree, it should be possible to define a single-player perfect information game on the same tree. The intuition is that the latter is specified by making a single player take all possible decisions in the tree under the best possible information. Also intuitively, the single-player perfect information game should be determined by the tree *alone*, in the sense that there should exist a unique form of defining this game for a given tree.

If the tree were discrete, choices available to the single player at a move $x \in X$ under perfect information would simply be the immediate successors of x. In the abstract framework used here a more general construction is required. For any move $x \in X$ and a play $w \in x$ define the *perfect information choice* $\gamma(x, w) \subseteq W$ as the set of plays

$$\gamma(x, w) = \cup \{z \,|\, w \in z \in \downarrow x \setminus \{x\}\}, \tag{11}$$

i.e. the union of all proper successors of x that contain w. Note that by Trivial Intersection, (3), these successors form a chain.

Example 7. *A typical example, where this construction goes beyond immediate successors is the differential game (Example 1). Recall that in this example moves are of the form* $x_t(f) = \{h \in W \,|h(\tau) = f(\tau), \forall \tau \in [0,t)\}$ *for* $f \in W$ *and* $t \in \mathbb{R}_+$*. Given* $g \in x_t(f) \in X$*, notice that* $x_t(g) = x_t(f)$ *and let* $g(t) = a \in A$*. Then,*

$$\begin{aligned}\gamma\left(x_t(f), g\right) &= \cup\left\{z \in N \,|g \in z \in\downarrow x_t(f) \setminus \{x_t(f)\}\right\} = \\ &= \cup\left\{x_{t'}(g) \left|t' > t\right.\right\} = \left\{h \in W \,|h(\tau) = g(\tau), \forall \tau \in [0,t]\right\} = \\ &= \left\{h \in x_t(g) \,|h(t) = a\right\},\end{aligned}$$

i.e., perfect information choices at $x_t\,(f) \in X$ *are* $c_t(f,a) = \{g \in x_t\,(f)\,|g(t) = a\} = \gamma\left(x_t\,(f)\,, g\right) \subset x_t\,(f)$ *for any* $g \in x_t\,(f)$ *with* $g\,(t) = a \in A$*. Alternatively, choices will also be denoted by* $c_t(g) = c_t(g, g(t)) = \{h \in x_t\,(g)\,|h(t) = g(t)\}$*. These correspond to the choices described in AR, Section 5.1, to illustrate that an EDP can be defined on the differential game tree.*[8]

The properties of perfect information choices as sets of plays determine when they can indeed be considered choices in a well-defined EDP. Intuitively, given a move x and a play $w \in x$, the corresponding perfect information choice models the decision to go along w. Thus, one expects that $\gamma\,(x, w) \subset x$, that is, something is indeed selected by this choice. As the next example illustrates, though, this is not always the case.

Example 8. ***(Inverse Infinite Centipedes)*** *Let* $W = \{1, 2, ...\}$ *be the set of natural numbers and consider the dual order* $\geq^D$*given by* $t \geq^D t'$ *if and only if* $t \leq t'$*. Consider the corresponding centipede (which is a game tree by Example 3, since* 1 *is a maximum for* $\geq^D$*), where nodes are given by* $N = \{W, (\{t\})_{t=2}^\infty\,, (x_t)_{t=1}^\infty\}$ *with* $x_t = \{\tau \in W\,|\tau \leq t\}$ *for all* $t = 1, 2, ...$

The set of plays for $(N, \supseteq)$ *consists of sets of the form* $\{W, (x_\tau)_{\tau=t}^\infty\,, \{t\}\}$ *for all* $t = 1, 2, ...$ *(since* $x_1 = \{1\}$*, the infinite play* $\{W, (x_\tau)_{\tau=1}^\infty\}$ *is included). Every play* $\{W, (x_\tau)_{\tau=t}^\infty\,, \{t\}\}$ *can be represented by the natural number* t*, for all* $t = 1, 2, ...$ *The infinite play is represented by the natural number* 1*. None of the chains* $\{x_\tau\}_{\tau=t}^\infty$ *has a maximum, however. As a consequence, the perfect information choice* $\gamma\,(W, 1)$ *coincides with the root itself, i.e.* $\gamma\,(W, 1) = \cup_{t=1}^\infty x_t = W$*.*

If it were possible to define an EDP on such a tree, then $J\,(W) \neq \emptyset$ *would imply* $W \in P\,(c)$ *for some* $c \in C_i$ *and some* $i \in I$*. Thus, there would have to exist some* $y \in\downarrow c$ *such that* $\{W\} =\uparrow W =\uparrow y \setminus \downarrow c$*. Since* $W \notin\downarrow c$*, there would be some* $w \in W \setminus c$*. If* t^* *were the smallest integer such that* $w \leq t^*$ *and* $y \subseteq x_{t^*}$*, then* $\{w\} \cup y \subseteq x_{t^*}$ *would hold by construction. As* t^* *would be finite, also* $x_{t^*} \in\uparrow y \setminus \{W\}$ *would hold and imply* $x_{t^*} \subseteq c$*, because* $\uparrow y \setminus \downarrow c = \{W\}$ *would hold by hypothesis. But that would mean* $\{w\} \cup y \subseteq c$ *in contradiction to* $w \notin c$*. Hence, it is impossible to define an EDP on such a tree.*

This shows that with the game tree from this example no EDP can be defined for which some choice is available at the root. This problem generalizes beyond the specific example.

[8] This is a different specification of the choice structure than the one proposed by Simon and Stinchcombe (1989), who use a discrete grid on the time axis.

Proposition 2. *Let* $(N, \supseteq)$ *be a game tree, let* $x \in X$*, and* $w \in x$*. Then:*

(a) if $w' \in x$ *and* $\gamma(x, w) \cap \gamma(x, w') \neq \emptyset$*, then* $\gamma(x, w) = \gamma(x, w')$*;*

(b) if $\gamma(x, w) \subset x$*, then* $P(\gamma(x, w)) = \{x\}$ *and there exists at least one* $w' \in x$ *such that* $\gamma(x, w) \cap \gamma(x, w') = \emptyset$*;*

(c) if $\gamma(x, w) = x$*, then the chain* $\{y \in \downarrow x \setminus \{x\} \,|\, w \in y\}$ *has no maximum, and there exists no EDP for which a choice is available at* x*.*

The last impossibility statement suggests that, in order to have an EDP defined on a tree, at the very least, a condition of the following sort is needed.

Definition 7. *A game tree* $(N, \supseteq)$ ***has available choices*** *if, for all* $x \in X$*,*

$$\gamma(x, w) \subset x \text{ for all } w \in x\,. \tag{12}$$

The terminology is motivated by the fact that, according to Proposition 2(c), if $\gamma(x, w) = x$ then there can be no choice available at x, i.e. $J(x) = \emptyset$. If, by contrast, $\gamma(x, w) \subset x$, then the sets $\gamma(x, w)$ can serve as choices—at least in a single-player perfect information game. Proposition 2(c) also provides a sufficient condition for available choices. It will be useful to give a name to this condition.

Definition 8. *A tree is* ***weakly up-discrete*** *if all maximal chains in* $\downarrow x \setminus \{x\}$ *have maxima, for all moves* $x \in X$*.*

This condition will play an important role for strategies to induce outcomes. It is clearly related to the existence of immediate successors. For the moment, however, the following consequence of Proposition 2(c) is the focal point.

Corollary 1. *If a game tree is weakly up-discrete, then it has available choices.*

4.3. Existence of EDPs. Denote the set of perfect information choices for a game tree $T = (N, \supseteq)$ by

$$\Gamma(T) = \{\gamma(x, w) \,|\, w \in x \in X\}\,. \tag{13}$$

By Proposition 2(b), if a game tree has available choices, then all choices in Γ are available at some node. The following theorem shows that availability of perfect information choices (in the sense of Definition 7) indeed *characterizes* game trees on which EDPs can be defined. Surprisingly, the presence of strange nodes does not prevent that, but they must be incorporated as (unavailable) choices.

Theorem 1. *Let* $T = (N, \supseteq)$ *be a game tree. The following assertions are equivalent:*

(a) Some EDP (T, C) *can be defined on* $T = (N, \supseteq)$*;*

(b) T *has available choices;*

(c) $\Pi(T) = (T, C_1)$ *is a well-defined single-player EDP, where* $C_1 = \Gamma(T) \cup S(N)$*.*

Proof. (c) trivially implies (a). To see that (a) implies (b), suppose T does not have available choices. Then, Proposition 2(c) implies that $J(x) = \emptyset$ for some $x \in X$. As Definition 6 requires $J(x) \neq \emptyset$ for all $x \in X$, no EDP can be defined on such a tree.

We now show that (b) implies (c). Suppose that $T = (N, \supseteq)$ has available choices. Let $I = \{1\}$ and $C_1 = \Gamma(T) \cup S(N)$. Then choices are unions of nodes by construction. Proposition 2(b) implies that $A_1(x) \neq \emptyset$, hence, $J(x) = \{1\} \neq \emptyset$ for all $x \in X$.

It remains to verify (EDP.i-iv). Property (EDP.ii) follows trivially from $J(x) = \{1\}$ for all $x \in X$. (EDP.iv) is also simple: consider $x \in X$ and $x \supset y \in N$. Choose $w \in y$ and $c = \gamma(x, w)$. Then $y \subseteq c$ and $c \in A_1(x)$ by Proposition 2(b).

Consider now (EDP.i). If $P(c) \cap P(c') \neq \emptyset$, then Proposition 2(b) and Lemma 3 imply that $P(c) = \{x\} = P(c')$ for some $x \in X$, where $c = \gamma(x, w)$ and $c' = \gamma(x, w')$ for some $w, w' \in x$. That $c \cap c' = \emptyset$ then follows from Proposition 2(a).

It remains only to establish (EDP.iii). Note that all strange nodes are actually choices in $C_1$by construction. We claim that finite nodes are also choices in C_1. Let y be a finite node and consider $x = \min \uparrow y \setminus \{y\}$. Let $c = \gamma(x, w)$ for some $w \in y$. By definition $y \subseteq c$. But, if $y \subset c \subset x$ (the latter by availability of perfect information choices), then there would be some $z \in \downarrow x \setminus \{x\}$ such that $w \in y \subset z$, in contradiction to $x = \min \uparrow y \setminus \{y\}$. Therefore, $y = c$ as desired. Hence, (EDP.iii) holds trivially for pairs of non-infinite nodes.

Consider now an infinite node y. Hence, $y = \inf \uparrow y \setminus \{y\}$. Consider any other node $y' \in N$ (infinite or not) such that $y \cap y' = \emptyset$. It follows that $y' \notin \uparrow y$. We claim that there exists $z \in \uparrow y \setminus \{y\}$ such that $z \cap y' = \emptyset$. For, if this were not the case, Trivial Intersection would imply that $y' \subseteq z$ for all $z \in \uparrow y \setminus \{y\}$. By the definition of the infimum, it would follow that $y \supseteq y'$, a contradiction. Take now $c = \gamma(z, w)$ for some $w \in y$. It follows that $y \subseteq c$ and $c \cap y' = \emptyset$. Hence, (EDP.iii) is verified if y is infinite and y' is not infinite (and analogously for the opposite case), since then y' itself must be a perfect-information choice.

If y, y' are infinite, disjoint nodes, then as above there exist a choice $c = \gamma(z, w)$ with $z \in \uparrow y \setminus \{y\}$, $z \cap y' = \emptyset$, $y \subseteq c \subseteq z$ and $c \cap y' = \emptyset$. Repeating the argument with the (disjoint) nodes z, y' we obtain a choice c' such that $y' \subseteq c'$ and $c' \cap z = \emptyset$, implying that $c \cap c' = \emptyset$. This completes the proof of (EDP.iii). ■

The strange nodes taken as choices in $C_1 = \Gamma(T) \cup S(N)$ are never available at any move. This makes the interpretation of Theorem 1 odd. Obviously, if the tree is regular, i.e. $S(N) = \emptyset$, the single-player game only requires the choices $C_1 = \Gamma(T)$, i.e. those that are always available.

Corollary 2. *For a regular game tree $T = (N, \supseteq)$ the following are equivalent:*
(a) Some EDP (T, C) can be defined on $T = (N, \supseteq)$;
(b) T has available choices;
(c) $\Pi(T) = (T, \Gamma(T))$ is a well-defined single-player EDP.

Say that an EDP (T, C) has *quasi-perfect information* if $P(c) \neq \emptyset$ implies that $P(c) = \{x\}$ for some $x \in X$, for all $c \in C_i$ and all $i \in I$. Quasi-perfect information differs from the traditional notion of "perfect information" in an extensive form in two respects. First, there may be choices that are never available at any move; second, if there are several players, they may choose at the same node.[9]

Theorem 1(c) (and Proposition 2(b)) reveals that the existence of an EDP with quasi-perfect information is necessary for any EDP to be definable on a tree.

[9] In other words, abstracting from choice unavailability, quasi-perfect information corresponds to the standard concept of perfect information, except for the possibility of simultaneous moves. Recall that, in the present framework, simultaneous moves do not require cascading information sets, because players, who choose simultaneously, may be active at the same node.

Corollary 3. *If an EDP (T, C) can be defined on the game tree $T = (N, \supseteq)$ at all, then there exists one with quasi-perfect information defined on T that is unique up to an assignment of decision points to players. In particular, it is unique if a single player is assumed.*

Given an EDP (T, C) with game tree $T = (N, \supseteq)$, the associated single player decision problem with perfect information is henceforth denoted $\Pi(T) = (T, C_1)$, as in Theorem 1(c).

5. Pure Strategies

This section introduces strategies formally and spells out the three desiderata. Moreover, we show that (A1)—that for every play there is a strategy that selects it—holds true without further restrictions on the tree.

5.1. Definition. A key object derived from an EDP is a *pure strategy* for a player $i \in I$. This is a function $s_i : X_i = \{x \in X \,|i \in J(x)\} \to C_i$ such that

$$s_i^{-1}(c) = P(c) \text{ for all } c \in s_i(X_i) \tag{14}$$

where $s_i(X_i) \equiv \cup_{x \in X_i} s_i(x)$. That is, the function s_i assigns to every move $x \in X_i$ a choice $c \in C_i$ such that (a) choice c is *available* at x, i.e. $s_i(x) = c \Rightarrow x \in P(c)$ or $s_i^{-1}(c) \subseteq P(c)$, and (b) to every move x in an *information set* $h = P(c)$ the same choice gets assigned, i.e. $x \in P(c) \Rightarrow s_i(x) = c$ or $P(c) \subseteq s_i^{-1}(c)$, for all $c \in C_i$ that are actually chosen somewhere, viz. $c \in s_i(X_i)$.

If chance (player $i = 0$) is part of the EDP, it is treated symmetrically to personal players in Definition 6, so there is no problem with defining pure strategies for chance as well. Let S_i denote the set of all pure strategies for player $i \in I$. A *pure strategy combination* is an element $s = (s_i)_{i \in I} \in S \equiv \times_{i \in I} S_i$.

If an EDP captures consistent rules, then there are a few basic desiderata that need to be satisfied, like that every strategy combination ought to induce an outcome/play. First, of course, it has to be clarified when a pure strategy combination "induces" a play. To this end define, for every $s \in S$, the correspondence $R_s : W \to W$ by

$$R_s(w) = \bigcap \{s_i(x) \,|w \in x \in X, i \in J(x)\}. \tag{15}$$

Say that the strategy combination s *induces* the play w if $w \in R_s(w)$, i.e., if w is a fixed point of R_s.

5.2. Desiderata. The following are the key desiderata on the mapping R_s, when applied in an EDP:

(A1) For every $w \in W$ there is some $s \in S$ such that $w \in R_s(w)$.

(A2) For every $s \in S$ there is some $w \in W$ such that $w \in R_s(w)$.

(A3) If for $s \in S$ there is $w \in W$ such that $w \in R_s(w)$, then R_s has no other fixed point and $R_s(w) = \{w\}$.

The first desideratum asks for every play to be reachable by specifying an appropriate pure strategy combination. This would fail, for instance, if the game had absent-mindedness (Piccione and Rubinstein (1997)). But this is ruled out by the current definition (see Proposition 13 of AR).[10] Therefore, the verification of (A1) derives directly from Theorem 4 of AR:

Theorem 2. *If (T, C) is an EDP, then for every play $w \in W$ there is a pure strategy combination $s \in S$ such that $w \in R_s(w)$.*

Proof. Given $w \in W$, we construct $s \in S$ in three steps. First, consider the moves $x \in X$ with $w \in x$ and let $i \in J(x)$. Let $y \in\downarrow x \setminus \{x\}$ with $w \in y$. By (EDP.iv), there exists a choice $c_i \in A_i(x)$ such that $y \subseteq c_i \subset x$, so $w \in c_i$. Define $s_i(x) = c_i$.

Second, consider the moves $x \in X$ with $w \notin x$. Suppose there exists some node x' with $w \in x'$ and $A_i(x) \cap A_i(x') \neq \emptyset$. Then, necessarily $A_i(x) = A_i(x')$, by (EDP.i). (Let $c^* \in A_i(x) \cap A_i(x')$, so that $x, x' \in P(c^*)$. Now let $c \in A_i(x)$, i.e. $x \in P(c)$. This means that x is a common predecessor of c and c^*, so by (EDP.i), $x' \in P(c) = P(c^*)$ and $c \in A_i(x')$.) Hence, by (EDP.iv) and (EDP.i) there exists a unique $c \in A_i(x) = A_i(x')$ with $w \in c$. Set $s_i(x) = c$. To see that this definition is consistent, suppose there exist two different nodes x', x'' with $w \in x', x''$, $A_i(x) \cap A_i(x') \neq \emptyset$ and $A_i(x) \cap A_i(x'') \neq \emptyset$. The latter imply $A_i(x') = A_i(x'')$. But, by Trivial Intersection x' and x'' are ordered, and then Proposition 13 in AR implies that $x' = x''$, a contradiction.[11]

Third, consider the moves $x \in X$ with $w \notin x$ such that for all nodes x' with $w \in x'$ it follows that $A_i(x) \cap A_i(x') = \emptyset$. Then, for every $i \in J(x)$ choose $s_i(x)$ arbitrarily. Obviously, $w \in R_s(w)$ holds by construction. ■

6. WHEN DO STRATEGIES INDUCE OUTCOMES?

This section is devoted to (A2). The focus is on properties of the tree. This is because the crucial condition on choices is already part of the definition of an EDP: Clearly, property (EDP.ii) is necessary for (A2) to hold. For, if (EDP.ii) were not true, players could choose such that the game cannot continue from some move.

But the restriction on choices incorporated in (EDP.ii) is not enough to fulfill (A2). Below two important properties of a tree are identified—"weak up-discreteness" and "coherence"—that turn out to characterize the class of trees for which every strategy induces outcomes *after every history*—a slightly stronger criterion than (A2). This, then, turns into a characterization of (A2) for the class of *regular* game trees.

6.1. Examples for Non-existence.

The following example provides a transparent illustration for what can go wrong with existence of outcomes.

Example 9. ***(Augmented Inverse Infinite Centipede)*** *No EDP can be defined on the inverse infinite centipede from Example 8. Construct, though, an* augmented

[10] A decision theory for games with absent-mindedness would require allowing the state that obtains to depend on the decision maker's choice. Since we do not know how to handle that, we find it worthwhile to follow Kuhn (1953) by excluding absent-mindedness in the definition of an EDP.

[11] The only delicate point in the proof is showing that this definition can be done consistently. This is made possible by Proposition 13 of AR, which in turn only depends on (EDP.iv). This is exactly the condition which prevents absent-mindedness.

inverse centipede by adding a new element ∞ *to the underlying set* W *(which was previously just the natural numbers), such that* $\infty > t$ *for any other* $t = 1, 2, ...,$ *and consider the corresponding* W*-centipede. Now the root has an "immediate successor,"* $\{\infty\}$*, and* $\gamma(W, 1) \subset W$*. It is not difficult to verify that this (regular) tree has available choices and, hence, admits an EDP. The corresponding single-player perfect information problem* $\Pi(T)$ *as in Theorem 1(c) is easy to construct:* $\gamma(x_t, t') = x_{t+1} \subset x_t$ *if* $t' > t$*, and* $\gamma(x_t, t') = \{t\} \subset x_t$ *if* $t' = t$*. The interpretation of these choices as "continue" or "stop" are obvious.*

Consider a strategy s *which prescribes to continue at the beginning and to stop at every other move, i.e.* $s(W) = \gamma(W, 1) = \{1, 2, ...\}$ *and* $s(x_t) = \gamma(x_t, t) = \{t\}$ *for all* $t = 1, 2, ...$ *There is no play that is consistent with this strategy, i.e.* R_s *has no fixed point. For,* $R_s(\infty) = s(W) = \{1, 2, ...\}$ *so that* $\infty \notin R_s(\infty)$*, and for any* $t = 1, 2, ...$ *we obtain* $R_s(t) = s(W) \cap [\cap_{\tau=t}^{\infty} s(x_\tau)] = \cap_{\tau=t}^{\infty} \{\tau\} = \emptyset$*. In other words, the strategy* s *induces no outcome at all.*

A similar point can be made with the continuous centipede (Example 5). Even though this problem is not peculiar to continuous time, it may well plague continuous-time decision problems. As already observed by Simon and Stinchcombe (1989) and Stinchcombe (1992), decision problems in continuous time may suffer from strategies that induce no outcomes. To see this, turn to the differential game (Example 1).

Example 10. *Consider the differential game with a single player and perfect information as described in Examples 1 and 7. Let the action space be* $A = \{0, 1\}$ *and specify a strategy* $s \in S$ *by* $s(W) = c_0(h, 1)$ *and*

$$s(x_t(f)) = \begin{cases} c_t(f, 0) & \text{if } f(r) = 1 \ \forall\, r < t\,, \\ c_t(f, 1) & \text{otherwise,} \end{cases}$$

for any $t > 0$ *and any* f*. Obviously, the constant function* $\mathbf{1}$ *(i.e.* $\mathbf{1}(t) = 1\ \forall t$*) is not a fixed point of* R_s*, because* $\mathbf{1}(s) = 1$ *for all* $s < t$ *for any* $t > 0$*, so that by the construction of* s *it would follow that* $\mathbf{1}(t) = 0$*, a contradiction.*

Suppose that R_s *has a fixed point* f*. It follows that* $f(0) = 1$ *but, since* $f \neq \mathbf{1}$*, there exists* $t > 0$ *such that* $f(t) = 0$*. Thus, the set of real numbers* $\{t \geq 0 \,|\, f(t) = 0\}$ *is nonempty and bounded below by* 0*. By the Supremum Axiom, this set has an infimum* t^**. If* $t^* > 0$*, consider* $t' = t^*/2$*. Then,* $f(t') = 1$*, but also* $f(r) = 1$ *for all* $r < t'$*. By the definition of* s *we should have* $f(t') = 0$*, a contradiction.*

It follows that $t^* = 0$*. But then, consider any* $t > 0$*. By definition of infimum, there exists* $0 < r < t$ *such that* $f(r) = 0$*. By the definition of* s *we have that* $f(t) = 1$*. Since* $t > 0$ *was arbitrary, it follows that* f *must be identically* 1*, a contradiction.*

6.2. Histories and Discarded Nodes. While desideratum (A2), that strategies induce outcomes, is clearly appealing, it is not always sufficient. It may well be sufficient for pure one-shot decisions among strategies. But for truly sequential decision making it is necessary to evaluate counter-factuals, that is, 'continuation' strategies after arbitrary histories (chains of nodes). This requires that strategies not only induce outcomes, but do so after every history. This is clearly a prerequisite for any "backwards induction" solution concept, like subgame perfection, for instance. But in the present framework such a criterion is pushed beyond subgames, as arbitrary—possibly infinite—histories need to be accounted for. This subsection introduces the associated notions.

Let $T = (N, \supseteq)$ be a game tree. A *filter* is a chain h in N such that $\uparrow x \subseteq h$ for all $x \in h$. A *history* is a filter that is not maximal in T, i.e. that is not a play. For a history h in T a *continuation* is the complement of h in a play that contains h.

In this section it is occasionally necessary to distinguish between an outcome $w \in W$ and the associated play (maximal chain of nodes), $\uparrow\{w\}$. Recall from AR (Theorem 3) that $\psi(w) = \uparrow\{w\}$ defines a bijection between the set of outcomes and the set of plays such that $w \in x$ if and only if $x \in \psi(w)$, for any $w \in W$ and $x \in N$. Define, for any chain h in N,

$$W(h) = \bigcap_{x \in h} x \,, \tag{16}$$

as the set of outcomes that have still not been discarded after history h. Clearly, $W(\{W\}) = W$ corresponds to the null history that consists only of the root.

Definition 9. *Let (T, C) be an EDP and s a pure strategy combination. Say that*

(a) s ***induces outcomes*** *if there exists $w \in W$ such that $w \in R_s(w)$, where R_s is defined in (15);*

(b) s ***induces outcomes after every history*** *if for every history h there exists $w \in W(h)$ such that $w \in R_s^h(w)$, where*

$$R_s^h(w) = \bigcap \{ s_i(x) \,|\, w \in x \subseteq W(h),\, x \in X,\, i \in J(x) \} \,. \tag{17}$$

An EDP is ***playable*** *if every strategy combination induces outcomes. It is* ***playable everywhere*** *if every strategy combination induces outcomes after every history.*

That is, an EDP is *playable* if the mapping R_s has a fixed point for every $s \in S$; it is *playable everywhere* if the mapping R_s^h has a fixed point for every $s \in S$ and for every history h. Histories take the role of subgames in a general EDP, but only correspond to subgames (under quasi-perfect information) when they have infima. This has a useful characterization in terms of the sets $W(h)$ from (16).

Lemma 4. *Let $T = (N, \supseteq)$ be a game tree and h a history in T. Then:*

(a) $\emptyset \neq W(h) = \{w \in W \,|\uparrow\{w\} = h \cup g$ for some continuation g of $h\}$, and

(b) $W(h) \in N$ if and only if h has an infimum.

Fix a history h and a strategy $s \in S$. Define the set of *discarded nodes at h*, denoted $D^h(s)$, as the set of nodes $y \in N$ that are properly contained in $W(h)$ and for which there are $x \in \uparrow y \setminus \{y\}$, $i \in J(x)$, and $c \in A_i(x)$ such that $x \subseteq W(h)$ and $y \subseteq c \neq s_i(x)$. The set of *undiscarded nodes at h*, denoted $U^h(s)$, is the set of nodes contained in $W(h)$ that are not discarded. The sets of *discarded* and *undiscarded nodes* are defined as $D(s) = D^{\{W\}}(s) \subseteq N \setminus \{W\}$ and $U(s) = U^{\{W\}}(s) = N \setminus D(s)$ respectively. Clearly, $W \in U(s)$ by construction. The following result is immediate.

Lemma 5. *Let $T = (N, \supseteq)$ be a game tree and h a history in N. Then:*

(a) $D^h(s)$ is an ideal, i.e. $x \in D^h(s)$ implies $\downarrow x \subseteq D^h(s)$, and

(b) every chain in $U^h(s)$ is a filter, i.e. $x \in U^h(s)$ implies $\uparrow x \subseteq U^h(s)$.

Existence of outcomes is equivalent to existence of plays consisting of undiscarded nodes, as the following result shows.

Proposition 3. *Consider an EDP, a history h, and a strategy combination $s \in S$. Then, there exists $w \in W$ such that $w \in R_s^h(w)$ if and only if $U^h(s)$ contains a maximal chain in $\{x \in N \,|x \subseteq W(h)\}$.*

This is, of course, almost a tautology. But it illustrates what goes wrong in the "augmented inverse infinite centipede," Example 9. There, a strategy that "continues" at the root, but "stops" everywhere else, generates a set of undiscarded nodes that consists only of the root. In the following example undiscarded nodes form a larger, but not a maximal chain.

Example 11. ***(Modified Hole in the Middle)*** *The following modification of Example 2 uses a regular game tree. Again, $W = [0,1]$ is the set of outcomes/plays and the set of nodes is given by $N = \left\{(x_t)_{t=1}^{\infty}, (y_t)_{t=1}^{\infty}, (y'_t)_{t=1}^{\infty}, [1/4, 3/4], (\{w\})_{w \in W}\right\}$, where x_t, y_t, and y'_t are as in Example 2. Make this into a single-player EDP with perfect information by assigning all nodes, except the root, plus $[1/4, 1/2) \cup (1/2, 3/4] = \gamma([1/4, 3/4], 1/4) \cup \gamma([1/4, 3/4], 3/4)$ as choices to the personal player. Then there is a strategy s that assigns to any move x_t the non-singleton choice x_{t+1}, to every move y_t or y'_t the corresponding singleton choice $\{(2t-1)/(4t)\}$ or $\{(2t+1)/(4t)\}$, and to $[1/4, 3/4]$ the choice $[1/4, 1/2) \cup (1/2, 3/4]$. This strategy induces no play. Still, this tree is regular (because $[1/4, 3/4] \in N$, see Example 6), it has available choices, and all choices are available at some move.*

6.3. Perfect Information, Up-Discreteness, and Coherence. Recall from Corollary 3 that the existence of an EDP with quasi-perfect information (and a single player) is necessary for any EDP to be definable on a tree. It will now be shown that for playability it is sufficient to consider the perfect information case.

The following statements, whose proofs are straightforward and omitted, concern an arbitrary EDP (T, C) and the associated single-player perfect information problem $\Pi(T) = (T, C_1)$, as in Theorem 1(c). Let $s \in S$ denote the strategy combinations in (T, C) and $s' \in S'$ the strategies in $\Pi(T)$.

Lemma 6. *For (T, C): If $x \in P(c)$, with $c \in C_i$ for some $i \in I$, and $w \in x \cap c$, then $\gamma(x, w) \subseteq x \cap c$.*

Lemma 7. *Let h be a history. If $s \in S$ and $s' \in S'$ are such that $s'(x) \subseteq s_i(x) \cap x$ for all $i \in J(x)$ for all $x \in X$, then $U^h(s') \subseteq U^h(s)$.*

Whether or not every strategy combination induces outcomes in a given EDP is purely a matter of the tree and, therefore, independent of the choice (information) structure (granted (EDP.ii) holds). This is the essence of the following result.

Proposition 4. *Fix a history h. If every strategy $s' \in S'$ for $\Pi(T)$ induces outcomes after h, then for any EDP (T, C) with the same tree every strategy combination $s \in S$ induces outcomes after h.*

Choosing $h = \{W\}$, for a fixed game tree, Proposition 4 implies the following.

Corollary 4. *If $\Pi(T)$ is playable, then any EDP (T, C) with the same tree is playable. Analogously, if $\Pi(T)$ is playable everywhere, then any EDP (T, C) with the same tree is playable everywhere.*

Even though playability is purely a matter of the tree according to the above results, it remains a surprisingly subtle problem. To clarify, a number of properties of a game tree are needed that concern the following two issues. First, a history (i.e. a non-maximal filter) may or may not have a minimum. (E.g., the chain of proper predecessors of an infinite or strange node do not.) Second, a given continuation of a fixed history may or may not have a maximum. In "classical" games, all histories have minima and all continuations have maxima. Large games, e.g. in continuous time, provide examples, where this is not the case.

Recall that T is *weakly up-discrete* if for every move $x \in X$ all maximal chains in $\downarrow x \setminus \{x\}$ have a maximum. The following result supplies two characterizations.

Lemma 8. *For a game tree $T = (N, \supseteq)$ the following statements are equivalent:*
(a) T is weakly up-discrete;
(b) for every history with a minimum, every continuation has a maximum;
(c) $x \supset \gamma(x, w) \in N$ for all $w \in x$ and all $x \in X$.

Statement (b) will be crucial for what follows. Weak up-discreteness implies available choices (Corollary 1), but Lemma 8(c) states the additional property that perfect information choices are, in fact, nodes. This is not the case, for instance, in the differential game (Example 1). Furthermore, for weakly up-discrete trees the converse of Proposition 4 holds too.

Proposition 5. *Let (T, C) be an arbitrary EDP with a weakly up-discrete game tree $T = (N, \supseteq)$ and consider a history h in T. If every strategy combination $s \in S$ for (T, C) induces outcomes after h, then every strategy $s' \in S'$ for the single-player perfect information problem $\Pi(T)$ induces outcomes after h.*

This result implies that if an EDP with a weakly up-discrete tree is playable resp. everywhere playable, then *every* EDP with the same tree is playable resp. everywhere playable. For, if (T, C) is a (everywhere) playable EDP with weakly up-discrete tree, then the single-player perfect information problem $\Pi(T)$ is (everywhere) playable; but then also *any other* EDP (T, C') is (everywhere) playable, by Proposition 4.

It is natural to introduce the following strengthening of weak up-discreteness which can be characterized in analogy with Lemma 8(b).

Definition 10. *A tree is* ***up-discrete*** *if all (nonempty) chains have a maximum.*

Lemma 9. *A game tree $T = (N, \supseteq)$ is up-discrete if and only if for every history every continuation has a maximum.*

Remark 2. *In order theory the posets that are here called trees are sometimes called "pseudotrees" (see Koppelberg and Monk (1992)). The word "tree" is then reserved for posets for which, additionally, the sets* $\uparrow x$ *are (dually) well-ordered: all their subsets have a first element according to* $\geq$ *(i.e. a maximum; see Koppelberg (1989), chp. 6). This condition is also common in theoretical computer science. It is equivalent to up-discreteness as defined here. For, if* $(N, \geq)$ *is an up-discrete tree, let* $x \in N$ *and consider any subset* $c \subseteq\uparrow x$*. Since* c *is a chain, it follows from up-discreteness that it has a maximum. Conversely, let* $(N, \geq)$ *be a tree such that all subsets of* $\uparrow x$ *have a maximum, for all nodes* x*. Let* c *be any chain in* N *and consider any* $x \in c$*. The chain* $c\cap \uparrow x$ *must have a maximum,* z*. But, if* $y \in c$ *and* $y \notin\uparrow x$*, then (since* c *is a chain),* $z \geq x \geq y$*, which shows that* z *is a maximum for the whole chain* c*.*

Finally, complementary to weak up-discreteness, the idea of coherent trees is introduced. Intuitively, a game tree is coherent if it has no "holes." A "hole" appears in the "hole in the middle," Example 2, and it amounts to a missing node. More formally, a "hole" amounts to a history without minimum, whose continuations have no maxima. Under regularity, the previous lemmata then yield an important relation between coherence and the two up-discreteness concepts.

Definition 11. *A game tree* $T = (N, \supseteq)$ *is* ***coherent*** *if every history without minimum has at least one continuation with a maximum.*

Corollary 5. *A regular game tree* $T = (N, \supseteq)$ *is up-discrete if and only if it is weakly up-discrete and coherent.*

6.4. Everywhere Playable EDPs. A one-shot decision theory that considers once-and-for-all decisions among strategies can do with playable EDPs. The purpose of this study is, however, to provide a domain for sequential decision theories (see the Introduction). For the latter it is essential that the decision maker can evaluate her 'continuation' strategies. That is, an adequate domain for sequential decision theories has to be playable everywhere. Therefore, we turn now to a characterization of the class of trees where this is fulfilled.

Theorem 3. *Let* $T = (N, \supseteq)$ *be a game tree with available choices. Then, any EDP* (T, C) *is playable everywhere if and only if* T *is coherent and weakly up-discrete.*

Proof. "if:" By Proposition 4 it suffices to consider $\Pi(T)$. Fix a history h and a strategy s. By Lemma 5 the set $D^h(s)$ of discarded nodes is an ideal and every chain in the set $U^h(s)$ of undiscarded nodes is a filter.

First, $U^h(s)$ is nonempty: If h has an infimum, then $W(h) \in N$ by Lemma 4 and, by definition, $W(h) \in U^h(s)$. If h has no infimum (and hence no minimum), by coherence there exists a continuation g of h which has a maximum, $z = \max g$. By definition, $z \in U^h(s)$, because there exists no node $x \in\uparrow z \setminus \{z\}$ such that $x \subseteq W(h)$.

Second, suppose that there is no $w \in W$ such that $w \in R^h_s(w)$. Since $U^h(s)$ is nonempty, there exists a maximal chain u in $U^h(s)$ by the Hausdorff Maximality Principle. Let $w \in W$ be such that $u \subseteq\uparrow \{w\}$. If $u =\uparrow \{w\} \setminus h$, then $\uparrow \{w\} = u \cup h$ and, by construction, $w \in R^h_s(w)$, a contradiction. Thus, $u \subset\uparrow \{w\} \setminus h$ and $u \cup h$ is a history.

Third, we claim that u has no minimum. If it had, say, $x = \min u$, then $x = \min u \cup h$. By weak up-discreteness and Lemma 8, every continuation of $u \cup h$ would then have a maximum. Let $w \in s(x)$, and let z be the maximum of the continuation $\uparrow\{w\} \setminus (u \cup h)$. Hence, $P(z) = \{x\}$. Since $z \subseteq s(x)$ by (EDP.i) and (EDP.iv), and $x \in U^h(s)$, it follows from the fact that $U^h(s)$ is a filter that $z \in U^h(s)$, a contradiction to maximality of u.

Since u has no minimum, coherence implies that there exists a continuation g of $u \cup h$ which has a maximum, $z' = \max g$. Let $w \in W$ be such that $\uparrow\{w\} = u \cup h \cup g$. Since $w \in x$ for all $x \in u$, it follows that $w \in s(x)$ for all $x \in u$. For, since u has no minimum, for any $x \in u$ there is $x' \in u$ such that $x' \subseteq x$. Since $x' \in u \subseteq U^h(s)$, it follows that $w \in x' \subset s(x)$, using (EDP.i) and (EDP.iv) again. But then that $w \in z'$ implies $z' \subseteq s(x)$ for all $x \in u$. Since $\uparrow z' \setminus \{z'\} \subseteq u \cup h$, it follows that $z' \in U^h(s)$, a contradiction.

"only if:" It has to be shows that if either weak up-discreteness or coherence fail, then there are a history and a strategy such that the strategy induces no outcome after the history. The EDP used in the construction is again $\Pi(T)$.

Suppose, first, that weak up-discreteness fails. Then, by Lemma 8, there exists a history h which has a minimum, $z = \min h = W(h)$ (by Lemma 4), and a continuation g of h which has no maximum. Let $w^* \in W$ be such that $\uparrow\{w^*\} = h \cup g$. Define a strategy s as follows. For every $x \in h$ (which includes $z = \min h$) set $s(x) = \gamma(x, w^*)$. For every $x \in g$, choose $s(x) \neq \gamma(x, w^*)$. Choose arbitrary choices at all other nodes. Obviously, $w^* \notin R_s^h(w^*)$. If $w \in z \setminus \gamma(z, w^*)$, also $w \notin R_s^h(w)$ because $s(z) = \gamma(z, w^*)$.

Let $w \in \gamma(z, w^*)$ be such that $w \neq w^*$, and consider the choice $\gamma(z, w)$. Since $w \in \gamma(z, w^*) = \bigcup\{x \in \downarrow z \setminus \{z\} \,|\, w^* \in x\}$, there exists $x \in N$ such that $x \subset z = W(h)$ and $w, w^* \in x$. Since $x \subset z$ and $w^* \in x$, hence, $x \in g$ (which has no maximum), there exists $y \in g$ such that $x \subset y \subset z$. Hence, $\gamma(y, w) \cap \gamma(y, w^*) \neq \emptyset$ and it follows from Proposition 2(a) that $\gamma(y, w) = \gamma(y, w^*)$. Thus, $s(y) \neq \gamma(y, w)$, implying that x is discarded at y and hence $w \notin R_s^h(w)$. Therefore, s does not induce an outcome after the history h.[12]

Suppose, second, that coherence fails. Then, there exists a history h that has no minimum, such that no continuation g of h has a maximum. Consider the set $W(h)$ and define the following relation on it. Given $w, w' \in W(h)$, say that wRw' if there exists a node $x \in N$ such that $x \subseteq W(h)$ and $w, w' \in x$. This relation is an equivalence (it is obviously reflexive and symmetric; transitivity follows from the fact that T is a tree) and, hence, induces a partition of the set $W(h)$ in the form of its quotient set $W(h)/R$. Further, given any $x \in N$ such that $x \subseteq W(h)$, there exists a unique $z \in W(h)/R$ such that $x \subseteq z$.

For each $z \in W(h)/R$, choose an element $w(z) \in z$ and let $g(z) = (\uparrow\{w(z)\}) \setminus h$. Define a strategy s as follows. For every $x \in h$, set $s(x) = \gamma(x, w(z))$ for any $w(z)$. For every $z \in W(h)/R$ and every $x \in g(z)$, choose $s(x) \neq \gamma(x, w^*)$.

Consider an arbitrary $z \in W(h)/R$. Clearly, $w(z) \notin R_s^h(w(z))$. Now consider any $w \in z$ such that $w \neq w(z)$. By definition of R, there exists a node $x \in N$ such that $x \subseteq W(h)$ and $w, w' \in x$. Since $x \in g(z)$ (which has no maximum), there exists $y \in g(z)$ such that $x \subset y \subset z$. Hence, $\gamma(y, w) \cap \gamma(y, w(z)) \neq \emptyset$ and it follows from Proposition 2(a) that $\gamma(y, w) = \gamma(y, w(z))$. Thus, $s(y) \neq \gamma(y, w)$, implying that x is discarded at y and, hence, $w \notin R_s^h(v)$. Therefore, s does not induce an outcome after the history h.[13] ■

[12] This does not mean that it does not induce an outcome in the whole game. The strategy could have selected a play which bifurcates from h before z, for instance if there is a strict subhistory of h which has a continuation which is simply a terminal node and no other continuation has a maximum.

[13] A similar comment as in the previous footnote applies.

The Theorem fails, if coherence is replaced by regularity. In the "hole in the middle" with $\{1/2\}$ removed from N and $1/2$ removed from W the strategy that "continues" at all x_t and "stops" at all y_t and y'_t fails to induce an outcome (see Example 2), but the tree is regular and weakly up-discrete. On the other hand, the Theorem applies to some non-regular trees, as the following example illustrates.

Example 12. *Consider again Example 2 ("hole in the middle"), where $\{1/2\}$ is a strange node, and add the (also strange) nodes $y = [1/4, 1/2)$ and $y' = (1/2, 3/4]$ to the tree. Choices are available in the sense of (11) and an EDP can be defined, but the three strange nodes are unavailable choices. The tree is clearly weakly up-discrete and coherent. Thus, any EDP defined on this tree is everywhere playable.*

6.5. Playable vs. Everywhere Playable EDPs. To be relevant for sequential decision theory an EDP has to be everywhere playable, and not merely playable. This distinction may first appear puzzling. It may seem that if every strategy induces outcomes, then every strategy should induce outcomes after every history. An intuitive (but false) argument in support of such a position could be as follows. Suppose every strategy induces an outcome, but there is a strategy that induces no outcome after a history h. Then, construct a new strategy that coincides with the first after history h, but is constructed so as to "select" h before. Since the new strategy must induce an outcome, and this outcome must necessarily "come after" h, it follows that the original strategy combination must induce an outcome after the history h.

This argument is an instance of an intuition that is guided by the finite case, but fails in the general case. The following example presents a *playable* EDP, where not every strategy induces an outcome after every history.

Example 13. ***(Lexicographic Centipede)*** *The idea is as follows. Start with an infinite centipede (Example 4). At the "end" of the infinite play of this centipede, replace the infinite terminal node by an augmented inverse infinite centipede (Example 9) but delete its root. At the end of this augmented inverse infinite centipede, and replacing its terminal node, add another augmented inverse infinite centipede, now with root. Then every strategy induces an outcome, but not after every history. For, roughly, if it does not end during the first centipede, it is forced to select the first augmented inverse infinite centipede. That the latter "has no beginning" generates an "unavoidable" outcome, but it does not impose anything on the rest of the tree. In particular, in the rest things can be arranged so as to violate playability. Formally,*

$$W = \{((-1)^{\tau} t, \tau) \,|t = 1, 2, ..., \tau = 0, 1, 3\} \cup \{(\infty, 0)\},$$

endow W with the natural lexicographic order, consider the associated W-centipede, and remove the node $\{(-t, \tau) \,|\tau = 1, 3, t = 1, 2, ...\} \cup \{(\infty, 0)\}$ viz. the root of the first augmented inverse infinite centipede.[14] *Note that the ordering of the nodes for $\tau = 1, 3$ is the reverse than for $\tau = 0$, and that $(\infty, 0)$ does not belong to any one of the moves $y_t = \{(-k, \tau) \in W \,|\tau = 1, 3, \tau = 1 \Rightarrow k = 1, ..., t\}$ or $z_t = \{(-k, 3) \in W \,|k = 1, ..., t\}$, but to all moves $x_t = \{(k, \tau) \in W \,|\tau = 0 \Rightarrow k \geq t\}$.*

[14] The resulting tree is technically not a centipede, because it not regular anymore.

At each move two choices (for $\Pi(T)$) are available, a singleton and the remainder of the plays in the move. That is, choices at x_t are $\{(t,0)\}$ and x_{t+1} for $t=1,2,...$, at y_t they are $\{(-t,1)\}$ and y_{t-1} for $t=2,3,...$, at y_1 they are $\{(-1,1)\}$ and $\{(-t,3)\in W\,|t=1,2,...\}=\gamma(y_1,(-1,3))$, at z_t they are $\{(-t,3)\}$ and z_{t-1} for $t=2,3,...$, and at z_1 they are $\{(-2,3)\}$ and $\{(-1,3)\}$.

The strategy s^* given by $s^*(x_t)=x_{t+1}$ for $t=1,2,...$, $s^*(y_t)=y_{t-1}$ for $t=2,3,...$, $s^*(y_1)=\gamma(y_1,(-1,3))$, and $s^*(z_t)=\{(-t,3)\}$ does not induce an outcome after the history $\{x_1,x_2,...,y_3,y_2\}$. This is so, because the remainder of the tree is an augmented inverse infinite centipede with a strategy like in Example 9: "continue" at the root, but "stop" everywhere else. So, this EDP is not everywhere playable.

Let, on the other hand, s be any strategy. If there is $t=1,2,...$ such that $s(x_t)=\{(t,0)\}$, let t^* be the smallest such t. Then, s induces the outcome $(t^*,0)$. If $s(x_t)=x_{t+1}$ for all $t=1,2,...$, consider the outcome $(\infty,0)$. Since $\uparrow\{(\infty,0)\}\setminus\{\{(\infty,0)\}\}=\{x_t\,|t=1,2,...\}$ and $(\infty,0)\in x_t$ for all $t=1,2,...$, the strategy s induces the outcome $(\infty,0)$.

The crux of the matter is that the node $\{(\infty,0)\}$ is strange. For a finite node x, a strategy that continues towards x along $\uparrow x\setminus\{x\}$ will reach the immediate predecessor of x and, hence, may or may not choose x. For an infinite node x, a strategy that continues towards x along $\uparrow x\setminus\{x\}$ must select x, but cannot select anything else. For a strange node, a strategy that continues towards x along $\uparrow x\setminus\{x\}$ must select x, but also selects anything else "after" (contained in the union over) $\uparrow x\setminus\{x\}$. If the infinite node is terminal, the strategy selects it and the game ends. If the strange node is terminal, as in the example, the strategy selects it, but the game goes on.

This is suggestive: If there are no strange nodes and a strategy induces no outcome after history h, then there exists another strategy which prescribes to go along the history h and *nothing else*. Hence, this new strategy does not induce an outcome. In other words, playability may indeed be equivalent to playability everywhere—but only for regular trees. The hypothesis of regularity is necessary for this, as the example above demonstrates.

That playability and everywhere playability are equivalent for regular game trees will now be formally demonstrated. The strategy of the proof is as follows. First, violations of either weak up-discreteness or coherence will be shown to allow for a strategy that induces no outcome. As a consequence, playability under regularity implies both weak up-discreteness and coherence. Those, in turn, imply that the EDP is playable everywhere by Theorem 3. The following preliminary results enable the construction of the first step.

Lemma 10. *Let $(N,\supseteq)$ be a game tree with available choices and h a history.*

(a) The sets $W(h,g)=\cup\{x\in N\,|x\in g\}$, where g is a continuation of h, form a partition of $W(h)$.

(b) If h has no minimum and no continuation of h has a maximum, then h has no infimum and the set $W(h)=\cap\{x\in N\,|x\in h\}$ is not a node in N.

The next Lemma shows how to construct a strategy that discards all the nodes of a continuation without maximum. The construction is similar to the one in the augmented inverse infinite centipede (Example 9) but, due to the arbitrariness of the game tree, has to invoke Zorn's Lemma (see e.g. Hewitt and Stromberg (1965)).

For a game tree with available choices $T = (N, \supseteq)$ and its associated single-player perfect information problem $\Pi(T)$, define a *partial strategy* on a set of nodes $Y \subseteq N$ to be the restriction of a strategy to $Y \cap X$, i.e. a mapping that assigns to every $y \in Y \cap X$ a choice available at y. The set of partial strategies on Y is denoted S_Y.

Lemma 11. *Let $T = (N, \supseteq)$ be a game tree with available choices and consider the associated single-player perfect information problem $\Pi(T)$. Let h be a history and g a continuation that has no maximum. Then, there exists a partial strategy s on*

$$N(h, g) = \{x \in N \,|x \subset W(h, g)\}$$

such that all nodes in $N(h, g)$[15] *are discarded under s, i.e. for all $x \in N(h, g)$ there is $y \in N(h, g)$ such that $x \subset y$ and $x \cap s(y) = \emptyset$.*

The next lemma summarizes how regularity will be used in the main proof.

Lemma 12. *Let $(N, \supseteq)$ be a regular game tree and h a history without minimum. Then, for any subhistory $h' \subset h$ which also has no minimum, no alternative continuation of h' (i.e. g' with $g' \cap (h \setminus h') = \emptyset$) has a maximum.*

Proof. Suppose h' is a subhistory of h such that one continuation g' with $g' \cap (h \setminus h') = \emptyset$ has a maximum, $x = \max g'$. Then, by construction $\uparrow x \setminus \{x\} = h'$ and, since $g' \cap (h \setminus h')$ is empty, h' has no infimum. Thus x is strange, in contradiction to regularity. ■

This completes the preparations for the proof that, for regular game trees, weak up-discreteness and coherence are necessary and sufficient for both playability and everywhere playability, so that the latter two are equivalent. Under regularity, though, weak up-discreteness and coherence are equivalent to up-discreteness.

Theorem 4. *For a regular game tree $T = (N, \supseteq)$ with available choices the following statements are equivalent:*

(a) Every EDP (T, C) is playable everywhere;
(b) Every EDP (T, C) is playable;
(c) T is weakly up-discrete and coherent;
(d) T is up-discrete.

Proof. First, (c) and (d) are equivalent by Corollary 5. Furthermore, (c) implies (a) by Theorem 3 and that (a) implies (b) is trivial. It remains to show that (b) implies (c).

Coherence. Start with coherence. Suppose, by contradiction, that coherence is violated. We construct a strategy that induces no outcome.

Let h be a history without minimum such that no continuation has a maximum. Define a strategy s as follows. Fix $w \in W(h)$. For any $x \in h$, let $s(x) = \gamma(x, w)$. Thus, $h \subseteq U(s)$.

[15] Note that the defining inclusion in $N(h, g)$ is strict. If h had a minimum, then the minimum would not be in this set. Note also that if h had an infimum but not a minimum, then there could not exist a continuation without maximum.

For the nodes in $N(h) = \{x \in N \,|\, x \subset W(h)\}$, proceed as follows. Partition $W(h)$ into the sets $W(h,g)$ (which can be done by Lemma 10(a)) where g are the possible continuations of h. Clearly, every node in $N(h)$ belongs to one and only one $N(h,g)$. For every $W(h,g)$, define s on $N(h,g)$ as the partial strategy identified by Lemma 11. Thus, $N(h) \subseteq D(s)$.

Consider any subhistory h' of h having no minimum. By Lemma 12 no continuation has a maximum. For all $W(h',g)$ such that g is a continuation of h' and such that $W(h',g) \neq W(h', \uparrow\{w\} \setminus h')$, define s on $N(h',g)$ as the partial strategy identified by Lemma 11. Thus $N(h',g) \subseteq D(h)$ for all such g. At all other nodes s is arbitrary.

Let $x \notin h$ such that $x \notin N(h)$. Choose any $w' \in x$. Define $g = \uparrow\{w'\} \setminus h$ and $h' = \uparrow\{w'\} \cap h$ (which is nonempty, because h is a history). Then, h' is a subhistory of h and g is a continuation of h'. If h' has no minimum, it follows by construction that $x \in D(s)$. Suppose, then, that there exists $z = \min h'$. Since $z \in h$, we have that $s(z) = \gamma(z,w)$. We claim that $\gamma(z,w') \cap \gamma(z,w) = \emptyset$. For, if not, $\gamma(z,w') = \gamma(z,w)$ by Proposition 2(a) and there would exist $y \in \uparrow\{w'\}$ with $y \subset z$ such that $w \in y$. It follows that $w \in y \in g = \uparrow\{w'\} \setminus h$. If $y \in h$, then $y \in \uparrow\{w'\} \cap h = h'$, in contradiction to $y \subset z = \min h'$. Thus $y \notin h$. Then $w \in y$ implies $y \in \uparrow\{w\} \setminus h$, thus $w' \in W(h)$, in contradiction to $x \notin N(h)$.

In conclusion, $s(z) = \gamma(z,w)$ and $x \subseteq \gamma(z,w')$ with $\gamma(z,w') \cap \gamma(z,w) = \emptyset$, i.e. $x \in D(s)$. We have shown that $U(s) = h$ and, hence, $U(s)$ contains no play, i.e. s induces no outcome by Proposition 3.

Weak Up-Discreteness. Next, turn to weak up-discreteness.. The proof again proceeds by contradiction. Let h be a history with a minimum such that one continuation g of h has no maximum (recall Lemma 8). By Lemma 10(b) $W(h,g)$ is not a node. The construction is similar to the one in the coherence part above. Let $z^* = \min h$, and define a strategy s as follows. Fix $w \in W(h,g)$. For any $x \in h$, including z^*, let $s(x) = \gamma(x,w)$. Thus all nodes in h are undiscarded, $h \subseteq U(s)$.

For the nodes in $N(h) = \{x \in N \,|\, x \subset W(h)\}$, proceed as follows. Define s on $N(h,g)$ as the partial strategy given by Lemma 11. For every $x \in N(h)$ not in $W(h,g)$, specify s arbitrarily. Since $s(z^*) = \gamma(z^*,w)$, it follows that $N(h) \subseteq D(s)$.

For subhistories h' of h having no minimum, proceed as in the proof of coherence to obtain $N(h',g') \subseteq D(h)$ for all alternative continuations g' of h'. At all other nodes s is specified arbitrarily. As in the coherence part above, for any $x \notin h$ such that $x \notin N(h)$, we obtain $x \in D(s)$. This shows that $U(s) = h$ and, hence, $U(s)$ contains no play, i.e. s induces no outcome by Proposition 3. ■

That not every playable EDP is necessarily everywhere playable may be regarded as an argument in favor of focusing on the class of regular games trees, banning strange nodes. If this position were taken, (A2) would lead to up-discrete game trees.

7. UNIQUENESS

This section is devoted to a characterization of the class of EDPs for which the uniqueness criterion (A3) holds: "extensive forms." Those satisfy both an extra condition on the choice (information) structure and one on the tree. The condition on choices becomes redundant, though, in the class of EDPs that satisfy (A2).

7.1. Examples with multiple outcomes. That the third desideratum, (A3), imposes further restrictions on the tree is illustrated by the example of "Twins" (AR, Example 13). For this example there is a strategy (for $\Pi(T)$) that assigns to each move its immediate successor among the moves (rather than an available singleton) and, thereby, induces two distinct plays. A similar defect may also occur "in the middle of the moves," as the following example illustrates.

Example 14. *Reconsider the "hole in the middle," Example 2. As argued in Section 2.2, there is a strategy s that assigns to any move the non-singleton choice. This strategy selects the plays $1/4, 1/2, 3/4 \in W$, i.e. $\{1/4\} = R_s(1/4)$, $\{3/4\} = R_s(3/4)$, and $1/2 \in R_s(1/2)$ (without equality in this case), because there is no move, where a decision between $\cup_{t=1}^{\infty} y_t = [1/4, 1/2)$ and $\cup_{t=1}^{\infty} y'_t = (1/2, 3/4]$ is taken.*

This would hold even if this tree would be turned regular by removing $1/2 \in W$ from the underlying set W of plays and removing the strange node $\{1/2\}$ from N. The absence of a node $[1/4, 3/4]$ would still dictate that the strategy described above induces multiple plays, viz. $\{1/4\} = R_s(1/4)$ and $\{3/4\} = R_s(3/4)$ in this case. Hence, even a regular tree may allow for strategies that induce multiple plays.

The defect in the "hole in the middle" example may appear to be the absence of a minimum for the chain $\uparrow y_t \cap \uparrow y'_t$, like in the "Twins" example (AR, Example 13, where $\uparrow \{0\} \cap \uparrow \{1\}$ has no minimum).[16] But in the following example *every* chain of the form $\uparrow x \cap \uparrow y$ with $x, y \in N$ has a minimum.[17] Still (A3) fails in this example, illustrating that the problem is deeper.

Example 15. *Reconsider the differential game from Example 1. Recall (from Example 7) that the set of perfect information choices is given by*

$$\Gamma(T) = \left\{ c_t(f, a) \in 2^W \,|\, f \in W,\, a \in A, t \in \mathbb{R}_+ \right\},$$

where $c_t(f, a) = \{g \in x_t(f) \,|\, g(t) = a\} = \gamma(x_t(f), g)$ for any $g \in x_t(f)$ with $g(t) = a \in A$. The choice $c_t(f, a)$ is properly contained in $x_t(f)$, thus available at this move. That is, the single-player problem $\Pi(T)$ does not require the introduction of unavailable choices as in the proof of Theorem 1(c), because the tree is regular and Corollary 2 applies.

In this tree every chain of the form $\uparrow x \cap \uparrow y$ with $x, y \in N$ has a minimum. To see this, let $x_r(g)$ and $x_s(h)$ be any two nodes. If they are not disjoint, the conclusion is trivial (by Trivial Intersection). Hence, they can be assumed disjoint. The infimum $t^* = \inf\{t \geq 0 \,|\, g(t) \neq h(t)\}$ exists, because the defining set of real numbers is bounded from below. It follows that $g(t) = h(t)$ for all $t < t^*$. Moreover, since the two nodes are disjoint, it follows that $t^* < \min\{r, s\}$. Define $x = x_{t^*}(g) = x_{t^*}(h)$. Obviously, $x \supseteq x_r(g)$ and $x \supseteq x_s(h)$. Suppose $x \supset x' \supseteq x_r(g) \cup x_s(h)$. It follows that $x' = x_t(g)$ for some $t \in (t^*, \min(r, s))$. But then there exists $t^* < t' < t$ such that $g(t') \neq h(t')$. It follows that $h \notin x'$, a contradiction which establishes that $x = \min \uparrow x_r(g) \cap \uparrow x_s(h)$.

[16] Recall that the present notion of a tree corresponds to the dual of the concept of a pseudotree in order theory. If the chain $\uparrow x \cap \uparrow y$ had a minimum for all $x, y \in N$, this would correspond to a "well-joined" tree, the dual of the concept of a "well-met" (pseudo)tree in Baur and Heindorf (1997).

[17] If there exists $\min \uparrow x \cap \uparrow y$ for all $x, y \in N$, then $N \cup \{\emptyset\}$ is a *lattice*, where $x \wedge y = x \cap y$ and $x \vee y = \min \uparrow x \cap \uparrow y$ for all $x, y \in N$. For, by Trivial Intersection $x \wedge y = x \cap y \in \{\emptyset, x, y\} \subseteq N \cup \{\emptyset\}$ and, by the hypothesis, $x \vee y = \min \uparrow x \cap \uparrow y \in N$, for all $x, y \in N$.

But the tree of the differential game still allows strategies that induce multiple outcomes. More concretely, let the action space be $A = \{0, 1\}$ *and consider again the single-player perfect information problem* $\Pi(T)$*. Define choices as above and let* $\mathbf{1}$ *denote the function that is constant* 1*. Note that* $g \in x_t(f)$ *if and only if* $f \in x_t(g)$ *for all* $x_t(f), x_t(g) \in N$ *and all* $f, g \in W$*. Define the pure strategy* $s \in S$ *by* $s_t(x_t(f)) = c_t(f, 1)$ *if* $f \in x_t(\mathbf{1})$ *and* $s(x_t(f)) = c_t(f, 0)$ *otherwise. Consider the function* $f_r \in W$ *defined by* $f_r(t) = 1$ *for all* $t \in [0, r]$ *and* $f_r(t) = 0$ *for all* $t > r$*, for any real number* $r > 0$*. Then,*

$$\begin{aligned} R_s(f_r) &= \cap_{f_r \in x_t(g)} s(x_t(g)) = \cap_{g \in x_t(f_r)} s(x_t(g)) \\ &= \left[\cap_{t \le r} s(x_t(f_r))\right] \cap \left[\cap_{r<t} \cap_{g \in x_t(f_r)} s(x_t(g))\right] \\ &= \left[\cap_{t \le r} s(x_t(\mathbf{1}))\right] \cap \left[\cap_{r<t} \cap_{g \in x_t(f_r)} s(x_t(g))\right] \\ &= \left[\cap_{t \le r} c_t(\mathbf{1}, 1)\right] \cap \left[\cap_{r<t} \{h \in x_t(f_r) \,|\, h(t) = 0\}\right] \\ &= \left[\cap_{t \le r} c_t(\mathbf{1}, 1)\right] \cap \left[\cap_{r<t} \{h \in x_t(f_r) \,|\, h(\tau) = 0, \forall \tau \in (r, t]\}\right] = \{f_r\} \end{aligned}$$

In other words, for every $r > 0$ *the play* $f_r \in W$ *constitutes a fixed point of* R_s*, so the strategy* $s \in S$ *induces a continuum of outcomes!*[18]

7.2. Extensive Forms. In order to understand the origin of the multiplicity of outcomes, we begin by identifying a necessary condition for (A3).

Proposition 6. *Suppose for the tree* $T = (N, \supseteq)$ *of an EDP* (T, C) *there are* $y, y' \in N$ *with* $y \cap y' = \emptyset$ *such that*

$$y \subseteq c,\ y' \subseteq c',\ \text{and}\ c \cap c' = \emptyset\ \text{imply}\ P(c) \cap P(c') = \emptyset \tag{18}$$

for all $c, c' \in C_i$ *and all* $i \in I$*. Then, for every pair* $(w, w') \in y \times y'$ *there is* $s \in S$ *such that* $w \in R_s(w)$ *and* $w' \in R_s(w')$*.*

According to this Proposition, if for an EDP every strategy combination ought to induce a *unique* play, then condition (EDP.iii) necessarily has to be strengthened.

Definition 12. *An* ***extensive form*** *(EF) with player set* I *is an EDP which (instead of (EDP.iii)) satisfies*

(EDP.iii') if $y \cap y' = \emptyset$*, then there are* $i \in I$ *and* $c, c' \in C_i$ *such that* $y \subseteq c$*,* $y' \subseteq c'$*,* $c \cap c' = \emptyset$*, and* $P(c) \cap P(c') \neq \emptyset$*, for all* $y, y' \in N$*.*

This condition is almost identical to (EDP.iii) except that, additionally, the two choices $c, c' \in C_i$ have to be available simultaneously at some move, i.e. $P(c) \cap P(c') \neq \emptyset$, in (EDP.iii'). Without this stronger property multiple fixed points of R_s for some $s \in S$ cannot be avoided. As with (A2), condition (EDP.iii') combines a restriction on the tree with one on choices. To make this precise the following is introduced.

Definition 13. *A game tree* $T = (N, \supseteq)$ *is* ***selective*** *if, for all* $w, w' \in W$*,*

$$\text{if } w \neq w' \text{ then } \exists\, x \in X \text{ such that } w, w' \in x \text{ and } \gamma(x, w) \neq \gamma(x, w'). \tag{19}$$

[18] We are grateful to Nicolas Vieille for suggesting this example.

That a game tree is selective is purely a condition on the tree, though a strong one, as the first part of the next result shows. Its second part identifies a sufficient condition for selectiveness that is reminiscent of the conditions for playability (in fact, it implies coherence, but not weak up-discreteness).

Proposition 7. *(a) If a game tree $T = (N, \supseteq)$ is selective, then it is regular.*

(b) If a game tree $T = (N, \supseteq)$ is regular and every history h has a continuation g with a maximum $z \in g$, then T is selective.

That a selective tree is necessary for (A3) is shown by the next result. It states that whenever the tree fails to be selective, there is a strategy for some EDP that induces multiple outcomes.

Lemma 13. *If for game tree $T = (N, \supseteq)$ there are $w, w' \in W$, $w \neq w'$, such that*

$$\text{if } w, w' \in x \in X \text{ then } \gamma(x, w) \cap \gamma(x, w') \neq \emptyset$$

then there is an EDP defined on T and $s \in S$ such that $w \in R_s(w)$ and $w' \in R_s(w')$.

There is also a necessary condition for (A3) on the *choices* that is identified next.

Lemma 14. *If for an EDP (T, C) there are a move $x \in X$, a choice combination $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$, and plays $w, w' \in x$ such that $\gamma(x, w) \neq \gamma(x, w')$, but*

$$\gamma(x, w) \cup \gamma(x, w') \subseteq x \cap \left[\cap_{i \in J(x)} c_i\right],$$

then there is $s \in S$ such that $w \in R_s(w)$ and $w' \in R_s(w')$.[19]

The defect captured by the previous lemma can occur even if the tree is selective.

Example 16. *Reconsider the modified "hole in the middle," Example 11, where the node $[1/4, 3/4]$ has been added. Make this into a single-player EDP by assigning all nodes except the root plus $[1/4, 1/2) \cup (1/2, 3/4]$ as choices to the personal player. The singleton $\{1/2\}$ is a choice available at $\bar{x} = [1/4, 3/4] \in X$, so the decision at $\bar{x}$ is nontrivial. Since $\gamma(\bar{x}, 1/4) = [1/4, 1/2)$ and $\gamma(\bar{x}, 3/4) = (1/2, 3/4]$, the tree is selective. Still the construction of choices fulfills the hypothesis of Lemma 14.*

This shows that choices need to satisfy an extra condition, on top of selectiveness of the tree, to fulfill (A3). The following characterization spells out such a requirement and relates it to (EDP.iii') in the definition of an EF (Definition 12).

Proposition 8. *The EDP (T, C) is an EF if and only if T is selective and for all $x \in X$ and all $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ there is $w \in x$ such that $x \cap \left[\cap_{i \in J(x)} c_i\right] \subseteq \gamma(x, w)$.*

In the above statement the weak inclusion may as well be written as an equality, by Lemma 6 and Proposition 2(b). That is, Proposition 8 states that an EF could equivalently be defined by requiring a selective tree and replacing (EDP.ii) by

[19] Note that $\gamma(x, w) \neq \gamma(x, w')$ implies that $w \neq w'$.

(EDP.ii') $x \cap \left[\cap_{i\in J(x)} c_i\right] = \gamma(x, w)$ for some $w \in x \cap \left[\cap_{i\in J(x)} c_i\right]$, for all $(c_i)_{i\in J(x)} \in \times_{i\in J(x)} A_i(x)$ and for all $x \in X$.

This disentangles what (A3) requires on the tree—that it is selective—and what on the choices—namely condition (EDP.ii'). Combining the "only if"-part of Proposition 8 with Proposition 7(a) yields the following conclusion:

Corollary 6. *The tree of an EF is selective and, hence, regular.*

Example 17. *Reconsider the single-player problem $\Pi(T)$ defined on the differential game tree, as above. Recall the notation $c_t(g) = c_t(g, g(t))$ for choices (see Example 7). Let us show directly that the differential game tree fulfills property (EDP.iii). Let $x_r(g)$ and $x_s(h)$ be two disjoint nodes and $t^* = \inf\{t \geq 0 \,|\, g(t) \neq h(t)\}$, as in Example 15, so that $t^* < \min\{r, s\}$. If $g(t^*) \neq h(t^*)$, then that $x_r(g) \subseteq c_{t^*}(g)$, $x_s(h) \subseteq c_{t^*}(h)$, and $c_{t^*}(g) \cap c_{t^*}(h) = \emptyset$ establishes the claim. If $g(t^*) = h(t^*)$, there must be $t' \in (t^*, \min\{r, s\})$ such that $g(t') \neq h(t')$, and $x_r(g) \subseteq c_{t'}(g)$, $x_s(h) \subseteq c_{t'}(h)$, and $c_{t'}(g) \cap c_{t'}(h) = \emptyset$ verify the claim. These choices might not be "simultaneous alternatives"—as $c_{t^*}(g)$ and $c_{t^*}(h)$ were in the first case—in the sense that g and h may already differ before time t'.*

Condition (EDP.iii') fails in the differential game from Example 15, though. For, consider $\mathbf{1}$, $f_r \in W$ for some $r > 0$, where $\mathbf{1}$ is the constant function and $f_r(\tau) = 1$ for all $\tau \in [0, r]$ and $f_r(\tau) = 0$ for all $\tau > r$. If (EDP.iii') were to hold, then for any $t > r$ there would be $c, c' \in C_1$ (since $I = \{1\}$) such that $x_t(\mathbf{1}) \subseteq c$, $x_t(f_r) \subseteq c'$, $c \cap c' = \emptyset$, and $P(c) \cap P(c') \neq \emptyset$, because $t > r$ implies $x_t(\mathbf{1}) \cap x_t(f_r) = \emptyset$. Choosing $x \in P(c) \cap P(c')$ would then yield $c, c' \in A_1(x)$ and $\mathbf{1} \in c$, $f_r \in c'$, and $c \cap c' = \emptyset$. If $x = x_\tau(f)$ and $c, c' \in A_1(x_\tau(f))$ for some $\tau \in \mathbb{R}_+$ and $f \in W$, then $c = c_\tau(g)$ and $c' = c_\tau(g')$ for some $g, g' \in x_\tau(f)$; furthermore, $\mathbf{1} \in c_\tau(g)$ implies $c_\tau(g) = c = c_\tau(\mathbf{1})$. Consequently, if $\tau \leq r$ would hold, then $\mathbf{1} \in c = c_\tau(\mathbf{1})$ would imply $f_r \in c$, in contradiction to $c \cap c' = \emptyset$. If, on the other hand, $\tau > r$ would obtain, then $\mathbf{1} \in c = c_\tau(\mathbf{1})$ would imply that $\mathbf{1} \in x_\tau(f)$, hence $x_\tau(f) = x_\tau(\mathbf{1})$, but also that $f_r \notin x = x_\tau(f)$ and, therefore, $f_r \notin c_\tau(g)$ for any $g \in x_\tau(f) = x_\tau(\mathbf{1})$, in contradiction to $f_r \in c' \in A_1(x_\tau(f))$. Thus, (EDP.iii') must fail in the differential game, because there is no move, where $\mathbf{1} \in W$ and $f_r \in W$ get "sorted out."

Still, by construction (EDP.ii') holds true in this version of the differential game. Therefore, (EDP.iii') fails, because the tree of the differential game is not selective. It is, thus, a property of the tree that is responsible for the failure of (A3) in the differential game, and not "misspecified" choices.

This demonstrates that the single-player perfect information problem $\Pi(T)$ defined on the tree of the differential game is not an EF, because the tree is not selective.

Incidentally, this also shows that the converse of Proposition 7(a) is false: The tree of the differential game is regular, but not selective. (To see that the tree of the differential game is regular it suffices to notice that all nodes except the root are infinite, i.e. $x_t(f) = \inf \uparrow x_t(f) \setminus \{x_t(f)\}$ for all $f \in W$ and all $t > 0$.)

Remark 3. *For an EF, i.e. with (EDP.iii') instead of (EDP.iii), the set of undiscarded nodes for a given strategy combination, $U(s)$, is a chain. For, let $y, y' \in U(s)$*

and assume that $y \cap y' = \emptyset$. Choose $w \in y$ and $w' \in y'$ and let $x \in X$ be such that $w, w' \in x$ and $\gamma(x, w) \neq \gamma(x, w')$ which exists, because the tree is selective by Corollary 6. By (EDP.iv), Lemma 6, and Proposition 8 there are choice combinations $(c_j)_{j \in J(x)}, \left(c'_j\right)_{j \in J(x)} \in \times_{j \in J(x)} A_j(x)$ such that

$$y \subseteq \gamma(x, w) = x \cap \left[\cap_{j \in J(x)} c_j\right] \text{ and } y' \subseteq \gamma\left(x, w'\right) = x \cap \left[\cap_{j \in J(x)} c'_j\right]$$

with $c_i \neq c'_i$ for at least one $i \in J(x)$ (the latter by the choice of x and selectiveness, (19)). By (EDP.i), $c_i \cap c'_i = \emptyset$. If $c_i = s_i(x)$ (or $c'_i = s_i(x)$), then $y' \in D(s)$ (or $y \in D(s)$), in contradiction to the assumption. Hence, if $y, y' \in U(s)$, then either $y \subset y'$ or $y' \subseteq y$ by Trivial Intersection. In other words, $U(s)$ is a chain.

Even though the tree of an EF need not be weakly up-discrete (e.g. $\Pi(T)$ for the "modified hole in the middle," Example 11), the converse of Proposition 4 is also true for any EF, as it is for any EDP with a weakly up-discrete tree by Proposition 5.

Proposition 9. *Fix a history h for a game tree $T = (N, \supseteq)$. If for an arbitrary EF (T, C) every strategy combination induces outcomes after h, then for the single-player perfect information problem $\Pi(T)$ every strategy induces outcomes after h.*

Like with Proposition 5, this result says that if an EF is playable resp. everywhere playable, then *every* EF with the same tree is playable resp. everywhere playable.

7.3. A Uniqueness Result. Up to this point it has been argued that an EF—replacing (EDP.iii) by (EDP.iii')—is *necessary* for (A3). It will now be shown that an EF is also sufficient for (A3) to hold. The following is the key technical step.

Lemma 15. *Consider an EF as in Definition 12 and a pure strategy combination $s \in S$. If $w \in R_s(w)$ and $w' \in W \setminus \{w\}$, then there are $x \in X$ and $i \in J(x)$ such that $w \in s_i(x)$ and $w' \notin s_i(x)$.*

Theorem 5. *Consider an EF as in Definition 12 and fix a pure strategy combination $s \in S$. If $w \in R_s(w)$, then (a) $R_s(w) = \{w\}$, and (b) if $w' \in R_s(w')$ then $w' = w$.*

Proof. (a) Let $s \in S$, assume that $w \in R_s(w)$, and consider any $w' \in R_s(w)$. If $w' \in W \setminus \{w\}$, then by Lemma 15 there are $x \in X$ and $i \in J(x)$ such that $w \in s_i(x)$ and $w' \notin s_i(x)$. But this contradicts $w' \in R_s(w)$. Hence, $w' = w$.

(b) Let again $s \in S$ and assume $w \in R_s(w)$. Consider any $w' \in W$ such that $w' \in R_s(w')$. If w' were not equal to w, then, again by Lemma 15, there would be $x \in N$ and $i \in J(x)$ such that $w \in s_i(x)$ and $w' \notin s_i(x)$. Since the latter would contradict $w' \in R_s(w')$, it follows that $w' = w$. ■

Since the strengthening of (EDP.iii) to (EDP.iii') is necessary for (A3), this result means that (EDP.iii') is *precisely* what is needed for strategies to induce unique outcomes. Recall, however, that (EDP.iii') is a combination of a condition on the tree, namely selectiveness, (19), and one on choices, namely (EDP.ii'). The condition on choices, (EDP.ii'), becomes redundant, though, when the tree is weakly up-discrete.

Proposition 10. *An EDP (T, C) with a weakly up-discrete tree $T = (N, \supseteq)$ is an EF if and only if T is selective.*

In the class of weakly up-discrete trees (A3) is, therefore, characterized by selective trees. Note that weak up-discreteness and selectiveness are independent properties. The "hole in the middle," Example 2, has a weakly up-discrete tree that is not selective: There is no $x \in X$ such that $1/4, 3/4 \in x$ and $\gamma(x, 1/4) \cap \gamma(x, 3/4) = \emptyset$, because a node $[1/4, 3/4]$ is absent from this tree. For, $1/4, 3/4 \in x$ implies $x \in \{x_t\}_{t=1}^{\infty}$, yet $\gamma(x_t, 1/4) = \gamma(x_t, 3/4) = x_{t+1}$ for all $t = 1, 2, ...$ But the "modified hole in the middle," Example 11, has a selective tree, because the node $[1/4, 3/4]$ has been added. It is not weakly up-discrete, though, because nodes of the form $[1/4, 1/2)$ and $(1/2, 3/4]$ are still missing, i.e., the chains $\{y_t\}_{t=1}^{\infty}$ and $\{y'_t\}_{t=1}^{\infty}$ have no maxima.

8. A Joint Characterization

Combining Theorems 2, 3, 4, and 5 finally yields a characterization of EDPs that satisfy the three desiderata (A1), (A2), and (A3).

Theorem 6. *An EDP (T, C) satisfies (A1), (A2), and (A3) if and only if the game tree $T = (N, \supseteq)$ is regular, weakly up-discrete, and coherent.*

Proof. "if": First, (A1) holds true for any EDP by Theorem 2. Second, by Corollary 1 that the tree is weakly up-discrete implies that it has available choices. Because the tree is weakly up-discrete and coherent by hypothesis, Theorem 3 implies that every strategy induces an outcome after every history. In particular, every strategy induces an outcome after the null history that consists only of the root. Hence, (A2) holds true.

To see (A3), observe first that every history without minimum has a continuation with a maximum by coherence. Second, for any history with a minimum all continuations have maxima by Lemma 8(b) and weak up-discreteness. Then regularity and Proposition 7(b) imply that the tree is selective. Consequently, the EDP (T, C) is an EF by (the "if" part of) Proposition 10. It follows from Theorem 5 that (A3) holds.

"only if": The uniqueness criterion (A3) implies by Proposition 6 that the EDP (T, C) must be an EF. Therefore, the tree is regular by Corollary 6. The playability criterion (A2) for regular trees implies that the tree is weakly up-discrete and coherent by Theorem 4. ■

The three characterizing properties are independent: Eliminate in the "hole in the middle," Example 2, the element $1/2$ from the underlying set $W = [0, 1]$ of plays (and $\{1/2\}$ from N); the resulting tree is regular and weakly up-discrete, but not coherent. Modify further by adding the node $[1/4, 3/4] \setminus \{1/2\}$; the resulting tree is regular and coherent, but not weakly up-discrete. Finally, in the original "hole in the middle" example (with $1/2$ and $\{1/2\}$) add the nodes $[1/4, 1/2)$ and $(1/2, 3/4]$; the resulting tree is weakly up-discrete and coherent, but not regular.

Recall that if the EDP is playable, then all EDPs with the same tree are playable by Propositions 3 and 9.

Corollary 7. *If an EDP satisfies (A1), (A2), and (A3), then so does every EDP with the same tree.*

Moreover, by Corollary 5 a regular tree is up-discrete if and only if it is weakly up-discrete and coherent. Combining this with Theorem 4 yields the following:

Corollary 8. *An EDP satisfies (A1), (A2), and (A3) if and only if its tree is regular and up-discrete. Furthermore, the EDP is then everywhere playable.*

Note, however, that all these results could also be stated for the combination of (A2) and (A3) alone, since (A1) comes for free by Theorem 2.

9. Discrete Trees

To the best of our knowledge all games that have been studied in the literature—with the exception of differential games—satisfy (A1)-(A3) from the joint characterization. This is because they usually satisfy even more stringent conditions than regularity and up-discreteness. In particular, the differential games and the transfinite cheap-talk game by Aumann and Hart (2003) seem to be the only ones which fail the following property that is complementary to up-discreteness.

Definition 14. *A game tree $(N, \supseteq)$ is* ***down-discrete*** *if the chain $\uparrow x \setminus \{x\}$ has an infimum in $E \cup \uparrow x \setminus \{x\}$ for all $x \in N \setminus \{W\}$. It is* ***discrete*** *if it is up-discrete and down-discrete.*

That is, a game tree is down-discrete if for all $x \in N \setminus \{W\}$ the chain $\uparrow x \setminus \{x\}$ has either a minimum or an infimum in the set of terminal nodes. In particular, $\uparrow x \setminus \{x\}$ has always an infimum, so there are no strange nodes. Moreover, if $\uparrow x \setminus \{x\}$ has no minimum, x must be its infimum and, therefore, x is terminal. If, conversely, a game tree is regular and all moves are finite, then for every $x \in X$ there exists $\min \uparrow x \setminus \{x\}$ by the definition of finite nodes; furthermore, any non-finite node $x \in N \setminus F(N)$ must then be terminal and the infimum of $\uparrow x \setminus \{x\}$ by Lemma 2(a) and (b), as there are no strange nodes by regularity. This demonstrates:

Proposition 11. *A game tree is down-discrete if and only if it is regular and all moves are finite (i.e. $X \subseteq F(N)$).*

Infinitely repeated games, infinite bilateral bargaining games (Rubinstein (1982)), or stochastic games (Shapley (1953)) all employ discrete trees, as do Aumann's (1964) infinite extensive form games. "Long" cheap-talk à la Aumann and Hart (2003) employs a tree that satisfies (A1)-(A3), but is not down-discrete.

Example 18. *($\omega + 1$-**centipede**) The latter cheap-talk game has a time axis of order $\omega + 1$, where ω denotes the first infinite (limit) ordinal and $\omega + 1$ its successor according to the usual well-order. A similar example is obtained by considering the W-centipede (Example 3) with $W = \{1, 2, ..., \omega, \omega + 1\}$. There, the infimum of the chain $\{x_t \,|1 \leq t < \omega\} = \uparrow x_\omega \setminus \{x_\omega\}$ is the move x_ω. Hence, this tree is not down-discrete, even though it is regular and up-discrete.*

An EDP defined on the tree of the previous example, like the "long" cheap-talk game, will satisfy the three desiderata (A1), (A2), and (A3), even though the tree is not discrete—in contrast to differential games that fail (A2) and (A3). So, discreteness is more than what is needed. But it has convenient implications. A particularly striking consequence of discrete trees is captured by the final theorem. It characterizes discrete trees, within the class of regular game trees, by the existence of an *immediate predecessor function* $p : F(N) \to X$.[20]

Theorem 7. *For a rooted and regular game tree $(N, \supseteq)$ the following three statements are equivalent:*

(a) $(N, \supseteq)$ is discrete;

(b) all infinite nodes are terminal ($X \subseteq F(N)$) and there exists a surjection $p : F(N) \to X$ such that for all $x \in F(N)$

$$x \subset p(x) \text{ and if } x \subset y \in N \text{ then } p(x) \subseteq y \subseteq \cup_{t=1}^{\infty} p^t(x) \tag{20}$$

where $p^1 = p$, $p^t = p \circ p^{t-1}$ for all $t = 2, 3, ...$, and $p(W) = W$ as a convention;

(c) $X \subseteq F(N)$ and, for all $x, y \in F(N)$, the chain $\uparrow x \cap \downarrow y = \{z \in N \,|\, x \subseteq z \subseteq y\}$ is finite.

Proof. "(a) implies (b)": Since discreteness entails down-discreteness, that $X \subseteq F(N)$ follows from Proposition 11. Define $p : F(N) \to X$ by $p(x) = \min \uparrow x \setminus \{x\}$ which exists if and only if $x \in F(N)$ by Lemma 3. Then, $x \subset p(x)$ and if $x \subset y \in N$, then $y \in \uparrow x \setminus \{x\}$ implies $p(x) \subseteq y$, for any $x \in F(N)$.

Moreover, for $x \in F(N)$ consider the chain $\{p^t(x)\}_{t=1}^{\infty}$. Since the tree is up-discrete, it has a maximum $z = p^k(x)$. Therefore, $p(z) = p^{k+1}(x) = p^k(x) = z$ which contradicts the construction of p, unless $z = W$. Hence, $\cup_{t=1}^{\infty} p^t(x) = W \supseteq y$ for any $y \in \uparrow x \setminus \{x\}$.

Because discreteness entails weak up-discreteness, for any $x \in X$ there exists a maximum y for a maximal chain in $\downarrow x \setminus \{x\}$ (where the latter is nonempty by $x \in X$). Since $y \subset x$, it follows that $p(y) \subseteq x$. If $p(y) \neq x$, then $p(y) \in \downarrow x \setminus \{x\}$ with $y \subset p(y)$ contradicts the construction of y. Therefore, $p(y) = x$ and, because $x \in X$ was arbitrary, p is surjective.

"(b) implies (c)": That $X \subseteq F(N)$ is immediate. Thus, let $x, y \in F(N)$ and $x \subset y$. Then $p(x) \subseteq y \subseteq \cup_{t=1}^{\infty} p^t(x)$ by (20). It follows that there must be some $k \geq 1$ such that $p^{k-1}(x) \subset y \subseteq p^k(x)$. But by (20) applied to $p^{k-1}(x)$ this implies $y = p^k(x)$. Hence, the chain $\uparrow x \cap \downarrow y$ is contained in the finite chain $\{x, p(x), p^2(x), ..., p^k(x) = y\}$ and, therefore, itself finite. If $x, y \in F(N)$ are such that $x = y$, then $\uparrow x \cap \downarrow y = \{x\}$ is finite as well. If $x, y \in F(N)$ are such that $y \subset x$ or $x \cap y = \emptyset$, then $\uparrow x \cap \downarrow y$ is empty and, thus, contains no element.

"(c) implies (a)": If the tree is trivial, $N = \{W\}$, there is nothing to prove. If it is nontrivial, then $W \in X$ and, therefore, $W \in F(N)$ by the hypothesis. If $x \in F(N) \setminus \{W\}$, then $\uparrow x \cap \downarrow W = \uparrow x$ is a finite nonempty chain by hypothesis and, thus, has a minimum. If $x \in N \setminus F(N)$, then x is terminal by the hypothesis that $X \subseteq F(N)$. It follows from regularity that $x \in E$ is then infinite and, therefore, the infimum of the chain $\uparrow x \setminus \{x\}$ by Lemma 2(a) and (b). Hence, the tree is down-discrete.

Next, verify up-discreteness. Let $h \in 2^N$ be any chain in N. The set $W \in N$ is an upper bound for h. Then, for any $x \in h$ the chain $\{y \in N \,|\, x \subseteq y \subseteq W\}$ is finite by hypothesis. Enumerate its elements, $x = x_0 \subset x_1 \subset x_2 \subset ... \subset x_m = W$, and let t be the largest

[20] Recall that a node has an immediate predecessor if and only if it is finite by Lemma 3.

integer such that $x_t \in h$. We claim that x_t is a maximum for h. Choose any $z \in h$. If $z \subseteq x$ then $z \subseteq x = x_0 \subseteq x_t$. If $x \subset z$ there is $0 \leq \tau < m$ such that $z = x_\tau$ and, therefore, $z = x_\tau \subseteq x_t$. It follows that x_t is a maximum for h. ■

The hypothesis of a regular game tree is necessary for Theorem 7. This is illustrated by the examples of "Twins" (see AR, Example 13) that satisfies both (b) and (c) of Theorem 7, but is not regular and, hence, not down-discrete (by the "only if"-part of Proposition 11). The following example illustrates that (weak) up-discreteness is responsible for the very last part of (20), viz. that the root can be reached from a move by iterating the immediate predecessor function.

Example 19. *Reconsider the "augmented inverse infinite centipede," Example 9. This tree is down-discrete. All nodes are finite (because* $\uparrow x_t \setminus \{x_t\} = \{x_\tau\}_{\tau=t+1}^{\infty}$ *and* $\uparrow \{t\} \setminus \{t\} = \{x_\tau\}_{\tau=t}^{\infty}$ *for all* t*) and the immediate predecessor function is given by* $p(\{t\}) = x_t$ *and* $p(x_t) = x_{t+1}$ *for all* $t = 1, 2, ...$ *But, if the predecessor function is iterated from any node, the root will never be reached, i.e.,* $\cup_{k=1}^{\infty} p^k(x) \subset W$, *even though* $p(\infty) = W$ *(that is, even though the immediate predecessor function is surjective). This is so, because no chain in* $\downarrow W \setminus \{W\}$ *has a maximum.*

Discrete trees are used in a textbook exposition by Ritzberger (2002). Another prominent example for discrete trees in a textbook treatment is provided by the trees that Osborne and Rubinstein (1994) use.

Example 20. ***(Osborne-Rubinstein trees)*** *Let* A *be an arbitrary set of "actions" and* Z *a set of (finite or infinite) sequences from* A *such that (a)* $\emptyset \in Z$, *(b) if* $(a_\tau)_{\tau=1}^{t} \in Z$ *(where* t *may be infinite) and* $k < t$, *then* $(a_\tau)_{\tau=1}^{k} \in Z$, *and (c) if an infinite sequence* $(a_\tau)_{\tau=1}^{\infty}$ *satisfies* $(a_\tau)_{\tau=1}^{t} \in Z$ *for every positive integer* t, *then* $(a_\tau)_{\tau=1}^{\infty} \in Z$. *The set* W *of plays is given by* $W = W_\infty \cup W_F$ *where*

$$W_F = \left\{ z \in Z \,\middle|\, \exists\, t = 1, 2, ... : z = (a_\tau)_{\tau=1}^{t} \text{ and } \nexists\, a_{t+1} \in A : (a_\tau)_{\tau=1}^{t+1} \in Z \right\}$$

and $W_\infty = \{z \in Z \,|\, z = (a_t)_{t=1}^{\infty}\}$. *The ordering on nodes is given by* $(a_\tau)_{\tau=1}^{t} \geq (a'_\tau)_{\tau=1}^{k}$ *if* $t \leq k$ *and* $a_\tau = a'_\tau$ *for all* $\tau = 1, ..., t$, *and* $\emptyset \geq z$ *for all* $z \in Z$. *Then,* $(Z, \geq)$ *is a (rooted) tree. It may not be a decision tree, though, because a given node may be followed only by one other node. This can be fixed by adding the following condition: (d) If* $(a_\tau)_{\tau=1}^{t} \in Z$, *then there exists* $a'_t \neq a_t$ *such that* $\left((a_\tau)_{\tau=1}^{t-1}, a'_t\right) \in Z$. *Under this condition,* $(Z \setminus \{\emptyset\}, \geq)$ *is a discrete tree. (If* $(a_\tau)_{\tau=1}^{t} \in Z \setminus \{\emptyset\}$ *implies* $t = \infty$, *then this is the tree of an infinitely repeated game, a "supergame.")*

Define N *as the collection of subsets of* W *given by* $\{\{w\}\}_{w \in W_\infty}$ *together with all sets of the form* $W(z) = \{w \in W \,|\, z \subseteq w\}$ *for all finite sequences* $z \in Z$. *Then* $(N, \supseteq)$ *is the set representation by plays of* $(Z, \geq)$ *and a discrete game tree (notice that* $W(\emptyset) = W$*).*

Most textbooks, of course, simply use finite trees. Call a game tree $(N, \supseteq)$ *finite* if every chain in N has a maximum and a minimum. This definition has no implications on the number of choices available at any given move (i.e. on the number of successors of a given move). But it does restrict the lengths of plays.

For, if there were an infinite play for a finite game tree $(N, \supseteq)$, then this play would have to have a minimum $x \in N$ by definition. Since a finite game tree is clearly discrete, Theorem 7(c) would then imply that the chain $\uparrow x \cap \downarrow W$ is finite; but, since x is the minimum of the play, $\uparrow x \cap \downarrow W$ equals the play—a contradiction. Hence, for a finite game tree all plays have finite length. Conversely, a game tree, where all plays have finite length, is clearly a finite game tree.

Moreover, for finite game trees all nodes must be finite. For, if all plays have finite length, then all chains of the form $\uparrow x \setminus \{x\}$ (for a node $x \in N$) also do. Hence, all these chains have minima, implying that $N = F(N)$.[21] Consequently, by Theorem 7(b), the immediate predecessor function is defined on *all* nodes for finite game trees.

10. Conclusions

The concept of a non-cooperative game—that is, a game with complete rules—extends well beyond the confines of finite games. Infinitely repeated games, stochastic games (Shapley (1953)), and even differential games can be rigorously defined as games in extensive form. This step verifies that the rules of these games are complete.

Whether the rules of such games are also consistent is a different matter, though. It may well be that at each "when" it is fully specified "who can do what," but at the same time a global specification of such instructions—that is, a strategy combination—may not yield an outcome at all or multiple outcomes. In this paper we seek, therefore, to characterize extensive form games that satisfy three global criteria: (A1) For every outcome/play there is a strategy combination that induces it; (A2) every strategy combination does induce some outcome/play; (A3) the outcome/play induced by a strategy combination is unique.

It is shown that the class of EDPs satisfying (A1), (A2), and (A3) is fully characterized by three properties of the underlying game tree: regularity, coherence, and weak up-discreteness. The latter two together characterize the class of "everywhere playable" EDPs. Those are the ones that allow for a truly dynamic analysis, in the sense that strategies induce outcomes after every possible history—a principle underlying any backwards induction procedure. Regularity of the tree is added by virtue of the uniqueness criterion (A3).

The characterization result allows us to draw a dividing line between those games in the literature that can be defined on game trees with the above three properties, and those that cannot. As a rule, almost all games in the literature turn up on the safe side of this line. The only exception concerns differential games. Though the tree of a differential game is regular, it fails to be up-discrete. As a consequence, differential games allow for strategy combinations that do not induce any outcome/play at all—differential games are not "playable." Furthermore, they allow for strategy combinations that induce continua of outcomes/plays—differential games are not "extensive forms."

[21] The converse is not true, not even in the class of discrete trees. Examples can be constructed of discrete game trees, where all nodes are finite, and the chains $\uparrow x \cap \downarrow y$ are finite for all $x, y \in N$, but the tree is still not finite.

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A. APPENDIX

A.1. Proofs of results in Section 3.

Proof of Lemma 1. The "if"-part is trivial. To see the "only if"-part, let $x \in E$ and assume that $v, w \in x$. If $v \neq w$, then by Irreducibility there are $x', y' \in N$ such that $v \in x' \setminus y'$ and $w \in y' \setminus x'$. Hence, $v \in x \cap x'$ and $w \in x \cap y'$. Since $v \notin y'$ and $w \notin x'$, neither $x \subseteq x'$ nor $x \subseteq y'$. Therefore, by Trivial Intersection $x' \subset x$ and $y' \subset x$. But this contradicts $x \in E$. Therefore, $v = w$. ■

Proof of Lemma 2. (a) The "if"-part is trivial. For the "only if" part, let $x = \inf h$. Obviously, $x \subseteq \cap_{y \in h} y$. Fix $w \in x$, and suppose there exists $w' \in \cap_{y \in h} y$ such that $w' \notin x$. By Irreducibility there are $z, z' \in N$ such that $w \in z \backslash z'$ and $w' \in z' \backslash z$.

Consider any $y \in h$. Since $w' \in z' \backslash z$ and $w' \in y$, it follows from Trivial Intersection that either $y \subseteq z'$ or $z' \subseteq y$. In the first case, it would follow that $x \subseteq z'$, a contradiction with $w \in z \backslash z'$. Hence, $z' \subseteq y$ for all $y \in h$, i.e., z' is a lower bound for the chain h. Since x is its infimum, we have that $z' \subseteq x$, a contradiction with $w' \notin x$. This shows that $x = \cap_{y \in h} y$.

(b) follows from (a).

(c) "if": Assume that $\uparrow x \backslash \{x\}$ has no minimum and let $w \in x$ and $w' \in \left(\cap_{y \in \uparrow x \backslash \{x\}} y\right) \backslash x$. By Irreducibility and Trivial Intersection there are $y, y' \in N$ such that $w \in y$, $w' \in y'$, and $y \cap y' = \emptyset$. Since $w, w' \in z$ for all $z \in \uparrow x \setminus \{x\}$, it must be the case that $y \cup y' \subseteq z$ for all $z \in \uparrow x \setminus \{x\}$ by Trivial Intersection. Moreover, $x \cap y' = \emptyset$, because otherwise $y' \subseteq x$ would imply $w' \in x$ and $x \subset y'$ would imply $w \in y'$, both in contradiction to the construction. Thus, both y and y' are lower bounds for $\uparrow x \setminus \{x\}$ which, therefore, has no infimum.

"only if": If $x \in N$ is strange, then $\uparrow x \setminus \{x\}$ has no infimum, in particular, no minimum. If $x = \cap_{y \in \uparrow x \backslash \{x\}} y$ would hold, then by (a) $x = \inf \uparrow x \setminus \{x\}$ and x could not be strange.

(d) "if": Let x be an infinite node. Consider the chain $h = \uparrow x \backslash \{x\}$. By (b), $x = \cap_{y \in h} y$, but $x \subset y$ for all $y \in h$, hence the stated property cannot hold.

"only if": Suppose x does not verify the stated property. There exists a chain h in N such that $x \supseteq \cap_{y \in h} y$, but there is no $y \in h$ such that $x \supseteq y$. Since $x \supseteq \cap_{y \in h} y$ implies $x \cap y \neq \emptyset$ (because $\cap_{y \in h} y \neq \emptyset$, since the chain h has a lower bound in W because the tree is bounded) for all $y \in h$, it follows from Trivial Intersection that $x \subset y$ for all $y \in h$, i.e., $y \in \uparrow x \backslash \{x\}$ for all $y \in h$. Thus, $x \supseteq \cap_{y \in h} y \supseteq \cap_{y \in \uparrow x \backslash \{x\}} y \supseteq x$. Hence, x is infinite. ■

Proof of Proposition 1. Recall that, since $(N, \supseteq)$ is a (complete) game tree, we can take the set W both as the set of plays and as the underlying set.

Trivial Intersection for $(N, \supseteq)$ is trivially inherited by $(N \setminus Y, \supseteq)$. If c is a chain in $N \setminus Y$ then it is a chain in N, therefore, has a lower bound in W because $(N, \supseteq)$ is bounded. Thus $(N \setminus Y, \supseteq)$ is bounded.

To verify Irreducibility, let $v, v' \in W$ be such that $v \neq v'$. By Irreducibility for $(N, \supseteq)$ there are $x, x' \in N$ such that $v \in x \setminus x'$ and $v' \in x' \setminus x$. If $x, x' \in N \setminus Y$, we are done. If $x \in Y \subseteq E_I$ then by Lemma 1 $x = \{v\}$. Since x is infinite, $\{v\} = \cap_{y \in \uparrow x \backslash \{x\}} y$ by Lemma 2(b). Hence, there is $z \in \uparrow x \setminus \{x\}$ such that $v \in z$ and $v' \notin z$. If $x' \in N \setminus Y$ then $v \in z \setminus x'$ and $v' \in x' \setminus z$, as required. If $x' \in Y \subseteq E_I$ then, analogously, $x' = \{v'\}$ and there is $z' \in \uparrow x' \setminus \{x'\}$ such that $v' \in z'$ and $v \notin z'$. Thus, $(N \setminus Y, \supseteq)$ is a bounded irreducible W-set tree, i.e. a game tree.[22]

[22] Since the underlying set has not changed, Theorem 3 in AR is enough to establish that the set of plays in the new tree must be bijective with the set of plays in the former one. The proof we give here, however, is constructive.

Let $w' \in W'$. If $w' \in W$, there is a unique play for $(N, \supseteq)$, namely w' itself, such that $\Upsilon(w') = w'$. If $w' \notin W$ then there is $y \in Y$ such that $w' \cup \{y\}$ is a chain for $(N, \supseteq)$. Because $y \in Y \subseteq E_I$, by Lemma 1 $y = \{v\}$ for some $v \in W$. Therefore, $v \in x$ for all $x \in w'$, since $w' \cup \{y\}$ is a chain. Since $(N \setminus Y, \supseteq)$ is a game tree, Theorem 3(c) of AR implies that $\uparrow \{v\} \setminus \{\{v\}\} \in W'$ is a play (because $\{v\} \notin N \setminus Y$), so that $w' \subseteq \uparrow y \setminus \{y\}$ and maximality imply $w' = \uparrow y \setminus \{y\}$. Because y is infinite, $y = \cap_{x \in w'} x$ by Lemma 2(b), and there can be no other $y' \in Y \setminus \{y\}$ such that $w' \cup \{y'\}$ is a chain for $(N, \supseteq)$. Therefore, $w = w' \cup \{y\}$ is the unique play for $(N, \supseteq)$ such that $\Upsilon(w) = w'$. Since this shows that every element of W' has a unique preimage under Υ, this mapping is bijective.

Finally, since all terminal nodes are singletons by Lemma 1, adding to $(N \setminus Y, \supseteq)$ all singletons must yield the (complete) game tree $(N, \supseteq)$ with all singletons present. ■

A.2. Proofs of results in Section 4.

Proof of Lemma 3. If x is a finite node, $P(x) = \{\min \uparrow x \setminus \{x\}\}$ follows from uniqueness of the minimum. To see the "only if"-part, assume that $x' \in P(x)$ for $x \in N$. Then there is $y \in \downarrow x$ such that $\uparrow x' = \uparrow y \setminus \downarrow x$. Since $\uparrow y \setminus \downarrow x = \uparrow x \setminus \{x\}$ for all $y \in \downarrow x$, that $\uparrow x' = \uparrow x \setminus \{x\}$ implies $x' = \min \uparrow x \setminus \{x\}$, thus x is finite. ■

The following Lemma is needed to prove Proposition 2.

Lemma 16. *Let $(N, \supseteq)$ be a game tree with set of plays W. If $w, w' \in x \in X$, then there are $z, z' \in \downarrow x \setminus \{x\}$ such that $w \in z$ and $w' \in z'$, where $w \neq w'$ implies $z \cap z' = \emptyset$.*

Proof. First, $x \in X$ implies $\downarrow x \setminus \{x\} \neq \emptyset$. Suppose $w \neq w'$. By Irreducibility and Trivial Intersection there are $z, z' \in N$ such that $w \in z$, $w' \in z'$, and $z \cap z' = \emptyset$. Since $w \in x \cap z$ ($w' \in x \cap z'$), by Trivial Intersection either $x \subseteq z$ ($x \subseteq z'$) or $z \subset x$ ($z' \subset x$). Since $x \subseteq z$ ($x \subseteq z'$) would contradict $w' \notin z$ ($w \notin z'$), it follows that $z \subset x$ ($z' \subset x$). If $w = w'$, the conclusion follows taking $w'' \neq w$, $w'' \in x$, which is possible because $x \in X$ (recall Lemma 1). ■

Proof of Proposition 2. (a) Let $w'' \in \gamma(x, w) \cap \gamma(x, w')$. Then there are nodes $z, z' \in \downarrow x \setminus \{x\}$ such that $w, w'' \in z$ and $w', w'' \in z'$. Since $w'' \in z \cap z'$, Trivial Intersection implies either $z \subset z'$ or $z' \subseteq z$. If $z \subset z'$, then $w \in z'$ implies $\gamma(x, w') = \cup_{y \in \uparrow z' \setminus \uparrow x} y = \gamma(x, w)$. If $z' \subseteq z$, then $w' \in z$ implies $\gamma(x, w) = \cup_{y \in \uparrow z \setminus \uparrow x} y = \gamma(x, w')$.

(b) Denote $c = \gamma(x, w)$. We know that $c \subset x$. Since $x \in X$, there is some $y \in \downarrow x \setminus \{x\}$ such that $w \in y$ by Lemma 16 and, therefore, $y \in \downarrow c$. If $z \in \uparrow x$, then $y \subseteq c \subset x \subseteq z$ implies $z \in \uparrow y$ and $z \notin \downarrow c$, so $z \in \uparrow y \setminus \downarrow c$. Since $z \in \uparrow x$ was arbitrary, $\uparrow x \subseteq \uparrow y \setminus \downarrow c$. If $z \in \uparrow y \setminus \downarrow c$, i.e. $y \subseteq z$ and $z \setminus c \neq \emptyset$, then $x \subseteq z$. For, $y \subseteq z \cap x$ implies $z \subset x$ or $x \subseteq z$ by Trivial Intersection; but $z \subset x$ would imply $z \subseteq c$, because $w \in y \subseteq z \in \downarrow x \setminus \{x\}$, in contradiction to $z \setminus c \neq \emptyset$, so that $x \subseteq z$ must obtain. Since $z \in \uparrow y \setminus \downarrow c$ was arbitrary, $\uparrow x \supseteq \uparrow y \setminus \downarrow c$. Hence, $x \in P(c)$.

We show now that $P(c) = \{x\}$. Let $x' \in P(c)$. Then there is $y' \in \downarrow c$ such that $\uparrow x' = \uparrow y' \setminus \downarrow c$. Since $y' \in \downarrow c$, there exists some $z \in \downarrow x \setminus \{x\}$ such that $w \in z$ and $y' \subseteq z$. Thus $z \in \downarrow c$. Since $y' \subseteq x'$ but $x' \notin \downarrow c$, Trivial Intersection implies $z \subseteq x'$, in particular $w \in x'$. Then, $w \in x' \cap x$ implies, again by Trivial Intersection, that $x \subset x'$ or $x' \subseteq x$. If $x \subset x'$, then $x \in \uparrow y' \setminus \downarrow c$ but $x \notin \uparrow x'$, a contradiction. If $x' \subset x$, that $w \in x'$ implies $x' \in \downarrow c$, a contradiction. Thus $x = x'$. It follows that $P(c) = \{x\}$.

To prove the second statement, choose $w' \in x \setminus c$ and let $c' = \gamma(x, w')$. If $c \cap c' \neq \emptyset$ would hold, part (a) would imply that $c = c'$, yielding the contradiction $w' \in c$. Therefore, $c \cap c' = \emptyset$. Since $c' \subseteq x \setminus c$, it also follows that $x \in P(c')$.

(c) We now suppose $\gamma(x, w) = x$. If the chain $h = \{z \in \downarrow x \setminus \{x\} \,|\, w \in z\}$ had a maximum $y \in \downarrow x \setminus \{x\}$, then $\cup_{w \in z \in \downarrow x \setminus \{x\}} z = \gamma(x, w) = y \subset x$, in contradiction to the hypothesis. This proves the first part of the claim. To see the second one, suppose that an EDP is defined on this tree. Because $x \in X$ and $w \in x$, there is $z \in N$ such that $w \in z \subset x$ by Lemma 16, hence, $\gamma(x, w) = \cup_{y \in \uparrow z \setminus \uparrow x} y$. Since $z \subset x$, there is $c_i \in A_i(x)$ such that $z \subseteq c_i$ for all $i \in J(x) \neq \emptyset$ by (EDP.iv). By $x \in P(c_i)$ there is $y_i \in \downarrow c_i$ such that $\uparrow x = \uparrow y_i \setminus \downarrow c_i$ for all $i \in J(x)$. Since $x \notin \downarrow c_i$, there is $w_i \in x \setminus c_i$ for all $i \in J(x)$. Yet, by hypothesis there is $x_i \in \uparrow z \setminus \uparrow x$ such that $\{w_i\} \cup y_i \subseteq x_i$ for all $i \in J(x)$. Because $x_i \in \uparrow y_i$, it follows from $\uparrow x = \uparrow y_i \setminus \downarrow c_i$ that $x_i \in \downarrow c_i$ (i.e. $x_i \subseteq c_i$), in contradiction to $w_i \notin c_i$, for all $i \in J(x)$. ■

A.3. Proofs of results in Section 6.

Proof of Lemma 4. (a) Let $w \in W(h)$. Then, $w \in x$ and hence $x \in \uparrow \{w\}$ for all $x \in h$. Hence $\uparrow \{w\}$ is a play containing h and $\uparrow \{w\} = h \cup g$ for the continuation $g = \uparrow \{w\} \setminus h$. Conversely, let $w \in W$ be such that $\uparrow \{w\} = h \cup g$ for some continuation g of h. Then, $x \in \uparrow \{w\}$ and hence $w \in x$ for all $x \in h$. Hence, $w \in W(h)$. This proves the equality. $W(h)$ is nonempty by definition of game tree.

(b) is an immediate consequence of Lemma 2(a). ■

Proof of Proposition 3. "if:" Since $T = (N, \supseteq)$ is a game tree, it is bounded. Let $u \in U^h(s)$ be a chain that is maximal in $\{x \in N \,|\, x \subseteq W(h)\}$. It follows that $u \cup h$ is maximal in N. Because $u \cup h$ is a chain, there is $w \in W$ such that $w \in \cap_{x \in u \cup h} x$ by boundedness (see (7)). The fact that $u \cup h$ is maximal in N and Theorem 3(c) of AR then imply that $\uparrow \{w\} = u \cup h$. Hence, $w \in R_s^h(w) = \cap_{x \in u} \cap_{i \in J(x)} s_i(x)$.

"only if:" Let $w \in W$ such that $w \in R_s^h(w)$. By construction, $u = \uparrow \{w\} \setminus h$ is contained in $U^h(s)$. Since $\uparrow \{w\}$ is a play and h is a history, it follows that u is a maximal chain in $\{x \in N \,|\, x \subseteq W(h)\}$. ■

Proof of Proposition 4. Pick any $s \in S$ and construct $s' \in S'$ as follows: By (EDP.ii) and the Axiom of Choice we can select $w_x \in x \cap \left[\cap_{i \in J(x)} s_i(x)\right]$ for every $x \in X$; set $s'(x) = \gamma(x, w_x)$ for all $x \in X$. Then $s'(x) \subseteq x \cap \left[\cap_{i \in J(x)} s_i(x)\right]$ for all $x \in X$ by Lemma 6, and $U^h(s') \subseteq U^h(s)$ by Lemma 7. By hypothesis and the "only if"-part of Proposition 3 $U^h(s')$ contains a maximal chain u that is also maximal in $\{x \in N \,|\, x \subseteq W(h)\}$. By $U^h(s') \subseteq U^h(s)$ this chain u is also contained in $U^h(s)$. Hence, the "if"-part of Proposition 3 implies the statement. ■

Proof of Lemma 8. "(a) implies (b)": If h is a history with a minimum x, then by Lemma 4, $W(h) = x$ and every continuation of h is a maximal chain in $\downarrow x \setminus \{x\}$, thus the result follows.

"(b) implies (c)": Let $w \in x \in X$. Since w is a play, $u = \{z \in N \,|\, w \in z \in \downarrow x \setminus \{x\}\}$ is a maximal chain in $\downarrow x \setminus \{x\}$. Obviously, $h = \uparrow x$ is a history with minimum x and any maximal chain in $\downarrow x \setminus \{x\}$ is a continuation of h. Hence u has a maximum by (b), $y = \max u$. Further, $y \subset x$. It follows that $\gamma(x, w) = \cup \{z \,|\, z \in u\} = y \in N$ and $\gamma(x, w) \subset x$.

"(c) implies (a)" Suppose that $\gamma(x, w) \in N$ for all $w \in x$ and all $x \in X$. Consider $x \in X$ and let u be a maximal chain in $\downarrow x \setminus \{x\}$ Since T is a game tree, u is contained in a maximal chain in N (play), i.e. there exists $w \in W$ such that $w \in y$ for all $y \in u$. By hypothesis $w \in \gamma(x,w) \in \downarrow x \setminus \{x\}$ and thus $\gamma(x,w) \in u$. By the construction of γ also $y \subseteq \gamma(x,w)$ for any $y \in u$. Therefore, $\gamma(x,w) = \max u$. ■

Proof of Proposition 5. Assume that for (T,C) every strategy combination $s \in S$ induces outcomes after h. Let $s' \in S'$ be a strategy for $\Pi(T)$. Then, for every $x \in X$ there is some $w_x \in W$ such that $s'(x) = \gamma(x,w_x)$. Choose a strategy combination $s \in S$ for (T,C) such that $\gamma(x,w_x) \subseteq x \cap \left[\bigcap_{i \in J(x)} s_i(x)\right]$ for all $x \in X$ with $x \subseteq W(h)$. This is always possible by Lemma 6. By hypothesis there is $w \in W(h)$ such that $w \in R_s^h(w)$. This implies that $w \in x \cap \left[\bigcap_{i \in J(x)} s_i(x)\right]$ for all $x \in X \cap \uparrow\{w\}$ with $x \subseteq W(h)$.

Suppose there is $x \in X \cap \uparrow\{w\}$ with $x \subseteq W(h)$ such that $w \notin \gamma(x,w_x)$. Then $\gamma(x,w) \cap \gamma(x,w_x) = \emptyset$ by Proposition 2(a), but $\gamma(x,w) \cup \gamma(x,w_x) \subseteq x \cap \left[\bigcap_{i \in J(x)} s_i(x)\right]$. By the hypothesis of a weakly up-discrete tree and Lemma 8(c) $y = \gamma(x,w) \in N$ and $y' = \gamma(x,w_x) \in N$. Moreover, the construction implies that $y, y' \in F(N)$, because $x = \min \uparrow y \setminus \{y\} = \min \uparrow y' \setminus \{y'\}$. By (EDP.iii) $y \cap y' = \emptyset$ implies that there are $i \in I$ and $c, c' \in C_i$ for the EDP (T,C) such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$.

We claim that $x \in P(c) \cap P(c')$. Consider any $x' \in \uparrow$; then $y' \subset x \subseteq x'$ resp. $y \subset x \subseteq x'$ implies $x' \notin \downarrow c$ resp. $x' \notin \downarrow c'$, because $y' \subseteq x' \setminus c$ resp. $y \subseteq x' \setminus c'$; together with $y \cup y' \subseteq x \subseteq x'$ this yields $x' \in (\uparrow y \setminus \downarrow c) \cap (\uparrow y' \setminus \downarrow c')$. Consider any $x' \in \uparrow y \setminus \downarrow c$ resp. $x' \in \uparrow y' \setminus \downarrow c'$; since $x = \min \uparrow y \setminus \{y\} = \min \uparrow y' \setminus \{y'\}$ and $y \in \downarrow c$ resp. $y' \in \downarrow c'$, this implies $x' \in \uparrow x$. Together with the previous argument this yields $\uparrow x = \uparrow y \setminus \downarrow c = \uparrow y' \setminus \downarrow c'$, hence, $x \in P(c) \cap P(c')$ by (10).

But then $w \in y$ and $w_x \in y'$ imply $c = s_i(x) = c'$, in contradiction to $c \cap c' = \emptyset$. Therefore, $w \in \gamma(x,w_x)$ for all $x \in X \cap \uparrow\{w\}$ with $x \subseteq W(h)$. But that implies by construction that $w \in R_{s'}^h(w) = \bigcap \{s'(x) \,|\, W(h) \supseteq x \in \uparrow\{w\} \cap X\}$, demonstrating that s' induces outcomes after h. As $s' \in S'$ was arbitrary, this completes the proof. ■

Proof of Lemma 9. "only if:" Obvious, because every continuation is a chain.

"if:" Suppose every continuation of every history has a maximum. Let g be an arbitrary (nonempty) chain in T. If the root W is in g, then W is the maximum of g. Suppose, hence, $W \notin g$. Let w be a play such that $g \subseteq \uparrow\{w\}$. Let $g^* = \{x \in \uparrow\{w\} \,|\, x \subseteq x' \text{ for some } x' \in g\}$. Then, $h = \uparrow\{w\} \setminus g^*$ is a history and g^* is a continuation of h.

To see that h is a history, let $x \in h$ and $y \in \uparrow x \subseteq \uparrow\{w\}$. If $y \notin h$, then $y \in g^*$, hence there exists $y' \in g$ such that $x \subseteq y \subseteq y'$ and thus $x \in g$, a contradiction.

By hypothesis, g^* has a maximum, $z = \max g^*$. Then, $z \in g^*$ and hence there exists $z' \in g$ such that $z \subseteq z'$. But, since $g \subseteq g^*$, it follows that $z = z'$, i.e. z is also the maximum of g. ■

Proof of Corollary 5. The result follows from Lemmata 8 and 9 if it can be shown that under regularity coherence is equivalent to the property that for every history without minimum *every* continuation has a maximum. The latter clearly implies coherence. To see the converse, let h be a history without minimum and suppose g, g' are two continuations such that g has a maximum, $x = \max g$, but g' has no maximum.

Obviously, $\uparrow x \setminus \{x\} = h$. By hypothesis, h has no minimum, but, by regularity, there exists $z = \inf h$. Since x is a lower bound for h, it follows that $x \subseteq z$. If $x \neq z$, then $z \in \uparrow x \setminus \{x\} = h$ and $z = \min h$, a contradiction. Thus, $x = z = \inf h$.

Let $y \in g'$. Again, since y is a lower bound for h and $x = \inf h$, it follows that $y \subseteq x$. This shows that $x \in g'$ and $x = \max g'$, a contradiction. ■

Proof of Lemma 10. (a) Let g be a continuation of h. Since $x \subseteq y$ for any $x \in g$ and $y \in h$, it follows that $W(h,g) \subseteq W(h)$. That $\cup \{W(h,g) \,|g \text{is a continuation of } h\} = W(h)$ follows from the fact that, for any $w \in W(h)$, $\uparrow \{w\} \setminus h$ is a continuation of h. Finally, we have to show that the union is disjoint. Let g, g' be continuations of h such that $W(h,g) \cap W(h,g') \neq \emptyset$. Let $w \in W(h,g) \cap W(h,g')$. Then ,there exist $x \in g$ and $x' \in g'$ such that $w \in x$ and $w \in x'$, hence $x \cap x' \neq \emptyset$, implying that either $x \subseteq x'$ or $x' \subseteq x$. In the first case (the second is analogous), since $h \cup g$ is a play, it follows that $x' \in g$. This implies that $g \cap \uparrow x' = g' \cap \uparrow x'$ and thus $W(h,g) = W(h,g')$.

(b) Recall from Lemma 4 that the set $W(h) = \cap \{x \in N \,| x \in h\}$ is a node if and only if h has an infimum. Let g be a continuation of h. If h had an infimum z, then $z \in g$, because $h \cup g$ is a play and h has no minimum by hypothesis. Thus, h has no infimum and $W(h)$ is not a node. ■

Proof of Lemma 11. The proof proceeds in several steps. First, we observe that for every $x \in N(h,g)$, there exists $y \in N(h,g)$ such that $x \subset y$. That is, every node in $N(h,g)$ has a strict predecessor in $N(h,g)$. To see this, suppose $x \in N(h,g)$ has no immediate predecessor in $N(h,g)$. Let $w \in x$. Then, $g' = \uparrow \{w\} \cap N(h,g)$ is a continuation of h with $\max g' = x$. Clearly, $W(h,g) = W(h,g') = x$. It follows that $x \in g$ (because $x \notin h$) and $\max g = x$, a contradiction.

Second, we define the set

$$A = \left\{ (Y,s) \,\middle|\, \begin{array}{l} Y \subseteq N(h,g), Y \cap X \neq \emptyset, \text{ and } s \in S_Y \text{ such that} \\ \forall x \in Y \; \exists y \in Y \text{ with } x \subset y \text{ and } x \cap s(y) = \emptyset \end{array} \right\}$$

and the partial order on A given by $(Y,s) \geq (Y',s') \Leftrightarrow Y \supseteq Y'$ and $s|_{Y'} = s'$ (reflexivity and transitivity are obvious; antisymmetry follows by construction).

Third, we claim that, for every $x \in N(h,g)$, there exists $(Y,s) \in A$ such that $x \in Y \subseteq \uparrow x$. To see this, let $x \in N(h,g)$. By the first step, there exists $x_1 \in N(h,g)$ such that $x \subset x_1$. By (EDP.iv), there exists a choice c, available at x_1, such that $x \subseteq c$. By available choices, there exists an available choice at x_1 which is disjoint with c. Thus we can define $s(x_1) \neq c$. Analogously we can find $x_2 \in N(h,g)$ such that $x_1 \subset x_2$ and an available choice at x_2 which discards x_1. Proceeding iteratively (i.e. by an induction argument)[23] we obtain the desired conclusion.

Fourth, we will apply Zorn's Lemma to the poset $(A, \geq)$. Observe that A is nonempty by the third step. We have to show that every chain in $(A, \geq)$ has an upper bound in A. Let, thus, C be a chain in A. That is, for every $(Y,s), (Y',s') \in C$, either $(Y,s) \geq (Y',s')$ or $(Y',s') \geq (Y,s)$. Define $Z = \cup \{Y \,|(Y,s) \in C\}$ and construct $\overline{s}$ as follows. Given $z \in Z$, define $\overline{s}(z) = s(z)$ for any $(Y,s) \in C$ such that $z \in Y$. Such a (Y,s) exists by construction of Z, and $\overline{s}$ is well-defined because C is a chain. Further, given $z \in Z$, taking $(Y,s) \in C$ such that $z \in Y$ shows that there exists $y \in Y$ with $x \subset y$ and $x \cap s(y) = \emptyset$, and hence $x \cap \overline{s}(y) = \emptyset$. Thus $(Z, \overline{s})$ is an upper bound for C in A.

Zorn's Lemma implies that the poset $(A, \geq)$ has a maximal element (Z^*, s^*). Then, for any $x \in N(h)$, there exists $z \in Z^*$ such that $x \cap s^*(z) = \emptyset$. That is, any node in $N(h,g)$ (and not only in Z^*) is discarded under the partial strategy given by s^*.

[23] More rigorously, we are using here the Axiom of Dependent Choices, which is a consequence of the Axiom of Choice.

If $\uparrow x \cap Z^* \neq \emptyset$, then let $z \in \uparrow x \cap Z^*$. Since $(Z^*, s^*) \in A$, there exists $z' \in Z^*$such that $z \cap s^*(z') = \emptyset$,and thus $x \cap s^*(z') = \emptyset$ (because $x \subseteq z$).

Suppose, then, $\uparrow x \cap Z^* = \emptyset$. By the third step above, there exists $(Y, s) \in A$ such that $x \in Y \subseteq \uparrow x$.

Define now $Z_1 = Z^* \cup Y$. Clearly, $Z^* \subset Z_1$. Define also $s_1 \in S_{Z_1}$ as follows. For every $y \in Y$, let $s_1(y) = s(y)$. For every $z \in Z^*$, define $s_1(z) = s^*(z)$. Since $Y \cap Z^* = \emptyset$, s_1 is well defined. It follows that $(Z_1, s_1) \in A$ and $(Z_1, s_1) \geq (Z^*, s^*)$ but $Z \subset Z_1$, a contradiction with the maximality of (Z^*, s^*).

The conclusion now follows by specifying $s^*(x)$ arbitrarily for any $x \in N(h, g) \setminus Z^*$. ■

A.4. Proofs of results in Section 7.

Proof of Proposition 6. First, under (18), for every $x \in \uparrow y \cap \uparrow y'$ and every $i \in J(x)$ there is $c \in A_i(x)$ such that $y \cup y' \subseteq c$. To see this, note that $y \cap y' = \emptyset$ and $x \in \uparrow y \cap \uparrow y'$ imply $y \subset x$ and $y' \subset x$. Therefore, by (EDP.iv), for every $i \in J(x)$ there are $c, c' \in A_i(x)$ such that $y \subseteq c$ and $y' \subseteq c'$. Since $c \cap c' = \emptyset$ would imply $P(c) \cap P(c') = \emptyset$ from (18) in contradiction to $x \in P(c) \cap P(c')$, it follows that $c \cap c' \neq \emptyset$. But then (EDP.i) and $x \in P(c) \cap P(c')$ imply $c = c'$, as desired.

Fix any pair $(w, w') \in y \times y'$. Construct a strategy profile $s \in S$, by specifying $s_i(x)$ for each $x \in X$ and $i \in J(x)$, as follows. If $w, w' \notin x$, specify $s_i(x)$ arbitrarily. If $w \in x$ but $w' \notin x$, define $s_i(x)$ such that $w \in s_i(x)$ (this is always possible by (EDP.iv)). If $w' \in x$ but $w \notin x$, define $s_i(x)$ such that $w' \in s_i(x)$. Last, suppose $w, w' \in x$. Then, $x \cap y \neq \emptyset \neq x \cap y'$. Since $y \cap y' = \emptyset$, Trivial Intersection, (3), implies that $y \subset x$ and $y' \subset x$. By the above, for every $i \in J(x)$ there is $c \in A_i(x)$ such that $y \cup y' \subseteq c$. Set $s_i(x) = c$.

Clearly, $w \in s_i(x)$ for all $i \in J(x)$ whenever $w \in x$, and analogously for w'. Thus $w \in R_s(w)$ and $w' \in R_s(w')$. ■

The following Lemma is needed in the proof of Proposition 7. Its (straightforward) proof is omitted..

Lemma 17. *For a game tree $T = (N, \supseteq)$, a history h has a continuation with a maximum if and only if there is $z \in N$ such that $h = \uparrow z \setminus \{z\}$.*

Proof of Proposition 7. (a) We first claim that, if a game tree $(N, \supseteq)$ is selective, every chain of the form $\uparrow x \cap \uparrow y$ for $x, y \in N$ has a minimum.[24] Let $x, y \in N$ be such that $x \cap y = \emptyset$ (else the result is obvious) and choose $w \in x$ and $w' \in y$. Because the tree is selective, there is $z \in X$ with $w, w' \in z$ such that $\gamma(z, w) \cap \gamma(z, w') = \emptyset$ by Proposition 2(a). Let $z' \in \uparrow x \cap \uparrow y$. Since $\{w, w'\} \subseteq z \cap z'$, either $z' \subset z$ or $z \subseteq z'$ by Trivial Intersection. If $z' \subset z$ would hold, that $w, w' \in z' \in \downarrow z \setminus \{z\}$ would imply $z' \subseteq \gamma(z, w) \cap \gamma(z, w')$ in contradiction to $\gamma(z, w) \cap \gamma(z, w') = \emptyset$. Therefore, $z \subseteq z'$ for all $z' \in \uparrow x \cap \uparrow y$ implies together with $z \in \uparrow x \cap \uparrow y$ that $z = \min \uparrow x \cap \uparrow y$.

Suppose $x \in S(N)$. Since $x \neq \inf \uparrow x \setminus \{x\}$, there exists a node x' which is a lower bound of $\uparrow x \setminus \{x\}$ but such that $x' \nsubseteq x$. If $x \subset x'$, then $x' = \min \uparrow x \setminus \{x\}$, a contradiction with $x \in S(N)$. By Trivial Intersection this implies $x \cap x' = \emptyset$. Since x' is a lower bound of $\uparrow x \setminus \{x\}$, it follows that $\uparrow x \setminus \{x\} \subseteq \uparrow x \cap \uparrow x'$. By the claim above there exists $z = \min \uparrow x \cap \uparrow x'$. In particular, $z \supseteq x'$ and, therefore, $z \in \uparrow x \setminus \{x\}$. It follows that $\uparrow x \cap \uparrow x' \subseteq$

[24]In other words, every selective game tree is well-joined (see footnote 16).

$\uparrow x \setminus \{x\}$ and, hence, we have equality. Therefore, $z = \min \uparrow x \setminus \{x\}$, in contradiction to $x \in S(N)$.

(b) Assume that T is regular and every history has at least one continuation with a maximum. Let $w, w' \in W$ be such that $w \neq w'$ and $h = \uparrow \{w\} \cap \uparrow \{w'\}$, so that h is a history. Suppose that h has no minimum. By Lemma 7 there is $z \in N$ such that $h = \uparrow z \setminus \{z\}$. If $z = \cap_{x \in h} x$, then $w, w' \in z$ imply $z \in h$, a contradiction. Therefore $z \subset \cap_{x \in h} x$. But then, that h has no minimum and Lemma 2(c) imply that z is strange, a contradiction.

Therefore, regularity of the tree implies that $h = \uparrow \{w\} \cap \uparrow \{w'\}$ has a minimum $x \in h$. If $\gamma(x, w) \cap \gamma(x, w')$ were nonempty, $\gamma(x, w) = \gamma(x, w')$ would hold by Proposition 2(a). But that would imply that there is a node $y \in \downarrow x \setminus \{x\}$ with $w, w' \in y$, so that $y \in h$ by the construction of h. Since this contradicts $x = \min h$, it follows that $\gamma(x, w) \cap \gamma(x, w') = \emptyset$. Since $w, w' \in W$ were arbitrary, the game tree T is selective. ■

Proof of Lemma 13. Let $w, w' \in W$ be as in the statement and consider the single-player perfect information problem $\Pi(T)$. Suppose $w, w' \in x \in X$. By hypothesis $\gamma(x, w) \cap \gamma(x, w') \neq \emptyset$. Then, Proposition 2(a) implies $\gamma(x, w) = \gamma(x, w')$.

Choose $s \in S$ such that $s(x) = \gamma(x, w) = \gamma(x, w')$ if $w, w' \in x$; $s(x) = \gamma(x, w)$ if $w \in x$ but $w' \notin x$; $s(x) = \gamma(x, w')$ if $w' \in x$ but $w \notin x$; and $s(x)$ arbitrary otherwise. Then, $w \in s(x)$ whenever $w \in x \in X$ and analogously for w'. Hence $w \in R_s(w)$ and $w' \in R_s(w')$. ■

Proof of Lemma 14. Let $x \in X$, $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$, and $w, w' \in W$ be as in the statement. Suppose $x' \in \downarrow x \setminus \{x\}$ with $w, w' \in x'$. Then by Trivial Intersection either $x \subseteq x'$ or $x' \subset x$. But in the latter case $x' \subseteq \gamma(x, w) \cap \gamma(x, w')$ by the construction of perfect information choices, in contradiction to $\gamma(x, w) \cap \gamma(x, w') = \emptyset$ by Proposition 2(a) and the hypothesis. Therefore, for $x' \in \downarrow x$ we have that $w, w' \in x'$ if and only if $x' = x$.

Construct $s \in S$ as follows. First, choose $s_i(x) = c_i$ for all $i \in I$. Second, for all $x' \in \uparrow x \setminus \{x\}$, choose $s_i(x)$ such that $x \subseteq s_i(x')$, for all $i \in J(x')$. (This is always possible by (EDP.iv)). Third, for all $x' \in \downarrow x \setminus \{x\}$ with $w \in x'$ but $w \notin x'$, choose $s_i(x')$ such that $\gamma(x', w) \subseteq s_i(x')$; analogously, for all $x' \in \downarrow x \setminus \{x\}$ with $w' \in x'$ but $w \notin x'$, choose $s_i(x')$ such that $\gamma(x', w') \subseteq s_i(x')$. (The latter is always possible by Lemma 6.) Otherwise s is arbitrary. Then, $w \in s_i(x)$ whenever $w \in x \in X$ and $i \in J(x)$, thus $w \in R_s(w)$. Analogously $w' \in R_s(w')$. ■

Proof of Proposition 8. "if": Suppose T is selective and the property in the statement holds. Let $y, y' \in N$ be such that $y \cap y' = \emptyset$ and choose $w \in y$ and $w' \in y'$. Because the tree is selective, there is $x \in X$ with $w, w' \in x$ such that $\gamma(x, w) \cap \gamma(x, w') = \emptyset$. For each $i \in J(x)$ choose $c_i, c_i' \in A_i(x)$ such that $y \subseteq c_i$ and $y' \subseteq c_i'$, which is always possible by (EDP.iv), since $w, w' \in x$, $w \in y$, $w' \notin y$, $w' \in y'$, and $w \notin y'$ imply $y, y' \in \downarrow x \setminus \{x\}$ by Trivial Intersection. By the hypothesis and Proposition 2(b) it follows that $\gamma(x, w) = x \cap \left[\cap_{i \in J(x)} c_i\right]$ and $\gamma(x, w') = x \cap \left[\cap_{i \in J(x)} c_i'\right]$, because $\gamma(x, w) \subseteq x \cap \left[\cap_{i \in J(x)} c_i\right]$ and $\gamma(x, w') \subseteq x \cap \left[\cap_{i \in J(x)} c_i'\right]$ by Lemma 6. In particular, for some $i \in J(x)$ it must be the case that $y \subseteq \gamma(x, w) \subseteq c_i$, $y' \subseteq \gamma(x, w') \subseteq c_i'$, and $c_i \cap c_i' = \emptyset$. For, if for all $i \in J(x)$ it would hold that $c_i \cap c_i' \neq \emptyset$, then (EDP.i) would imply that $c_i = c_i'$ for all $i \in J(x)$ in contradiction to what has just been shown, because $x \in P(c_i) \cap P(c_i')$ by construction.

"only if": Suppose that (EDP.iii') holds. Let $w, w' \in W$ be such that $w \neq w'$. By Irreducibility there are $y, y' \in N$ such that $w \in y$, $w' \in y'$, and $y \cap y' = \emptyset$. By (EDP.iii') there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, $c \cap c' = \emptyset$, and $x \in P(c) \cap P(c') \neq \emptyset$, say. By Lemma 6 $\gamma(x, w) \subseteq x \cap c$ and $\gamma(x, w') \subseteq x \cap c'$. Since $c \cap c' = \emptyset$, it follows that $\gamma(x, w) \cap \gamma(x, w') = \emptyset$. Because $x \in P(c) \cap P(c')$, it follows that $\uparrow x = \uparrow y \setminus \downarrow c = \uparrow y' \setminus \downarrow c'$ by (10), in particular, $w, w' \in x$. Thus, the tree is selective.

Let $x \in X$ and $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$. By (EDP.ii) there is $w \in x \cap \left[\cap_{i \in J(x)} c_i\right]$ and by Lemma 6 $\gamma(x, w) \subseteq x \cap c_i$ for all $i \in J(x)$. Choose $w' \in x \cap \left[\cap_{i \in J(x)} c_i\right] \setminus \gamma(x, w)$. By Irreducibility there are $y, y' \in N$ such that $w \in y$, $w' \in y'$, and $y \cap y' = \emptyset$, so that $y, y' \in \downarrow x \setminus \{x\}$ by Trivial Intersection. By (EDP.iii') there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, $c \cap c' = \emptyset$, and $x' \in P(c) \cap P(c')$, say. Therefore, $w, w' \in x' \cap x$ implies either $x \subseteq x'$ or $x' \subset x$ by Trivial Intersection. In the latter case $w \in x'$ implies $x' \subseteq \gamma(x, w)$ in contradiction to $w' \in x'$. Therefore, $x \subseteq x'$. If $x \subset x'$ would holds, then $y \cup y' \subseteq x \subseteq \gamma(x', w) \subseteq x' \cap c$ (by Lemma 6) would contradict $c \cap c' = \emptyset$. Thus, $x' = x$. But then $c' \cap c_i = \emptyset$ implies $w' \notin c_i$, a contradiction. Hence, $x \cap \left[\cap_{i \in J(x)} c_i\right] \setminus \gamma(x, w) = \emptyset$ yields the property in the statement. ■

Proof of Proposition 9. Let (T, C) be an EF and assume that every strategy combination for (T, C) induces outcomes after h. Pick any strategy $s' \in S'$ for $\Pi(T)$ and choose $s \in S$ for (T, C) as follows: If $s'(x) = \gamma(x, w)$ for $w \in x$, then for (T, C) pick the choice combination $(c_i^w)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$ such that $x \cap \left[\cap_{i \in J(x)} c_i^w\right] \subseteq \gamma(x, w) = s'(x)$ (which exists by (EDP.iv) in the definition of an EDP and by the "only if"-part of Proposition 8) and declare this to be the value of s at x, i.e. $s_i(x) = c_i^w$ for all $i \in J(x)$; doing this for all $x \in X$ with $x \subseteq W(h) \equiv \cap_{y \in h} y$ determines s for the relevant part of the tree. It is then straightforward to see that $U^h(s) \subseteq U^h(s')$. By the "only if"-part of Proposition 3 the hypothesis implies that $U^h(s)$ contains a chain u that is maximal in the set of nodes $x \in N$ that are contained in $W(h)$; by the previous finding $u \subseteq U^h(s')$. Therefore, the "if"-part of Proposition 3 yields that s' induces outcomes after h. Since the strategy s' for $\Pi(T)$ was arbitrary, the desired conclusion follows. ■

Proof of Lemma 15. Let $s \in S$, assume that $w \in R_s(w)$, and consider $w' \in W \setminus \{w\}$. By Irreducibility there are $y, y' \in N$ such that $w \in y$, $w' \in y'$, and $y \cap y' = \emptyset$. By (EDP.iii') from the definition of an EF (Definition 12) there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, $c \cap c' = \emptyset$, and there is $x \in P(c) \cap P(c')$. Because $w \in R_s(w)$, it must be the case that $s_i(x) = c$ and, therefore, $w' \in y' \subseteq c' \neq s_i(x)$ implies that $w' \notin s_i(x)$. ■

Proof of Proposition 10. Let $x \in X$ and $(c_i)_{i \in J(x)} \in \times_{i \in J(x)} A_i(x)$. By (EDP.ii) there is $w \in x \cap \left[\cap_{i \in J(x)} c_i\right]$. By Lemma 6 $\gamma(x, w) \subseteq x \cap \left[\cap_{i \in J(x)} c_i\right]$ and, because T is weakly up-discrete, $y \equiv \gamma(x, w) \in N$ by Lemma 8(c). Suppose there is $w' \in x \cap \left[\cap_{i \in J(x)} c_i\right] \setminus \gamma(x, w)$. Then, $y' \equiv \gamma(x, w') \in N$ by Lemma 8(c), $y' \subseteq x \cap \left[\cap_{i \in J(x)} c_i\right]$ by Lemma 6, and $y \cap y' = \emptyset$ by Proposition 2(a). By (EDP.iii) there are $i \in I$ and $c, c' \in C_i$ such that $y \subseteq c$, $y' \subseteq c'$, and $c \cap c' = \emptyset$. From $y' \subseteq c' \cap x$ it follows that $x \setminus c \neq \emptyset$ and from $y \subseteq c \cap x$ it follows that $x \setminus c' \neq \emptyset$, i.e. $x \notin \downarrow c$ and $x \neq \downarrow c'$. Therefore, $\uparrow x \subseteq \uparrow y \setminus \downarrow c$ and $\uparrow x \subseteq \uparrow y' \setminus \downarrow c'$. By the construction of y and y' these inclusions are equalities, however, so that $\uparrow x = \uparrow y \setminus \downarrow c$ and $\uparrow x = \uparrow y' \setminus \downarrow c'$, i.e. $x \in P(c) \cap P(c')$. But then $i \in J(x)$ and $c = c_i \supseteq y' = \gamma(x, w')$ contradicts $c \cap c' = \emptyset$. Thus, $x \cap \left[\cap_{i \in J(x)} c_i\right] \setminus \gamma(x, w) = \emptyset$ yields, together with Lemma 6, $\gamma(x, w) = x \cap \left[\cap_{i \in J(x)} c_i\right]$. Since x and the choice combination were arbitrary, condition (EDP.ii') is fulfilled and the statement follows from Proposition 8. ■

B. Omitted Proofs - For the Referees' Perusal

This section collects the simple proofs which have been omitted from the paper. They are intended for the referees' convenience only and not for publication.

Proof of Proposition 5. Part (a) follows by definition. Then, every chain in $U^h(s)$ is a filter. For, if $x \in U^h(s)$ and $y \in D^h(s)$ with $y \supseteq x$, then $x \in D^h(s)$, a contradiction. ■

Proof of Lemma 6. If $x \in P(c)$, for $c \in C_i$ and $i \in I$, and $x \supset y \in N$, then by (EDP.iv) there is $c_i \in A_i(x)$ such that $y \subseteq c_i$ for all $i \in J(x)$. Therefore, if $c \in A_i(x)$ and $w \in x \cap c$, then $w \in y \in \downarrow x \setminus \{x\}$ implies $y \subseteq c$, because $c \neq c_i$ would imply $c \cap c_i = \emptyset$ by (EDP.i) for all $i \in J(x)$. It follows that $\gamma(x, w) \subseteq x \cap c$, as desired. ■

Proof of Lemma 7. Let $y \in D^h(s)$, i.e., there are $x \in \uparrow y \setminus \{y\}$ with $x \subseteq W(h)$, $i \in J(x)$, and $c \in A_i(x)$ such that $y \subseteq c \neq s_i(x)$. Then $y \cap s_i(x) = \emptyset$. Because $s'(x) \subseteq x \cap \left[\cap_{i \in J(x)} s_i(x)\right]$ by hypothesis, $s'(x) \cap y = \emptyset$. Hence, $y \in D^h(s')$. Since $y \in D^h(s)$ was arbitrary, $D^h(s) \subseteq D^h(s')$ or, equivalently, $U^h(s') \subseteq U^h(s)$. ■

Proof of Lemma 17. To see the sufficiency part of this statement, assume that $h = \uparrow z \setminus \{z\}$ for some $z \in N$ and choose $y \in \downarrow z$. By the Hausdorff Maximality Principle there is a maximal chain g in $\downarrow z$ that contains $\uparrow y \cap \downarrow z$, and $z = \max g$ by construction. It remains to show that $h \cup g$ is a play. If not, there is $x \in N \setminus (h \cup g)$ such that $h \cup g \cup \{x\}$ is a chain. Then, by $z \in g$ either $z \subset x$ or $x \subseteq z$. The latter would imply $x \in g$ by $x \in \downarrow z$ and maximality of g in $\downarrow z$; therefore, $z \subset x$. But that means $x \in \uparrow z \setminus \{z\} = h$, again a contradiction. Thus, $h \cup g$ is maximal in N, hence, a play.

To see the necessity part, let g be a continuation with maximum $z \in g$ of a history h, so that $z \notin h$. If $x \in h$, then $z \subset x$, because $h \cup g$ is a chain and $x \subseteq z$ would imply $z \in h$, since h is a history. Hence, $h \subseteq \uparrow z \setminus \{z\}$. If $x \in \uparrow z \setminus \{z\}$, then $x \notin g$, because $z = \max g$. On the other hand, $z \subset x \in N$ implies $x \in h \cup g$, because $h \cup g$ is a maximal chain in N, so that $x \in h$ follows. Hence, $\uparrow z \setminus \{z\} \subseteq h$ combines with the previous inclusion to $h = \uparrow z \setminus \{z\}$. ■