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On Core-Walras Equivalence in Banach Spaces when Feasibility is defined by the Pettis Integral *

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Abstract

The paper studies the core-Walras equivalence problem in the commodity space framework of Banach spaces, allocations being defined as Pettis integrable functions. In particular, a core-Walras equivalence result for a certain class of commodity spaces is established, without requiring that the commodity space be separable. The class covered by this result includes the $L_p(\mu)$ spaces, $1 \leq p < \infty$, μ being σ-finite. On the other hand, responding to objections made against some recent core-Walras non-equivalence results in the Bochner integrable allocations setting, it is shown that these latter results carry over to the Pettis integrable allocations setting, unless additional restrictions on the heterogeneity of agents' preferences are in force.

1 Introduction

This paper deals with the core-Walras equivalence problem in infinite dimensional commodity spaces; in particular with the impact of the heterogeneity of preferences which may appear in an economy with a continuum of agents when the commodity space is large.

Several extensions of Aumann's (1964) classical core-Walras equivalence theorem to infinite dimensional commodity spaces have been established in the literature. See, e.g., Bewley (1973), Gabszewicz (1968), Mas-Colell (1975), Mertens (1970), Ostroy and Zame (1994), Rustichini and Yannelis (1991), Tourky and Yannelis (2001), Zame (1986). In most of these results, the commodity space is separable (at least in a topology for which preferences are continuous), and may thus be interpreted as being "not too large" relative to the size of an economy with a continuum of agents; in particular, agents' preferences cannot be "too dispersed."

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In the seminal contribution of Tourky and Yannelis (2001), it was shown that having a commodity space that is “not too large" indeed matters for the core-Walras equivalence problem. In fact, these authors showed that given any non-separable Hilbert space as commodity space, one can find an atomless economy such that, when feasibility of allocations is defined in terms of the Bochner integral, core-Walras equivalence fails even though the usual standard assumption are met. Subsequently, it was shown in Podczeck (2003) that a core-Walras non-equivalence result like that of Tourky and Yannelis (2001) actually holds in any non-separable Banach space, and in Podczeck (2002) related results for the commodity space setting of Banach lattices were established.[1]

The interpretation of these results of Tourky and Yannelis (2001) and Podczeck (2003, 2002) is that a large number of agents does not guarantee perfect competition unless there are in fact “many more agents than commodities;” if this latter condition does not hold, then a large number of agents means that agents' characteristics may be extremely dispersed, so that the standard theory of perfect competition fails.[2]

The reason underlying core-Walras non-equivalence in non-separable Banach spaces when feasibility is defined in terms of the Bochner integral can be viewed as follows. Since Bochner integrable allocations must be essentially separably valued, the property of an allocation being in the core is separably determined in the sense that a feasible allocation is a core allocation already when it is a core allocation relative to every separable subspace of the commodity space. On the other hand, across the separable subspaces of the commodity space the profile of agents' preferences may be extremely dispersed. As a consequence, since the property of an allocation being Walrasian is determined relative to the entire commodity space, the core may be larger than the set of Walrasian allocations—even when the economy in question is atomless.

This intuition, however, leads to an objection that has been made against the analysis in Tourky and Yannelis (2001) and Podczeck (2003, 2002): Since allocations are essentially separably valued, blocking possibilities are very limited when the commodity space is non-separable, which makes the core “large” in some sense, thus implying a bias in favor of core-Walras non-equivalence; therefore a notion of integrability weaker than Bochner integrability should be used to define feasibility of allocations.

In this note we take up this objection and consider the core-Walras equivalence problem in the Pettis integrable allocations setting. Our main result is

[1]In the non-equivalence results of Tourky and Yannelis (2001) and Podczeck (2003), the ordering of the commodity space is not taken as a priori given. Rather, it is constructed in the proofs; in particular, it is not a lattice ordering.

[2]We refer to Tourky and Yannelis (2001) and Podczeck (2003) for a more detailed discussion of this point.

that, in this setting, core-Walras equivalence indeed holds for some class of commodity spaces regardless of whether or not the actual space is separable. (See Theorem 6 in Section 3.2.) However it turns out that this is not due to the Pettis integral by itself, but rather due to the interplay between defining allocations to be Pettis integrable functions and measurability assumptions on the profile of agents' preferences.

It is well known that without such a measurability assumption core-Walras equivalence can fail even in the setting of finitely many commodities. (See e.g. the example in Tourky and Yannelis (2001).) Now measurability of the profile of agents' preferences can be defined in several ways. Two of them are—where $(T, \mathcal{T}, \nu)$ is the measure space of agents of an economy, and $\succ_t$ denotes the strict preference relation of agent $t \in T$:

(M1) If x and y are any two consumption bundles then $\{t \in T \colon x \succ_t y\}$ is a measurable set, i.e. it belongs to $\mathcal{T}$.

(M2) If f and g are any two allocations then $\{t \in T \colon f(t) \succ_t g(t)\}$ is a measurable set, i.e. it belongs to $\mathcal{T}$.

If the commodity space is a separable Banach space, then, regardless of whether allocations are defined to be Bochner integrable functions or to be just Pettis integrable, (M1) and (M2) amount to the same condition (provided, of course, that certain standard assumptions on preferences are in force).[3] If allocations are defined to be Bochner integrable functions, then regardless of whether the commodity space is a separable or a non-separable Banach space, (M1) and (M2) amount to the same condition, too. However, if allocations are defined to be Pettis integrable and the commodity space is a non-separable Banach space, then (M1) and (M2) need no longer be equivalent, and this indeed has consequences in regard to the core-Walras equivalence problem. In fact, we show:

(1) The core-Walras non-equivalence results of Tourky and Yannelis (2001) and Podczeck (2003, 2002) continue to hold when allocations are defined to be Pettis integrable but only (M1) is required to be satisfied by the profile of agents' preferences. Thus, defining feasibility in terms of the Pettis integral has, by itself, no effect for the core-Walras equivalence problem compared with defining feasibility in terms of the Bochner integral.

(2) Even if (M2) is required to hold instead of only (M1), and allocations are defined to be Pettis integrable, one can find non-separable commodity spaces in which core-Walras equivalence fails.

(3) However, together with requiring (M2), defining allocations to be Pettis integrable may have an effect: As will be shown, under these conditions, in the

[3]For this and the following sentence, see the proposition in Section 3.2.

commodity space setting of Banach lattices with an order continuous norm and a weak unit, core-Walras equivalence holds, regardless of whether the commodity space is separable or not.

Taken together, (1)–(3) say that defining allocations to be Pettis integrable may indeed lead to different conclusions for the core-Walras equivalence problem, compared with the results in Tourky and Yannelis (2001) and Podczeck (2003, 2002), but only in connection with a strong version of a measurability assumption concerning the profile of agents' preferences.

The interpretation is that it is crucial for core-Walras equivalence to hold in a large commodity space that preferences are not too dispersed across agents, and that whether allocations are defined to be Bochner or just Pettis integrable matters only in connection with this. If the restriction on the allowed heterogeneity of a profile of agents' preferences is only as incorporated in the measurability assumption (M1), then the core-Walras non-equivalence results for the Bochner integrable allocations setting carry over to the Pettis integrable allocations setting. On the other hand, for a non-separable commodity space, the measurability assumption (M2) may imply a restriction on the allowed heterogeneity of preferences which, in the Pettis integrable allocations setting, goes beyond that implied by (M1). Therefore, in that setting, core-Walras equivalence may hold under (M2) even when the commodity space is non-separable.

We close the introduction by noting that non-separable Banach spaces, in particular non-separable Banach lattices, indeed appear as commodity spaces in the economic literature. An example is the model by Khan and Sun (1997) of financial trading under uncertainty. In that model, the space of asset returns is an L_p space over a sample space of uncertain states that is taken to be an atomless Loeb probability space. However, L_p spaces on such probability spaces are non-separable. (See, e.g., Jin and Keisler, 2000.) In Sun (1996) it was argued that atomless Loeb probability spaces are indeed the most appropriate infinite idealizations of a large finite set of uncertain states. But then non-separability has to be taken into account in the context of $L_p(\mu)$ spaces as models for economic situations involving uncertainty. Note that the $L_p(\mu)$ spaces, for $1 \leq p < \infty$ and μ σ-finite, are covered by our core-Walras equivalence result in Theorem 6.

Another example are models of commodity differentiation where the commodity space is $M(\Omega)$, the space of all regular bounded Borel measures on a compact Hausdorff space Ω. If Ω is uncountable, then this space is non-separable. The theory of thick and thin markets developed by Ostroy and Zame (1994) uses this framework of commodity differentiation. As shown by these authors, in order to have examples of thin markets, preferences must not be weak* continuous (as frequently assumed in models of commodity differentiation), but just norm continuous, so that (norm) non-separability of $M(\Omega)$ actually matters.

Let us remark here that, for the space $M(\Omega)$, core-Walras equivalence holds in the Pettis integrable allocations setting under assumption (M2). This can be deduced from Theorem 6 in the present paper together with some arguments from the proof of Theorem 2 in Podczeck (2002).[4] Actually, if Ω is such that for every regular Borel measure μ on Ω, $L_1(\mu)$ is separable (e.g. if $\Omega = [0,1]$) then core-Walras equivalence holds in $M(\Omega)$ already in the Pettis integrable allocations setting under (M1) as well as in the Bochner integrable allocations setting.[5]

Finally, we remark that if the commodity space is actually a dual Banach space, then the notion of the Gelfand integral may be more appropriate than that of the Pettis integral to define feasibility of allocations. This is so in particular for the space $M(\Omega)$ as model of commodity differentiation. An investigation of the core-Walras equivalence problem for dual Banach spaces in the Gelfand integrable allocations setting will be the topic of future research. Note, though, that for reflexive Banach spaces, the Gelfand and the Pettis integral coincide. Thus, if only the measurability assumption (M1) is required to hold for the preference mapping of an economy, then—for a dual Banach space—simply replacing "Pettis" by "Gelfand" in the definition of allocations will not eliminate the possibility of core-Walras non-equivalence.

2 Notation and Terminology

(1) If E is a Banach space, then E^* denotes the dual space of E, i.e. the space of all continuous linear functions from E into $\mathbb{R}$. If $x \in E$ and $p \in E^*$, the value $p(x)$ of p at x will often be denoted by $\langle p, x\rangle$ for notational convenience. E^* is always regarded as endowed with the dual norm. We write $\|\cdot\|$ for both the

[4] By applying Theorem 6 to the restriction of an economy to the norm closure of the order ideal generated by the aggregate endowment, followed by applying Assumption (A9) (bounded marginal rates of substitution) to get an equilibrium with respect to the entire commodity space. For the first step one has to note that, for the space $M(\Omega)$, if allocations are Pettis integrable and consumption sets are the positive cone of the commodity space, then a feasible allocation takes almost all of its values in the norm closure of the order ideal generated by the aggregate endowment. For the second step, see the proof of Podczeck (2002, Theorem 2(i)⇒(ii)) for details.

[5] For the Bochner integrable allocations setting, see Podczeck (2002, Theorem 2(i)⇒(ii)). The arguments in the proof of that result can be adapted to deal, for the space $M(\Omega)$, with the Pettis integrable allocations setting under (M1). As shown in Podczeck (2002) for the Bochner integrable allocations setting, if the commodity space is a Banach lattice, and the ordering considered is the given lattice ordering, then separability properties of order ideals are relevant for core-Walras equivalence, and not separability of the entire commodity space. This is not in contradiction with the non-equivalence results in Tourky and Yannelis (2001) and Podczeck (2003). For as noted above, in those non-equivalence results the ordering of the commodity space is not taken as a priori given, but is constructed in the proofs, and in particular is not a lattice ordering.

norm of E and the norm of E^*. We write $\sigma(E, E^*)$ for the weak topology of E, and $\sigma(E^*, E)$ for the weak* topology of E^*. Finally, for a subset A of E:
$\operatorname{int} A$ denotes the (norm) interior of A.

(2) Let E be a Banach space, and let $(T, \mathcal{T}, \nu)$ be a complete finite measure space. A function $s \colon T \to E$ is called a *measurable simple function* if there are $x_1, x_2, \dots, x_n \in E$ and $S_1, S_2, \dots, S_n \in \mathcal{T}$ such that $s = \sum_{i=1}^n x_i 1_{S_i}$. Here and later on, if $S \subset T$ then 1_S denotes the characteristic function of S, i.e. $1_S(t) = 1$ if $t \in S$ and $1_S(t) = 0$ if $t \in T \setminus S$. If $s = \sum_{i=1}^n x_i 1_{S_i}$ is a measurable simple function from T into E and $S \in \mathcal{T}$ then the integral of s over S is defined as $\int_S s\, d\nu = \sum_{i=1}^n \nu(S_i \cap S) x_i$. A function $f \colon T \to E$ is said to be *weakly measurable* if the function $t \mapsto \langle q, f(t) \rangle$ is measurable for every $q \in E^*$. The function $f \colon T \to E$ is said to be *strongly measurable* if f is the pointwise limit almost everywhere of a sequence of measurable simple functions. Recall that according to Pettis's measurability theorem, f is strongly measurable if and only if f is weakly measurable and essentially separably valued; the latter means that there is a separable subspace F of E such that $f(t) \in F$ for almost all $t \in T$. A weakly measurable function $f \colon T \to E$ is said to be *Pettis integrable* if for each $S \in \mathcal{T}$ there is an $x_S \in E$ such that $\langle q, x_S \rangle = \int_S \langle q, f(t) \rangle\, d\nu(t)$ for all $q \in E^*$. In this case we write $x_S = \int_S f(t)\, d\nu(t)$ or $x_S = \int_S f\, d\nu$ or simply $x_S = \int_S f$ and call x_S the Pettis integral of f over S. A strongly measurable function $f \colon T \to E$ is said to be *Bochner integrable* if there exists a sequence (s_n) of measurable simple functions from T into E such that $\int \| f(t) - s_n(t) \|\, d\nu(t) \to 0$ as $n \to \infty$. In this case for each $S \in \mathcal{T}$, $\lim \int_S s_n\, d\nu$ exists (and is independent of the special choice of the sequence (s_n)) and is called the Bochner integral of f over S. Note that if f is Bochner integrable then f is Pettis integrable, and the Pettis integral and the Bochner integral of f coincide over any $S \in \mathcal{T}$. Thus if f is Bochner integrable, we may also write $\int_S f$ to denote the Bochner integral of f over S.

(3) By an *ordered Banach space* we mean a Banach space endowed with a vector ordering such that the positive cone is closed. Let E be an ordered Banach space.

(a) As usual, the ordering of E is denoted by $\geq$, and E_+ denotes the positive cone of E, i.e. $E_+ = \{x \in E \colon x \geq 0\}$.

(b) E^* will always be regarded as endowed with the dual ordering; thus, in particular:
$E_+^* = \{q \in E^* \colon q(x) \geq 0 \text{ for all } x \in E_+\}$.

(c) A linear functional $q \in E^*$ is said to be *strictly positive* if $q(x) > 0$ whenever $x \in E_+ \setminus \{0\}$.

(4) Let F be a Riesz space (i.e. vector lattice).

(a) The ordering of F is again denoted by $\geq$, and F_+ denotes the positive cone of F, i.e. $F_+ = \{x \in F \colon x \geq 0\}$. For $x, y \in F$ the expressions x^+, x^-, $|x|$, $x \vee y$,

$x \wedge y$, and $x \perp y$ have the usual lattice theoretical meaning.

(b) Let $x, y \in F$. Then:

$[x, y]$ denotes the order interval $\{z \in F\colon x \le z \le y\}$.

A_x denotes the order ideal in F generated by x. Thus, if $x \in F_+$ then

$$A_x = \bigcup_{n=1}^{\infty} [-nx, nx] = \{z \in F\colon |z| \le nx \text{ for some } n \in \mathbb{N}\}.$$

(5) (a) $C(\Omega)$ stands for the space of all continuous real valued functions on some compact Hausdorff space Ω, endowed with the supremum norm and the usual pointwise ordering; thus $C(\Omega)$ is a Banach lattice.

(b) By a "$C(\Omega)$ space" we mean a Banach lattice that is isomorphic as a Banach lattice to a concrete space $C(\Omega)$. Recall that every Banach lattice whose positive cone has a non-empty interior is a $C(\Omega)$ space.

(6) Let E be any Banach lattice.

(a) A point $x \in E_+$ is said to be a *quasi-interior point* of E_+ if A_x is dense in E. Recall that this can be equivalently expressed by saying that x is a quasi-interior point of E_+ if $q(x) > 0$ whenever $q \in E_+^* \setminus \{0\}$.

(b) For a strictly positive $q \in E_+^*$:

$\sigma(E, A_q)$ denotes the weak topology of E with respect to the order ideal A_q. (Note that when q is strictly positive, A_q separates the points of E.)

3 The model and the results

3.1 The model

Let E be an ordered Banach space. An *economy* $\mathcal{E}$ with commodity space E is a pair $[(T, \mathcal{T}, \nu), (X(t), \succ_t, e(t))_{t \in T}]$ where

- $(T, \mathcal{T}, \nu)$ is a complete positive finite measure space of agents;
- $X(t) \subset E$ is the consumption set of agent t;
- $\succ_t \subset X(t) \times X(t)$ is the (strict) preference relation of agent t;
- $e(t) \in E$ is the initial endowment of agent t;

and where the endowment mapping $e\colon T \to E$, given by $t \mapsto e(t)$, is assumed to be Pettis integrable. The economy $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succ_t, e(t))_{t \in T}]$ is said to be *atomless* if the measure space $(T, \mathcal{T}, \nu)$ is atomless.

An *allocation* for the economy $\mathcal{E}$ is a Pettis integrable function $f\colon T \to E$ such that $f(t) \in X(t)$ for almost all $t \in T$. An allocation f is said to be *feasible* if

$$\int_T f(t)\, d\nu(t) = \int_T e(t)\, d\nu(t)\,.$$

A *Walrasian equilibrium* for the economy $\mathcal{E}$ is a pair (p, f) where f is a feasible allocation and $p \in E^* \setminus \{0\}$ is a price system such that for almost every $t \in T$:

(i) $\langle p, f(t)\rangle \le \langle p, e(t)\rangle$ and

(ii) if $x \in X(t)$ satisfies $x \succ_t f(t)$ then $\langle p, x\rangle > \langle p, e(t)\rangle$.

A feasible allocation f is said to be a *Walrasian allocation* if there is a $p \in E^* \setminus \{0\}$ such that (p, f) is a Walrasian equilibrium. An allocation f is a *core allocation* if it is feasible and if there does not exist a coalition $S \in \mathcal{T}$ with $\nu(S) > 0$ and a Pettis integrable function $g\colon T \to E_+$ such that

(i) $\int_S g(t)\, d\nu(t) = \int_S e(t)\, d\nu(t)$, i.e. g is feasible for S, and

(ii) $g(t) \succ_t f(t)$ for almost all $t \in S$.

We denote by $C(\mathcal{E})$ the set of all core allocations of the economy $\mathcal{E}$, and by $\mathcal{W}(\mathcal{E})$ the set of Walrasian allocations.

The following assumptions on agents' characteristics are standard.

(A1) $e(t) \in E_+ \setminus \{0\}$ for every $t \in T$.

(A2) $X(t) = E_+$ for every $t \in T$.

(A3) $\succ_t$ is irreflexive and transitive for every $t \in T$.

(A4) For every $t \in T$, $\succ_t$ is continuous, i.e. for each $x \in E_+$ the sets $\{y \in E_+ \colon y \succ_t x\}$ and $\{y \in E_+ \colon x \succ_t y\}$ are (norm) open in E_+.[6]

(A5) For every $t \in T$, $\succ_t$ is strictly monotone, i.e. whenever $x, x' \in E_+$ with $x \ge x'$ and $x \ne x'$ then $x \succ_t x'$.

For our core-Walras non-equivalence results we will consider the following strengthening of (A3).

(A6) For every $t \in T$, $\succ_t$ is the asymmetric part of a reflexive, transitive, and complete preference/indifference relation $\succcurlyeq_t$.

Moreover, we will take into consideration the assumption that preferences are convex.

(A7) For every $t \in T$, $\succ_t$ is convex, i.e. for each $x \in E_+$ the set $\{y \in E_+ \colon y \succ_t x\}$ is convex.

[6]For convenience of reference later on, (A4) as well as the following assumptions on preferences are formulated for consumption sets that are equal to E_+ since these assumptions will be considered only in conjunction with Assumption (A2).

In the case where the commodity space E has the property that $\operatorname{int} E_+ \neq \emptyset$, in particular if E is actually a $C(\Omega)$ space, i.e. a Banach lattice with $\operatorname{int} E_+ \neq \emptyset$, we will take the following strengthening of (A1) into consideration.

(A8) $e(t) \in \operatorname{int} E_+$ for every $t \in T$.

In the general case where E is a Banach lattice whose positive cone E_+ may have an empty interior, we will consider a condition on marginal rates of substitution, which is taken from Zame (1986); see also Ostroy and Zame (1994).

(A9) There are strictly positive linear functionals α, $\beta \in E^*$ with $\alpha \leq \beta$ such that for every $t \in T$, whenever $x, u, v \in E_+$ satisfy $u \leq x$ and $\alpha(v) > \beta(u)$ then $x - u + v \succ_t x$.

Note that this is a requirement on preferences that is uniform over agents as well as over the consumption set E_+. We refer to Zame (1986) for a discussion of this condition as well as for corresponding examples. (It may be seen that (A9), together with (A3) and the convexity assumption (A7), is equivalent to the following statement: "There are strictly positive elements α, β in E^*, with $\alpha \leq \beta$, such that given any $t \in T$ and $x \in E_+$ there is a p in the order interval $[\alpha, \beta]$ such that $p(x) \leq p(y)$ for all $y \in E_+$ with $y \succ_t x$." Thus, since supporting price systems are measures of marginal rates of substitution, (A9) is indeed a condition putting bounds on these rates.)

It is well known that if the commodity space is infinite dimensional and consumption sets have empty interior, then—regardless of whether or not the commodity space is separable, and regardless of whether allocations are defined to be Bochner or just Pettis integrable—one way in which core-Walras equivalence can fail is through preferences displaying marginal rates of substitution that are not properly bounded; cf. the example of a failure of core-Walras equivalence described in Rustichini and Yannelis (1991). This reflects the general fact that if consumption sets in an infinite dimensional commodity space have empty interior, then continuity of preferences by itself does not provide the appropriate bounds on marginal rates of substitution in order for preferred sets to admit supporting price systems. By requiring economies to satisfy (A9), we will rule out this sort of failure of core-Walras equivalence, which is not the focus of this note.

It should be remarked that some of the above assumptions (in combination) may amount to an assumption on the commodity space E; or, to say it the other way round, some of these assumptions can be satisfied only if E^*, and hence E, has certain properties. This is the case for (A9), which can hold only if E^* indeed possesses strictly positive elements. Similarly, if E is a $C(\Omega)$ space then (A2) to (A5) together with (A7) can hold simultaneously only when strictly positive linear functionals on E do exist. (Indeed, when these assumptions hold and

$\operatorname{int} E_+ \neq \emptyset$, then, given any $t \in T$ and $x \in E_+$, the set of all $y \in E_+$ preferred to x by t is supported by a positive $p \in E^* \setminus \{0\}$, and when x actually belongs to $\operatorname{int} E_+$ then p must in fact be strictly positive, by the usual argument.) Let us remark here that strictly positive linear functionals exist on any separable Banach lattice as well as on any order continuous Banach lattice E whose positive cone E_+ contains quasi-interior points.

3.2 Results

We are going to present results showing that the crucial point for core-Walras equivalence to hold in Banach spaces is not in the first line whether allocations are defined to be Bochner or just Pettis integrable; rather, what matters are restrictions on the heterogeneity of preferences across agents, as embodied in measurability conditions on the profile of these characteristics.

As already noted in the introduction, without measurability assumptions on the profile of agents' preferences, core-Walras equivalence may fail even in the setting of finitely many commodities. In this note we will consider the following two measurability conditions, both being well known from the literature. Let E be an ordered Banach space, and let $\mathcal{E}$ be an economy with commodity space E satisfying Assumption (A2), i.e. consumption sets are equal to E_+.

(M1) If x and y are any two consumption bundles then $\{t \in T : x \succ_t y\}$ is a measurable set, i.e. it belongs to $\mathcal{T}$.

(M2) If f and g are any two allocations then $\{t \in T : f(t) \succ_t g(t)\}$ is a measurable set, i.e. it belongs to $\mathcal{T}$.

We first summarize some more or less well known facts concerning the formal relationship between these two conditions. Obviously (M2) implies (M1) under (A2), regardless of how allocations are being defined. The following proposition addresses the reverse implication. (See Section 4.1 for the proof.)

Proposition. *Let E be any ordered Banach space and let $\mathcal{E}$ be an economy with commodity space E satisfying assumptions* (A2) *to* (A5)*. Then* (M1) *implies that* (M2) *holds relative to the set of all allocations that are strongly measurable.*[7]

Thus, under some standard assumptions, in the setting where allocations are defined to be Bochner integrable, (M1) and (M2) are equivalent. According to the Pettis measurability theorem, a weakly measurable function taking values in a

[7] If for each $t \in T$, $\succ_t$ is the asymmetric part of a reflexive, transitive, and complete preference/indifference relation $\succcurlyeq_t$, then the strict monotonicity assumption (A5) can be dropped from the statement of this proposition. We have not checked whether this is possible in general.

separable Banach space is actually strongly measurable. Thus if E is separable, then (M1) and (M2) are also equivalent in the Pettis integrable allocations setting.

Let us turn to the core-Walras equivalence problem. As noted in the introduction, an objection that was made against the core-Walras non-equivalence results for non-separable Banach spaces by Tourky and Yannelis (2001) and Podczeck (2003, 2002) is that these results are artifacts of the Bochner integrable allocations setting, because Bochner integrable functions must be essentially separably valued and thus coalitional blocking possibilities are very limited when the commodity space is non-separable. Indeed, one could conjecture that if allocations are defined to be Pettis integrable, so that blocking is not restricted to separable subspaces of the commodity space, then these non-equivalence results would break down. However, as we will show now, this conjecture is false if there is no restriction on the profile of agents' preferences beyond that incorporated in the measurability assumption (M1). (Actually, in Tourky and Yannelis (2001) and Podczeck (2003, 2002), (M2) is required to hold, but by what has been noted in the previous paragraph, in the Bochner integrable allocations setting (M1) and (M2) are equivalent given that certain standard assumptions are met.)

Our first theorem shows, in particular, that the core-Walras non-equivalence results of Tourky and Yannelis (2001) and Podczeck (2003) carry over to the Pettis integrable allocations setting when only (M1) is required to hold for an economy. (For this theorem and the subsequent theorems and corollaries, note that according to the definitions in the previous subsection, the Pettis integrable allocations setting is in force in this paper.)

Theorem 1. *Let E be any non-separable Banach space. Assume the continuum hypothesis. Then there is an ordering $\geq$ on E, under which E is an ordered Banach space with* $\operatorname{int} E_+ \neq \emptyset$, *and an atomless economy $\mathcal{E}$ with commodity space E such that assumptions* (A2), (A4) *to* (A8), *and* (M1) *hold but such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$.*

(See Section 4.2 for the proof. The continuum hypothesis which is assumed in this theorem is also assumed in Tourky and Yannelis (2001) and Podczeck (2003, 2002).)

Let us turn to the commodity space setting of Banach lattices. (In particular, the ordering of the commodity space is taken to be the given lattice ordering, and is not an object of construction as in the previous theorem.) We will first consider the case where the commodity space E is actually a $C(\Omega)$ space, i.e. a Banach lattice with $\operatorname{int} E_+ \neq \emptyset$.

For such spaces E, and the context of atomless economies satisfying assumptions (A2), (A4) to (A8), and (M1), the following condition on E, called property (CD), was identified in Podczeck (2002) as the decisive condition on the commodity space in order for core-Walras equivalence to hold in the Bochner integrable allocations setting.

(CD) Given any $q \in E^*_+$ there is a countable subset D of E such that whenever $q' \in E^*_+$ and $q'(d) = q(d)$ for all $d \in D$ then $q' = q$.

(Note that this is a condition concerning only *positive* elements $q, q' \in E^*$, and that the set D in its statement may depend on q.) Clearly, every separable Banach lattice has property (CD), but there are also non-separable Banach lattices satisfying (CD) (e.g. $C(\Omega)$ where Ω is the so called split interval; see Podczeck (2002) for details). Examples of $C(\Omega)$ spaces that do not satisfy (CD) are provided by any infinite dimensional space $L_\infty(\mu)$ and, in particular, by ℓ_∞. (Again, see Podczeck (2002) for details.) The following theorem points out that also in the Pettis integrable allocations setting, core-Walras equivalence fails in $C(\Omega)$ spaces not satisfying property (CD) if the profile of agents' preferences has to satisfy only (M1).

Theorem 2. *Let E be a $C(\Omega)$ space with E^* containing strictly positive elements. Suppose that E fails property* (CD), *and assume the continuum hypothesis. Then there is an atomless economy $\mathcal{E}$ with commodity space E such that assumptions* (A2), (A4) *to* (A8), *and* (M1) *are satisfied but such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$.*

(See Section 4.3 for the proof.) As just noted, infinite dimensional $L_\infty(\mu)$ spaces fail property (CD), and when the measure μ is σ-finite then the duals of these spaces possess strictly positive elements. Thus:

Corollary 1. *Assume the continuum hypothesis. Then there exist* (*non-separable*) *$C(\Omega)$ spaces E such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$ holds for some atomless economy $\mathcal{E}$ with commodity space E satisfying assumptions* (A2), (A4) *to* (A8), *and* (M1).

Let us turn to the general case where the commodity space is a Banach lattice E whose positive cone E_+ may have an empty interior. In the context of atomless economies satisfying assumptions (A1) and (A2), (A4) to (A7), and (A9) as well as (M1), the decisive condition on E for core-Walras equivalence to hold in the Bochner integrable allocations setting was identified in Podczeck (2002) to be the following condition, called here (SI).

(SI) For every $e \in E_+$ and every strictly positive $q \in E^*$, the relativization of the topology $\sigma(E, A_q)$ to A_e is separable.[8]

(See Podczeck (2002) for an intuition for this condition, as well as for corresponding examples.) The following theorem for the Pettis integrable allocations setting holds.

[8]Recall from Section 2 that given $e \in E_+$ and $q \in E^*_+$, A_e denotes the order ideal in E generated by e, and A_q the order ideal in E^* generated by q; recall also that $\sigma(E, A_q)$ denotes the weak topology of E with respect to A_q.

Theorem 3. *Let E be a Banach lattice with E^* containing strictly positive elements. Suppose that E fails condition SI, and assume the continuum hypothesis. Then there is an atomless economy $\mathcal{E}$ with commodity space E such that assumptions* (A1) *and* (A2), (A4) *to* (A7), *and* (A9) *as well as* (M1) *are satisfied but such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$.*

(See Section 4.4 for the proof.) As shown in Podczeck (2002, Lemma 1), if E is a σ-Dedekind complete Banach lattice such that E_+ contains quasi-interior points and E^* contains strictly positive elements, then condition (SI) holds if and only if E is separable. Thus Theorem 3 implies:

Theorem 4. *Let E be any non-separable σ-Dedekind complete Banach lattice such that E_+ contains quasi-interior points and such that E^* contains strictly positive elements. Assume the continuum hypothesis. Then there is an atomless economy $\mathcal{E}$ with commodity space E such that assumptions* (A1) *and* (A2), (A4) *to* (A7), *and* (A9) *as well as* (M1) *are satisfied but such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$.*

Every order continuous Banach lattice is σ-Dedekind complete, and when the positive cone of an order continuous Banach lattice contains quasi-interior points then its dual contains strictly positive elements. Thus the following corollary of Theorem 4 holds.

Corollary 2. *Let E be any non-separable order continuous Banach lattice such that E_+ contains quasi-interior points. Assume the continuum hypothesis. Then there is an atomless economy $\mathcal{E}$ with commodity space E such that assumptions* (A1) *and* (A2), (A4) *to* (A7), *and* (A9) *as well as* (M1) *are satisfied but such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$.*

Note that all the $L_p(\mu)$ spaces, $1 \leq p < \infty$, the measure μ σ-finite, belong to the class of order continuous Banach lattices with a positive cone containing quasi-interior points. Thus for an important class of commodity spaces, core-Walras equivalence fails also in the Pettis integrable allocations setting when the commodity space is non-separable and just (M1) is required to hold for the profile of agents' preferences.

The results so far mean that defining allocations to be Pettis integrable, rather than Bochner integrable, can have an effect in regard to the core-Walras equivalence problem, compared with the non-equivalence results in Tourky and Yannelis (2001) and Podczeck (2003, 2002), only in conjunction with a restriction of preference heterogeneity across individuals that goes beyond that implied by the measurability assumption (M1). Thus let us consider (M2).

It turns out that even (M2) not necessarily does the job. Indeed, by a result due to Kunen (see Negrepontis, 1984, pp. 1123–1128) there exists, under the continuum hypothesis, a compact Hausdorff space Ω with the following properties: (a) Ω is separable; (b) there is a point in Ω at which Ω is not first countable, so that, in particular, $C(\Omega)$ is non-separable; but (c) given any finite measure space $(T, \mathcal{T}, \nu)$, every weakly measurable function from T into $C(\Omega)$, hence every Pettis integrable function from T into $C(\Omega)$, is in fact strongly measurable.[9] Because of (c), given an economy with commodity space $C(\Omega)$ satisfying assumptions (A2) to (A5), if (M1) holds then (M2) holds as well according to the proposition above. On the other hand, (a) means that $C(\Omega)^*$ possesses strictly positive elements, while (b) implies that $C(\Omega)$ fails property (CD) as may readily be seen. Consequently Theorem 2 implies:

Theorem 5. *Assume the continuum hypothesis. Then there exist* (*non-separable*) $C(\Omega)$ *spaces* E *such that* $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$ *holds for some atomless economy* $\mathcal{E}$ *with commodity space* E *satisfying assumptions* (A2), (A4) *to* (A8), *as well as* (M2).

However, there are some non-separable commodity spaces for which (M2) indeed leads to core-Walras equivalence in the Pettis integrable allocations setting. In fact, we have the following result.

Theorem 6. *Let* E *be any order continuous Banach lattice with* E_+ *containing quasi-interior points. Then* $C(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ *holds for every atomless economy* $\mathcal{E}$ *with commodity space* E *satisfying assumptions* (A1) *to* (A5), (A9), *and* (*M2*).

Thus, in the Pettis integrable allocations setting, core-Walras equivalence holds in particular for the important class of the $L_p(\mu)$ spaces, $1 \le p < \infty$, the measure μ σ-finite, regardless of whether the actual space under consideration is separable or not, provided that (M2) and some other standard assumptions are in force. (See Section 4.5 for the proof of Theorem 6. Note that in this latter theorem, preferences are not assumed to be complete or convex, and that no set theoretical assumption is involved.)

What drives Theorem 6, compared with Theorem 5, is the fact that for the commodity spaces of Theorem 6 there is a plenty of allocations that are not strongly measurable, so that (M2) indeed imposes a restriction on the heterogeneity allowed for a profile of agents' preferences in an atomless economy, which goes beyond the restriction implied by (M1).[10]

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[9]Properties (b) and (c) are not explicitly stated in Negrepontis (1984). However, Ω is the one-point compactification $\Omega' \cup \omega_0$ of a locally compact Hausdorff space Ω' that is not Lindelöf, which implies that Ω is not first countable at ω_0. On the other hand, $C(\Omega)$ is Lindelöf in the weak topology, which implies (c) because Ω being separable means that $C(\Omega)^*$ contains a countable set separating the points of $C(\Omega)$; for this latter implication, see Lemma 1 in Section 4.5.1.

[10]Of course, this is so only when the measure space of agents has non-measurable subsets. However, it is consistent with ZFC that there is no non-trivial atomless measure on the power set of any set, i.e. that every (non-trivial) atomless measure space has many non-measurable subsets. In the proof of Theorem 6 we will take care of the possibility of an atomless measure space of agents where every subset is measurable.

4 Proofs

4.1 Proof of the Proposition in Section 3.2

Note for the following that all consumption sets are equal to E_+ according to Assumption (A2). Let f, $g\colon T \to E_+$ be any two strongly measurable functions.

By definition of "strongly measurable," there exist sequences (f'_n) and (g'_n) of measurable simple functions from T into E such that for almost all $t \in T$, $f'_n(t) \to f(t)$ and $g'_n(t) \to g(t)$. Since f'_n, g'_n are measurable simple functions, there are, for each $n \geq 1$, measurable simple functions $f_n, g_n\colon T \to E_+$ such that

$$\|f'_n(t) - f_n(t)\| \leq \operatorname{dist}(f'_n(t), E_+) + 1/n$$

for all $t \in T$ as well as

$$\|g'_n(t) - g_n(t)\| \leq \operatorname{dist}(g'_n(t), E_+) + 1/n$$

for all $t \in T$, where $\operatorname{dist}(x, E_+) = \inf\{\|x - y\| : y \in E_+\}$. Consider the sequence (f_n). For each $t \in T$,

$$\begin{aligned} \|f(t) - f_n(t)\| &\leq \|f(t) - f'_n(t)\| + \|f'_n(t) - f_n(t)\| \\ &\leq \|f(t) - f'_n(t)\| + \operatorname{dist}(f'_n(t), E_+) + 1/n \\ &\leq \|f(t) - f'_n(t)\| + \|f'_n(t) - f(t)\| + 1/n \end{aligned}$$

the latter inequality holding because $f(t) \in E_+$. Thus $f_n(t) \to f(t)$ for almost all $t \in T$ (because $f'_n(t) \to f(t)$ for almost all $t \in T$). Analogously it follows that $g_n(t) \to g(t)$ for almost all $t \in T$.

Let T' be the set of all $t \in T$ for which $f_n(t) \to f(t)$ as well as $g_n(t) \to g(t)$. Then $T \setminus T'$ is a null set, and since $(T, \mathcal{T}, \nu)$ is complete, it suffices to show that $\{t \in T' : g(t) \succ_t f(t)\}$ belongs to $\mathcal{T}$. Thus we may as well assume that $T' = T$. Fix any $v \in E_+ \setminus \{0\}$. Using transitivity, continuity, and strict monotonicity of preferences, it is straightforward to check that

$$\{t \in T : g(t) \succ_t f(t)\} = \bigcup_k \bigcup_m \bigcap_{n \geq m} \{t \in T : (1 - (1/k))g_n(t) \succ_t (1/k)v + f_n(t)\}$$

where $k, m, n \in \mathbb{N} \setminus \{0\}$. (To see that the set on the left is contained in that on the right, note that continuity and strict monotonicity imply, in particular, that whenever $g(t) \succ_t f(t)$ there is a $z \in E_+$ such that $g(t) \succ_t z \succ_t f(t)$.)

Evidently (M1) implies that (M2) holds relative to the set of all allocations that are simple functions. Hence, since $(1 - (1/k))g_n$ and $(1/k)v1_T + f_n$, $k \geq 1$, are measurable simple functions, the sets

$$\{t \in T : (1 - (1/k))g_n(t) \succ_t (1/k)v + f_n(t)\}$$

are in $\mathcal{T}$, and it follows that $\{t \in T : g(t) \succ_t f(t)\} \in \mathcal{T}$ as well. This completes the proof of the proposition.

4.2 Proof of Theorem 1

We first construct an ordering on E, in the same way as in Tourky and Yannelis (2001) and Podczeck (2003). Let B_E denote the closed unit ball in E. Pick some $u \in E$ with $\|u\| = 3$ and let C be the cone generated by $\{u\} + B_E$, i.e.

$$C = \{x \in E\colon x = \lambda(u + y), y \in B_E, \lambda \geq 0\}.$$

Then C is convex, and since $u \notin B_E$, C is closed and $C \cap -C = \{0\}$. Thus C generates a vector ordering on E under which E becomes an ordered Banach space with positive cone E_+ equal to C.[11] Evidently, $\operatorname{int} E_+ \neq \emptyset$.

Next, using the Hahn Banach theorem, select a $\hat{q} \in E^*$ with $\|\hat{q}\| = 3$ and $\hat{q}(u) = 9$ (as is possible since $\|u\| = 3$). Then for each $q \in E^*$ with $\|q\| \leq 1$ and each $y \in B_E$,

$$(\hat{q} + q)(u + y) = 9 + \hat{q}(y) + q(u) + q(y) \geq 9 - 3 - 3 - 1 > 0.$$

That is, for each $q \in E^*$ with $\|q\| \leq 1$, $\hat{q} + q$ is a strictly positive element of E^*. In particular, $\hat{q}$ is strictly positive.

Since E is non-separable, and since the continuum hypothesis is assumed, we may appeal to Podczeck (2003, Section 4.1, Proposition)[12] to find a family $(q'_\alpha)_{\alpha<\omega_1}$ of elements of E^*, denoting by ω_1 the first uncountable ordinal number, such that $q'_\alpha \neq 0$ and $\|q'_\alpha\| \leq 1$ for every ordinal $\alpha < \omega_1$, but such that given any $x \in E$ there is an ordinal $\alpha_x < \omega_1$ such that for each $\alpha \in [\alpha_x, \omega_1)$, $q'_\alpha(x) = 0$. For each $\alpha < \omega_1$ set $q_\alpha = q'_\alpha + \hat{q}$. Then, by what has been noted in the previous paragraph, each q_α is a strictly positive element of E^*. Also, $q_\alpha \neq \hat{q}$ for each $\alpha < \omega_1$, but given any $x \in E$ there is an ordinal $\alpha_x < \omega_1$ such that for each $\alpha \in [\alpha_x, \omega_1)$, $q_\alpha(x) = \hat{q}(x)$.

Let $(T, \mathcal{T}, \nu)$ be any non-trivial, atomless, complete, finite measure space. We will construct an economy $\mathcal{E}$ with $(T, \mathcal{T}, \nu)$ as measure space of agents and E as commodity space such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$ but such that all the assumptions listed in the statement of Theorem 2 hold.

The continuum hypothesis, which is assumed, implies that there is no non-trivial atomless measure on the power set of any set. Thus there must be an $S \subset T$ with $\nu_*(S) < \nu^*(S)$.[13] Evidently this implies that there is an $\overline{S} \subset T$ such that actually $0 = \nu_*(\overline{S}) < \nu^*(\overline{S})$. Choose and fix such a set $\overline{S}$.

[11]It is easily seen that the cone E_+ so constructed is normal, i.e. has not an excessive "width;" in particular, order intervals are norm bounded, and every element of E^* is the difference of two elements of E^*_+. Cf. Kelley and Namioka (1976, pp. 227 and 228).

[12]This proposition in Podczeck (2003) relies on a result due to Juhász and Szentmiklóssy (1992) about transfinite sequences in compact spaces.

[13]If A is any subset of T, then $\nu_*(A)$ denotes the inner measure of A and $\nu^*(A)$ denotes the outer measure of A.

Again since the measure space (T, S, ν) is atomless, and since the continuum hypothesis is in force, we can write $T = \bigcup_{\alpha<\omega_1} N_\alpha$ where $(N_\alpha)_{\alpha<\omega_1}$ is a family of pairwise disjoint null sets in T, again denoting by ω_1 the first uncountable ordinal number. (Cf. Proposition 5.2 in Tourky and Yannelis, 2001.) Denote by $\phi\colon \overline{S} \to [0, \omega_1)$ the mapping that takes a $t \in \overline{S}$ to that ordinal number α for which $t \in N_\alpha$.

For each $t \in \overline{S}$ set $q_t = q_{\phi(t)}$. Then $(q_t)_{t\in\overline{S}}$ is a family of strictly positive elements of E^* such that

$$q_t \neq \hat{q} \text{ for all } t \in \overline{S} \tag{1}$$

but

$$\text{for any } x \in E,\ q_t(x) = \hat{q}(x) \text{ for almost all } t \in \overline{S} \tag{2}$$

because for each ordinal number $\alpha < \omega_1$ we have $\phi^{-1}([0, \alpha)) = \overline{S} \cap \bigcup_{\alpha'<\alpha} N_{\alpha'}$, each $N_{\alpha'}$ is a null set, and for each $\alpha < \omega_1$ the set $[0, \alpha)$ is countable.

It is straightforward to verify that (1) and (2) together with the fact that $\nu^*(\overline{S}) > 0$ imply:

$$\begin{aligned} &\text{There is no } p \in E^* \text{ such that for almost every } t \in \overline{S}, \\ &\qquad\qquad q_t = \lambda_t p \text{ for some real number } \lambda_t. \end{aligned} \tag{3}$$

We now construct an economy with $(T, \mathcal{T}, \nu)$ as measure space of agents in the following way. Fix any interior point $\overline{e}$ of E_+. For each agent t in T, we let the consumption set be equal to E_+ and the endowment $e(t)$ be equal to $\overline{e}$. Then assumptions (A2) and (A8) are met. Further, since the measure ν is finite, the endowment mapping $t \mapsto \overline{e}$ is Pettis integrable, as required in our definition of an economy.

Concerning preferences, for each $t \in \overline{S}$ let a utility function $u_t\colon E_+ \to \mathbb{R}$ be defined by

$$u_t(x) = q_t(x),\ x \in E_+,$$

and for each $t \in T \setminus \overline{S}$, let a utility function $u_t\colon E_+ \to \mathbb{R}$ be defined by

$$u_t(x) = \hat{q}(x),\ x \in E_+.$$

Clearly the family of preferences so defined satisfies all the assumptions from (A4) to (A7). Moreover, using (2), given any $x \in E_+$ we have $u_t(x) = \hat{q}(x)$ for almost all $t \in T$. Evidently this implies that (M1) holds because the measure space $(T, \mathcal{T}, \nu)$ is complete.

We have thus constructed an atomless economy $\mathcal{E}$ with commodity space E such that the assumptions listed in the statement of Theorem 2 all hold.

Consider the initial allocation $t \mapsto \overline{e}$. Since $\overline{e} \in \operatorname{int} E_+$ and consumption sets are equal to E_+, a glance at (3) shows that this allocation is not Walrasian. Thus to finish the proof, it suffices to show that the initial allocation $t \mapsto \overline{e}$ is in $C(\mathcal{E})$.

To this end, fix any coalition $S \in \mathcal{T}$ with $\nu(S) > 0$ and let $f: T \to E_+$ be any allocation (i.e. Pettis integrable function) such that $u_t(f(t)) > u_t(\overline{e})$ for almost all $t \in S$. By the definition of the u_t this means that for some null set $N \subset (S \setminus \overline{S})$ we have $\langle \hat{q}, f(t) \rangle > \langle \hat{q}, \overline{e} \rangle$ for all $t \in (S \setminus \overline{S}) \setminus N$. Let

$$S' = \{t \in S \colon \langle \hat{q}, f(t) \rangle \le \langle \hat{q}, \overline{e} \rangle\}.$$

Then $S' \setminus N \subset S \cap \overline{S}$. By definition of Pettis integrability, f is weakly measurable and thus $S' \in \mathcal{T}$ since $(T, \mathcal{T}, \nu)$ is complete. Hence $S' \setminus N \in \mathcal{T}$ as well, and because $\nu_*(\overline{S}) = 0$ we must have $\nu(S' \setminus N) = 0$ whence $\nu(S') = 0$. That is, $\langle \hat{q}, f(t) \rangle > \langle \hat{q}, \overline{e} \rangle$ for almost all $t \in S$ whence f is not feasible for S. We conclude that the initial allocation $t \mapsto \overline{e}$ indeed belongs to $C(\mathcal{E})$. This completes the proof of the theorem.

4.3 Proof of Theorem 2

Since E fails property (CD) by hypothesis, and since the continuum hypothesis is assumed, is follows by arguments from the proof of Theorem 1 in Podczeck (2002) that there are a $\hat{q} \in E^*_+$ and a family $(q_\alpha)_{\alpha < \omega_1}$ of elements of E^*_+—as earlier denoting by ω_1 the first uncountable ordinal number—such that $q_\alpha \neq \hat{q}$ for each α but such that given any $x \in E$ there is an ordinal $\alpha_x < \omega_1$ such that for each $\alpha \in [\alpha_x, \omega_1)$, $q_\alpha(x) = \hat{q}(x)$. Because E^* contains strictly positive elements by hypothesis, it may be assumed that $\hat{q}$ and each q_α are actually strictly positive (by adding, if necessary, a common strictly positive element of E^* to $\hat{q}$ and to each q_α). The arguments of the proof of Theorem 1 from the fourth paragraph upwards now verbatim apply to establish the theorem.

4.4 Proof of Theorem 3

Let $(T, \mathcal{T}, \nu)$ be any non-trivial, atomless, complete, finite measure space. The hypotheses about E together with the continuum hypothesis guarantee, according to the proof of Theorem 2 in Podczeck (2002), that there are an $\overline{e} \in E_+ \setminus \{0\}$, strictly positive elements α, β, $\hat{q} \in E^*$, with $\alpha \le \beta$, and for each $t \in T$ a $q_t \in E^*$ such that:

(a) $\hat{q} \in [\alpha, \beta]$ and for each $t \in T$, $q_t \in [\alpha, \beta]$.

(b) For each $t \in T$ there is a $z \in A_{\overline{e}}$ (depending on t) such that $q_t(z) \neq \hat{q}(z)$. (Recall: $A_{\overline{e}}$ denotes the order ideal generated by $\overline{e}$.)

(c) For each $x \in E$, $q_t(x) = \hat{q}(x)$ for almost all $t \in T$.

By the arguments from the proof of Theorem 1, choose a set $\overline{S} \subset T$ with $0 = \nu_*(\overline{S}) < \nu^*(\overline{S})$. Define an economy $\mathcal{E}$ with $(T, \mathcal{T}, \nu)$ as space of agents and E as commodity space in the following way. For each $t \in T$, let the consumption set be equal to E_+ and the endowment be equal to $\overline{e}$. For each $t \in \overline{S}$, let a utility function $u_t \colon E_+ \to \mathbb{R}$ be given by

$$u_t(x) = q_t(x),\ x \in E_+,$$

and for $t \in T \setminus \overline{S}$ let a utility function $u_t \colon E_+ \to \mathbb{R}$ be given by

$$u_t(x) = \hat{q}(x),\ x \in E_+.$$

The atomless economy $\mathcal{E}$ so defined satisfies all the assumptions listed in the statement of Theorem 3. Indeed, this is clear for (A1), (A2), and (A4) to (A7). (For (A5), recall that q_t, $t \in \overline{S}$, and $\hat{q}$ are strictly positive.) Since $\hat{q}$ and q_t, $t \in \overline{S}$, belong to the order interval $[\alpha, \beta]$ and α, β are strictly positive, (A9) is also satisfied as may readily be verified. (See the proof of Theorem 2 in Podczeck, 2002, for the details.) Finally, given any $x \in E_+$ we have $u_t(x) = \hat{q}(x)$ for almost all $t \in T$, and this implies that (M1) holds since the measure space $(T, \mathcal{T}, \nu)$ is complete.

Now by virtue of the fact that $\nu_*(\overline{S}) = 0$, it follows as in the proof of Theorem 1 that the initial allocation $t \mapsto \overline{e}$ is in $C(\mathcal{E})$. Thus it remains to see that this allocation is not Walrasian. To this end, let $\hat{q}|A_{\overline{e}}$ and $q_t|A_{\overline{e}}$, $t \in \overline{S}$, denote the restrictions to $A_{\overline{e}}$ of $\hat{q}$ and q_t, respectively. Then, from above,

$$q_t|A_{\overline{e}} \neq \hat{q}|A_{\overline{e}} \text{ for each } t \in \overline{S}$$

but

$$\text{for any } x \in A_{\overline{e}},\ q_t|A_{\overline{e}}(x) = \hat{q}|A_{\overline{e}}(x) \text{ for almost all } t \in \overline{S}.$$

By virtue of the fact that $\nu^*(\overline{S}) > 0$, this implies that there is no $p \in E^*$ such that for almost every $t \in \overline{S}$, $q_t|A_{\overline{e}} = \lambda_t p|A_{\overline{e}}$ for some real number λ_t, as may easily be verified.

Now suppose, if possible, that for some $p \in E^*$ the pair $(p, t \mapsto \overline{e})$ were a Walrasian equilibrium. Note that given any $z \in A_{\overline{e}}$, for some real number $\lambda > 0$ we have $\overline{e} + \lambda z \geq 0$. Thus the equilibrium conditions would imply that for almost every $t \in \overline{S}$, $A_{\overline{e}} \cap \ker p \subset \ker q_t$,[14] or, equivalently, $q_t|A_{\overline{e}} = \lambda_t p|A_{\overline{e}}$ for some real number λ_t. However, this contradicts the conclusion of the previous paragraph. Hence the theorem has been established.

[14] $\ker p$ denotes the kernel of p, i.e. $\ker p = \{x \in E \colon p(x) = 0\}$; similarly for $\ker q_t$.

4.5 Proof of Theorem 6

4.5.1 Preliminaries

In this subsection, E is a (real) Banach space and $(T, \mathcal{T}, \nu)$ is a complete finite measure space. We first fix some additional notation and terminology, and collect some facts which will be used in the following proofs.

If $A \subset E$ and $p \in E^*$ then
- $\langle p, A\rangle$ denotes the set $\{p(x) : x \in A\}$;
- $\mathrm{c}\ell\, A$ denotes the (norm) closure of A;
- $\mathrm{co}\, A$ denotes the convex hull of A.

Recall that two weakly measurable functions $f, g : T \to E$ are said to be *weakly equivalent* if for each $q \in E^*$, $\langle q, f(t)\rangle = \langle q, g(t)\rangle$ for almost all $t \in T$, the exceptional set of measure zero possibly depending on q.

Recall that a Banach space E is said to be *weakly compactly generated* if it contains a weakly compact subset whose linear span is dense in E. A Banach space E is *measure-compact* if given any finite measure space $(T, \mathcal{T}, \nu)$, every weakly measurable function $f : T \to E$ is weakly equivalent to a strongly measurable function $g : T \to E$.[15] The Banach space E is said to have the *PIP* ("Pettis integral property") if given any finite measure space $(T, \mathcal{T}, \nu)$, every norm bounded and weakly measurable function $f : T \to E$ is Pettis integrable.

We will use the following facts. If the Banach space E is weakly compactly generated then E is measure-compact, and if E is measure-compact then E has the PIP. Further, if E is weakly compactly generated then E^* is *angelic* in the weak* topology $\sigma(E^*, E)$; that is, for a bounded set $A \subset E^*$, the $\sigma(E^*, E)$-closure of A is the set of $\sigma(E^*, E)$-limits of sequences from A. (For these facts, see Edgar, 1979, pp. 563.)

Finally, note that if E is an order continuous Banach lattice such that E_+ contains quasi-interior points, then E is weakly compactly generated. (Indeed, e being a quasi-interior point of E_+ means, by definition, that the linear span of the order interval $[-e, e]$ is dense in E, and if E is order continuous then order intervals in E are weakly compact.)

The following lemma was invoked in the discussion preceding the statement of Theorem 5. We remark for that context that a Banach space that is Lindelöf in the weak topology is measure-compact. (Again, see Edgar, 1979, pp. 563.)

Lemma 1. *Suppose E is measure-compact and that E^* contains a countable set separating the points of E. Then every weakly measurable function from T into E is strongly measurable.*

[15] The original definition of "measure-compact" for a Banach space is that every probability measure on the Baire σ-algebra generated by the weak topology is τ-smooth; according to Edgar (1977, Proposition 5.4), this definition is equivalent to the one presented here.

Proof. Suppose $f: T \to E$ is weakly measurable. Since E is measure-compact, there is a strongly measurable function $g: T \to E$ which is weakly equivalent to f. But since E^* contains a countable set separating the points of E, the fact that f and g are weakly equivalent means that we must have $f(t) = g(t)$ for almost all $t \in T$. Thus, since g is strongly measurable, f is strongly measurable as well. □

The following lemmata will be needed in the sequel.

Lemma 2. *Let Λ be a closed convex cone in E and let $f: T \to E$ be a Pettis integrable function with $f(t) \in \Lambda$ for almost all $t \in T$. Let $g: T \to E$ be a strongly measurable function and suppose that g is weakly equivalent to f. Then also $g(t) \in \Lambda$ for almost all $t \in T$.*

Proof. Since g is strongly measurable, and since $(T, \mathcal{T}, \nu)$ is complete, g is $\mathcal{T}$–$\mathcal{B}(E)$ measurable[16] and therefore we can consider the image measure on $(E, \mathcal{B}(E))$ of ν under g; let us denote this measure by μ. Another appeal to the fact that g is strongly measurable shows that μ has a support, denoted by $\operatorname{supp}\mu$ in the sequel. (Indeed, select a closed separable subspace F of E such that $g(t) \in F$ for almost all $t \in T$. In particular, then, $\mu(E \setminus F) = 0$. Note that $\mathcal{B}(F) = \{B \in \mathcal{B}(E) : B \subset F\}$ and let μ_F be the restriction of μ to $\mathcal{B}(F)$. Since F is separable, μ_F has a support. It follows that μ has a support, too.)

For each $p \in E^*$, let H_p denote the closed halfspace $\{x \in E : p(x) \geq 0\}$. Further, let $\Lambda^* = \{p \in E^* : p(x) \geq 0 \text{ for all } x \in \Lambda\}$ and note that $\Lambda = \bigcap_{p \in \Lambda^*} H_p$ by the Hahn-Banach theorem.

Now since g is weakly equivalent to f and f is Pettis integrable, g is also Pettis integrable, and in particular, $\int_S g(t)\, d\nu(t) = \int_S f(t)\, d\nu(t)$ for each $S \in \mathcal{T}$. By hypothesis, for each $p \in \Lambda^*$, $f(t) \in H_p$ for almost all $t \in T$ and hence $\int_S f(t)\, d\nu(t) \in H_p$ for all $S \in \mathcal{T}$. Thus, for each $p \in \Lambda^*$, $\int_S g(t)\, d\nu(t) \in H_p$ for all $S \in \mathcal{T}$ whence $g(t) \in H_p$ for almost all $t \in T$. But since each set H_p is closed, this implies that $\operatorname{supp}\mu \subset H_p$ for each $p \in \Lambda^*$, that is, $\operatorname{supp}\mu \subset \Lambda$. Thus $\mu(E \setminus \Lambda) = 0$ whence $g(t) \in \Lambda$ for almost all $t \in T$. □

Lemma 3. *Suppose E is weakly compactly generated and let G be a total subset of E^*. Let A be any non-empty subset of T (*not necessarily measurable*) and let $f: A \to E$ be any function (*also not necessarily measurable*). Suppose that for each $p \in G$, $\langle p, f(t)\rangle = 0$ for almost all $t \in A$. Then for each $p \in E^*$, $\langle p, f(t)\rangle = 0$ for almost all $t \in A$.*

Proof. Consider the set $F \subset E^*$ defined as

$$F = \{p \in E^* : \langle p, f(t)\rangle = 0 \text{ for almost all } t \in A\}.$$

[16] $\mathcal{B}(E)$ denotes the (norm) Borel σ-algebra of E; similarly for $\mathcal{B}(F)$ below.

Evidently, F is a linear subspace of E^* containing G. Hence F is weak* dense in E^* because G is total. Observe next that if (p_n) is any sequence in F that is weak* convergent to some $p \in E^*$ then p must be in F, too. Consequently, since the dual of a weakly compactly generated Banach space is angelic in the weak* topology, $F \cap B$ is weak* closed for each weak* compact subset B of E^*. By the Krein-Smulian theorem, it follows that F is weak* closed, whence $F = E^*$ since F is weak* dense in E^*. □

For the presentation of the next three lemmata we introduce the following definition. Given a measurable space $(T, \mathcal{T})$ and any vector space X, we say that a set A of functions from T into X has *Property* $(*)$ if whenever $f, g \in A$ and $S \in \mathcal{T}$ then also $1_S f + 1_{T \setminus S} g \in A$.

Lemma 4. *Let A be a set of strongly measurable Pettis integrable functions from T into E and let $B = \{z \in E \colon z = \int f$ for some $f \in A\}$. Suppose that $(T, \mathcal{T}, \nu)$ is atomless and that A has Property $(*)$. Then the* (*norm*) *closure of B is convex.*

Proof. As shown by Zame (1986, Lemma D, p. 9-13), the (norm) closure of the range of a vector measure defined by a Pettis integrable but strongly measurable function[17] on an atomless measure space is convex. With this fact substituted for the corresponding fact about the vector measure defined by a Bochner integrable function, the arguments in the proof of Theorem 6.2 in Yannelis (1991, p. 22) apply to yield the claim of the lemma. □

Lemma 5. *Let $\varphi \colon T \to 2^E$ be a correspondence, A be the set of all strongly measurable Pettis integrable selections of φ, and $B = \{z \in E \colon z = \int f$ for some $f \in A\}$. If $(T, \mathcal{T}, \nu)$ is atomless then $c\ell\, B$ is convex.*

Proof. Evidently A has Property $(*)$. Thus the lemma follows from the previous one. □

Lemma 6. *Let $\varphi \colon T \to 2^E$ be a correspondence, let C be the set of* all *Pettis integrable selections of φ, and let $D = \{z \in E \colon z = \int g$ for some $g \in C\}$. Suppose that $(T, \mathcal{T}, \nu)$ is atomless and that E is measure-compact. Then $c\ell\, D$ is convex.*

Proof. Let A be the set of all strongly measurable functions f from T into E such that f is weakly equivalent to some $g \in C$. Then every element of A is Pettis integrable since every element of C is. Moreover, A has Property $(*)$ since C does. Let $B = \{z \in E \colon z = \int f$ for some $f \in A\}$. A glance at Lemma 4 shows that $c\ell\, B$ is convex.

Now since E is measure-compact, *every* $g \in C$ is weakly equivalent to some strongly measurable function f from T into E. Thus $D = B$ and we may conclude that $c\ell\, D$ is convex. □

[17] I.e. the vector measure W defined by $W(S) = \int_S f\, d\nu$, $S \in \mathcal{T}$, if f is the function in question.

For convenience of reference, we also state the following trivial modification of the previous lemma.

Lemma 7. *Let $\varphi\colon T \to 2^E$ be a correspondence, let $S \in \mathcal{T}$, and let C be the set of all Pettis integrable selections g of φ such that $1_S g$ is strongly measurable. Let $D = \{z \in E\colon z = \int g$ for some $g \in C\}$. Suppose that $(T, \mathcal{T}, \nu)$ is atomless and that E is measure-compact. Then $c\ell\, D$ is convex.*

Proof. The arguments of the proof of the previous lemma apply. □

Lemma 8. *Let $f\colon T \to E$ be a Pettis integrable but strongly measurable function. Then given any $\epsilon > 0$ and any $S \in \mathcal{T}$ with $\nu(S) > 0$ there is an $S' \in \mathcal{T}$ with $S' \subset S$ such that the mapping $1_{S'} f$ is Bochner integrable and such that*

$$\left\| \int_S f(t)\, d\nu(t) - \int_{S'} f(t)\, d\nu(t) \right\| < \epsilon.$$

(*In particular, the Pettis integral and the Bochner integral of $1_{S'} f$ coincide.*) *Moreover, given $\delta > 0$, S' can be chosen so that $\nu(S \setminus S') < \delta$.*

Proof. Let $S \in \mathcal{T}$, with $\nu(S) > 0$. For each integer $n > 0$, set

$$S_n = \{t \in S\colon \|f(t)\| \le n\}.$$

Then $S_n \uparrow S$. Also, since f is strongly measurable, the mapping $t \mapsto \|f(t)\|$ is measurable, so $S_n \in \mathcal{T}$ for each n. Now by Diestel and Uhl (1977, Theorem 5, p.53), the indefinite Pettis integral of the Pettis integrable function f is a ν-continuous vector measure, so $\int_{S \setminus S_n} f(t)\, d\nu(t) \to 0$ since $\nu(S \setminus S_n) \to 0$. Hence, by the additivity of the indefinite Pettis integral, $\int_{S_n} f(t)\, d\nu(t) \to \int_S f(t)\, d\nu(t)$.

Finally, by definition of the sets S_n, for each n the mapping $1_{S_n} f$ is norm bounded, and therefore Bochner integrable since it is strongly measurable. □

Lemma 9. *Suppose E is an order continuous Banach lattice such that E_+ contains quasi-interior points. Then there is a family $(x_i, p_i)_{i \in I}$ of elements of $E \times E^*$ such that:*

(i) $\langle p_i, x_j \rangle \ne 0$ *if and only if* $i = j$.

(ii) *The set $\{p_i\colon i \in I\}$ is a total subset of E^* (i.e. separates the points of E).*

(iii) *Let Q denote the set of all finite linear combinations of the x_i such that the coefficients are rational. Then $Q \cap E_+$ is dense in E_+.*

(*Thus the family $(x_i, p_i)_{i \in I}$ is a Markushevich basis for E with a special property. Note that it is* not *claimed that the elements x_i themselves belong to E_+.*)

Proof.[18] By a well known representation theorem (see Lindenstrauss and Tzafriri, 1979, p. 25, Theorem 1.b.14) we may assume that for some probability space (Ω, Σ, μ):

(I)

(a) $L_\infty(\mu) \subset E \subset L_1(\mu)$ and the ordering of E is that induced from $L_1(\mu)$ (i.e. is the "pointwise almost everywhere" ordering).

(b) $L_\infty(\mu) \subset E^* \subset L_1(\mu)$.

(c) $\langle p, x\rangle = \int px\, d\mu$ for all $p \in E^*$ and $x \in E$.

In particular, then, the subspace $L_\infty(\mu)$ of E separates the points of E^* (because $L_\infty(\mu)$ separates the points of $L_1(\mu)$). That is, $L_\infty(\mu)$ is $\|\cdot\|_E$-dense in E. By the continuity of the lattice operations in E this implies (since the ordering of E is the "pointwise almost everywhere" ordering):

(Id) $L_\infty(\mu)_+$ is a $\|\cdot\|_E$-dense subset of E_+.

Assume (for the time being) that there is a family $(x_i, p_i)_{i\in I}$ of elements of $L_\infty(\mu) \times L_\infty(\mu)$ such that:

(II)

(1) $\int p_i x_j\, d\mu \neq 0$ if and only if $i = j$.

(2) The set $\{p_i : i \in I\}$ separates the points of $L_1(\mu)$.

(3) Let Q be the set of all finite linear combinations of x_i's with rational coefficients. Then $Q \cap [0, 1_\Omega]$ is $\|\cdot\|_1$-dense in $[0, 1_\Omega]$ (where $[0, 1_\Omega]$ is the order interval $\{x \in L_\infty(\mu) : 0 \leq x \leq 1_\Omega\}$).

Then by (Ia) and (Ib), $(x_i, p_i)_{i\in I}$ is actually a family of elements of $E \times E^*$. Because of (Ic) and (II1), it satisfies (i) of the theorem. Also, from (Ic) and (II2), it is clear that (ii) of the theorem holds. To see that (iii) holds as well, consider the order interval $[0, 1_\Omega] \subset L_\infty(\mu) \subset L_1(\mu)$ and set $Q^1 = Q \cap [0, 1_\Omega]$. By (II3), Q^1 is $\sigma(L_1, L_\infty)$-dense in $[0, 1_\Omega]$. Now by (Ia), $[0, 1_\Omega]$ is also an order interval in E. In particular, $[0, 1_\Omega]$ is $\sigma(E, E^*)$-compact since E is order continuous. But from (Ib) and (Ic), the topology $\sigma(E, E^*)$ is, on $[0, 1_\Omega]$, at least as strong as the topology $\sigma(L_1, L_\infty)$. It follows that both topologies agree on $[0, 1_\Omega]$ and hence that Q^1 is $\sigma(E, E^*)$-dense in $[0, 1_\Omega]$. Evidently the $\|\cdot\|_E$-closure of Q^1 is convex, and is therefore the same as the closure of Q^1 for the topology $\sigma(E, E^*)$. Thus Q^1 is in fact $\|\cdot\|_E$-dense in $[0, 1_\Omega]$. It follows that $Q \cap L_\infty(\mu)_+$ is $\|\cdot\|_E$-dense in $L_\infty(\mu)_+$,

[18] In this proof, $\|\cdot\|_E$ will refer to the norm of E; $\|\cdot\|_1$ will refer to the usual $L_1(\mu)$ norm, and $\|\cdot\|_\infty$ to the usual $L_\infty(\mu)$ norm.

and from this combined with (Id) that $Q \cap E_+$ is $\|\cdot\|_E$-dense in E_+. Thus (iii) of the theorem is satisfied by the family $(x_i, p_i)_{i\in I}$.

Thus we must show that a family $(x_i, p_i)_{i\in I}$ of elements of $L_\infty(\mu) \times L_\infty(\mu)$ satisfying (1) to (3) of (II) does exist.

Before proceeding with this task, we introduce some notational conventions. Let $S \in \Sigma$ with $\mu(S) > 0$. Then μ_S denotes the restriction of μ to (S, Σ_S) where $\Sigma_S = \{S' \in \Sigma\colon S' \subset S\}$. Also, to avoid confusion, in $L_1(\mu_S)$ the characteristic function of S is denoted by $\tilde{1}_S$, while in $L_1(\mu)$ the characteristic function of S is denoted, following our general notation, by 1_S. Further, let $\{-1,1\}$ denote the two point measure space, each point in which has measure $1/2$, and let $\{-1,1\}^K$ denote the product measure space of K copies of $\{-1,1\}$, where K is an arbitrary non-empty index set.

Suppose we have a partition π of Ω into sets $S \in \Sigma$ with $\mu(S) > 0$—in particular, π is (at most) countable—such that for each $S \in \pi$ there is a family $(\tilde{x}_i, \tilde{p}_i)_{i\in I_S}$ in $L_\infty(\mu_S) \times L_\infty(\mu_S)$ for which (II) holds with $L_1(\mu_S)$ in place of $L_1(\mu)$ and $[0, \tilde{1}_S]$ in place of $[0, 1_\Omega]$. For each $S \in \pi$, set $x_i = 1_S\tilde{x}_i$ and $p_i = 1_S\tilde{p}_i$ for all $i \in I_S$. Further, set $I = \bigcup_{S\in\pi} I_S$ (disjoint union). Then $(x_i, p_i)_{i\in I}$ is a family of elements of $L_\infty(\mu) \times L_\infty(\mu)$, which satisfies (II) as may readily be checked.

(For (1) of (II), note that $\int p_i x_j \, d\mu = 0$ if $i \in I_S$ and $j \in I_{S'}$ with $S \neq S'$, and that for i, j belonging to the same I_S, $\int p_i x_j \, d\mu = \int \tilde{p}_i\tilde{x}_j \, d\mu_S$. For (2) note that if $z \in L_1(\mu)$ satisfies $\int p_i z \, d\mu = 0$ for all $i \in I$, then for each $S \in \pi$, the restriction of z to S is (almost everywhere in S) equal to zero since $\{\tilde{p}_i\colon i \in I_S\}$ separates the points of $L_1(\mu_S)$, whence $z = 0$ because π is at most countable. Concerning (3), note first that if $z \in [0, 1_\Omega]$ then for each $S \in \pi$, $z_S \in [0, \tilde{1}_S]$ where z_S is the restriction of z to S. Next note that if π is finite and for each $S \in \pi$ we have a finite linear combination of $\tilde{x}_i$'s, $i \in I_S$, say $\sum_{k=1}^{n_S} \alpha_{k_S}\tilde{x}_{i_{k_S}}$, which is in $[0, \tilde{1}_S]$, then $\sum_{S\in\pi}\sum_{k=1}^{n_S} \alpha_{k_S} x_{i_{k_S}}$ is a finite linear combination of x_i's which is an element of $[0, 1_\Omega]$; moreover, for any given $z \in [0, 1_\Omega]$, $\|z - \sum_{S\in\pi}\sum_{k=1}^{n_S} \alpha_{k_S} x_{i_{k_S}}\|_1 = \sum_{S\in\pi}\|1_S z - \sum_{k=1}^{n_S} \alpha_{k_S} x_{i_{k_S}}\|_1 = \sum_{S\in\pi}\|z_S - \sum_{k=1}^{n_S} \alpha_{k_S}\tilde{x}_{i_{k_S}}\|_1$. For the case of an infinite π, note that the set of all $x \in [0, 1_\Omega]$ with $1_S x = 0$ for all but finitely many $S \in \pi$ is $\|\cdot\|_1$-dense in $[0, 1_\Omega]$.)

Now by Maharam's theorem, there is a partition π of Ω into sets $S \in \Sigma$, with $\mu(S) > 0$, such that for each $S \in \pi$ there is Banach lattice isomorphism T_S from $L_1(\mu_S)$ onto either $\mathbb{R}$ or $L_1(\{-1,1\}^{K_S})$ for some infinite K_S, which satisfies $T_S(\tilde{1}_S) = 1_{\{-1,1\}^{K_S}}$ in case it has range $L_1(\{-1,1\}^{K_S})$.[19] Thus it is enough to show that, given an arbitrary non-empty index set K, there is a family $(x_i, p_i)_{i\in I}$ in $L_\infty(\{-1,1\}^K) \times L_\infty(\{-1,1\}^K)$ such that (II) holds, with $L_1(\{-1,1\}^K)$ in place of $L_1(\mu)$ and $[0, 1_{\{-1,1\}^K}]$ in place of $[0, 1_\Omega]$. (Indeed, if $(x_i, p_i)_{i\in I_S}$ is such a

[19]Of course, with the specification $T_S(\tilde{1}_S) = 1_{\{-1,1\}^{K_S}}$, the isomorphism T_S need not be an isometry since μ_S is not a probability measure unless $\pi = \{\Omega\}$.

family in $L_\infty(\{-1,1\}^{K_S}) \times L_\infty(\{-1,1\}^{K_S}) \subset L_1(\{-1,1\}^{K_S}) \times L_\infty(\{-1,1\}^{K_S})$, then setting $\widetilde{x}_i = T_S^{-1}(x_i)$ and $\widetilde{p}_i = T_S^*(p_i)$, $i \in I$—where T_S^* is the adjoint operator of T_S—evidently provides a family $(\widetilde{x}_i, \widetilde{p}_i)_{i\in I_S}$ in $L_\infty(\mu_S) \times L_\infty(\mu_S)$ for which (II) holds with $L_1(\mu_S)$ in place of $L_1(\mu)$ and $[0, \widetilde{1}_S]$ in place of $[0, 1_\Omega]$.)

Thus let $\{-1,1\}^K$ be given. In the following, t stands for a generic element of $\{-1,1\}^K$, and t_k, $k \in K$, stands for the kth coordinate of t. Let $\mathcal{F}$ be the set of all finite subsets of K. For each non-empty $F \in \mathcal{F}$ let $w_F\colon \{-1,1\}^K \to \{-1,1\}$ be the function given by

$$w_F(t) = \prod_{k\in F} t_k\,, \quad t \in \{-1,1\}^K,$$

and let $w_\varnothing = 1_{\{-1,1\}^K}$. That is, $(w_F)_{F\in\mathcal{F}}$ is the family of Walsh functions on $\{-1,1\}^K$.

According to a well known fact (see e.g. Negrepontis, 1984, p. 1076)

$$\int w_F w_{F'} = \begin{cases} 1 & \text{if } F = F' \\ 0 & \text{if } F \neq F' \end{cases}$$

and thus, identifying each w_F with its "equal almost everywhere" equivalence class, $(w_F, w_F)_{F\in\mathcal{F}}$ is a family of elements of $L_\infty(\{-1,1\}^K)\times L_\infty(\{-1,1\}^K)$ which satisfies (1) of (II).

Let W be the linear span of $\{w_F\colon F \in \mathcal{F}\}$ and let Y be the set of all elements of $L_1(\{-1,1\}^K)$ that are equivalence classes, modulo "equal almost everywhere," of functions depending on only finitely many coordinates.

Again by a standard fact, Y is $\|\cdot\|_1$-dense in $L_1(\{-1,1\}^K)$. Moreover, Y is a sublattice of $L_1(\{-1,1\}^K)$ containing $1_{\{-1,1\}^K}$. Hence, considering the order interval $[0, 1_{\{-1,1\}^K}]$ in $L_1(\{-1,1\}^K)$, we have that $Y \cap [0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_1$-dense in $[0, 1_{\{-1,1\}^K}]$ by virtue of the continuity of the lattice operations in $L_1(\{-1,1\}^K)$.

By another well known fact, every function on $\{-1,1\}^K$ that depends on only finitely many coordinates can be written as a finite linear combination of Walsh functions. Hence, from the previous paragraph, $W \cap [0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_1$-dense in $[0, 1_{\{-1,1\}^K}]$.

Now since order intervals in the dual of a Banach lattice are weak* compact, $[0, 1_{\{-1,1\}^K}]$ is compact for the weak* topology $\sigma(L_\infty(\{-1,1\}^K), L_1(\{-1,1\}^K))$, and it follows that this latter topology coincides on $[0, 1_{\{-1,1\}^K}]$ with the weak topology $\sigma(L_1(\{-1,1\}^K), L_\infty(\{-1,1\}^K))$. Thus $W \cap [0, 1_{\{-1,1\}^K}]$, being $\|\cdot\|_1$-dense and hence weakly dense in $[0, 1_{\{-1,1\}^K}]$, is actually weak* dense in $[0, 1_{\{-1,1\}^K}]$.[20] But $[0, 1_{\{-1,1\}^K}]$ separates the points of $L_1(\{-1,1\}^K)$, and consequently so does $W \cap [0, 1_{\{-1,1\}^K}]$, being weak* dense in $[0, 1_{\{-1,1\}^K}]$, whence so does $\{w_F\colon F \in \mathcal{F}\}$. Thus (2) of (II) is satisfied by the family $(w_F, w_F)_{F\in\mathcal{F}}$.

[20] I.e. $\sigma(L_\infty(\{-1,1\}^K), L_1(\{-1,1\}^K))$-dense in $[0, 1_{\{-1,1\}^K}]$.

Let Q be the set of all linear combinations of w_F's with rational coefficients. Note that Q is $\|\cdot\|_\infty$-dense in W and hence $Q \cap \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_\infty$-dense in $W \cap \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}]$. Now $W \cap \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}] \neq \emptyset$; e.g. $(1/2)1_{\{-1,1\}^K}$ belongs to this intersection. Hence $W \cap \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_\infty$-dense in $W \cap [0, 1_{\{-1,1\}^K}]$ (because if $x \in [0, 1_{\{-1,1\}^K}]$ and $y \in \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}]$ then $(1-\lambda)x + \lambda y \in \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}]$ for $0 < \lambda < 1$). Therefore the fact that $Q \cap \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_\infty$-dense in $W \cap \|\cdot\|_\infty\text{-}\operatorname{int}[0, 1_{\{-1,1\}^K}]$ implies that $Q \cap [0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_\infty$-dense in $W \cap [0, 1_{\{-1,1\}^K}]$. Consequently, $Q \cap [0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_1$-dense in $[0, 1_{\{-1,1\}^K}]$, since $W \cap [0, 1_{\{-1,1\}^K}]$ is $\|\cdot\|_1$-dense in $[0, 1_{\{-1,1\}^K}]$ and since $\|x\|_1 \leq \|x\|_\infty$ for $x \in L_\infty(\{-1,1\}^K)$. Thus the family $(w_F, w_F)_{F \in \mathcal{F}}$ satisfies (3) of (II). This completes the proof of the lemma. □

The following lemmata are needed only to cover the case of an economy where the space of agents is a (non-trivial) atomless measure space which has no non-measurable subset. Recall that it is (relatively) consistent with ZFC that no such measure space exists. However it is not known whether the existence of such a measure space is inconsistent with ZFC. Thus, for sake of generality, we do not want to exclude the possibility of such a measure space.

Lemma 10. *Let $(T, \mathcal{T}, \nu)$ be a finite measure space and let $g\colon T \to E$ be Pettis integrable. Let $S \in \mathcal{T}$ and suppose that $2^S \subset \mathcal{T}$. Then given $\epsilon > 0$ there are an $S' \subset S$ and an integer $n > 0$ such that $\|\int_{S'} g - \int_S g\| < \epsilon$ and $\|g(t)\| \leq n$ for all $t \in S'$. Moreover, given $\delta > 0$, S' can be chosen so that $\nu(S \setminus S') < \delta$.*

Proof. For every integer $n > 0$ let $S_n = \{t \in S\colon \|g(t)\| \leq n\}$. Then the sequence (S_n) is increasing and $\bigcup_{n=1}^\infty S_n = S$. By the hypothesis about S, each S_n belongs to $\mathcal{T}$; in particular, $\nu(S \setminus S_n)$ is well defined for each n, and we have $\nu(S \setminus S_n) \to 0$ as $n \to \infty$. Thus, according to the additivity and continuity properties of the indefinite Pettis integral, $\int_{S_n} g \to \int_S g$. □

Lemma 11. *Let $(T, \mathcal{T}, \nu)$ be a finite measure space and suppose that E is an order continuous Banach lattice with the PIP. Let $g\colon T \to E_+$ be a Pettis integrable function, let $S \in \mathcal{T}$, and let $x \in E$ with $0 \leq x \leq \int_S g$. Suppose that $2^S \subset \mathcal{T}$ and that $1_S g$ is norm bounded. Then there is a Pettis integrable function $h\colon T \to E_+$ with $\int_S h = x$ and $h(t) \leq g(t)$ for all $t \in S$.*

Proof. Clearly, we may assume without loss of generality that $S = T$ (because if h is a Pettis integrable function from the subspace $(S, 2^S, \nu_S)$ into E_+—where ν_S is the restriction of ν to 2^S—then $1_S h\colon T \to E_+$ is Pettis integrable as well). Now combine the next two lemmata, and recall for this that if X and Y are Riesz spaces, then a positive linear operator $\theta\colon X \to Y$ is called *interval preserving* if $\theta([0, x]) = [0, \theta(x)]$ for all $x \in X_+$, and, if Y is endowed with some topology, is called *almost interval preserving* if $\theta([0, x])$ is dense in $[0, \theta(x)]$ for all $x \in X_+$.

Note that if θ is interval preserving, then, in fact, $\theta([a,b]) = [\theta(a), \theta(b)]$ for any a, $b \in X$ with $a \leq b$, and that if θ is almost interval preserving then, for any a, $b \in X$ with $a \leq b$, $\theta([a,b])$ is dense in $[\theta(a), \theta(b)]$. Recall also that any order continuous Banach lattice is σ-Dedekind complete. □

Lemma 12. *Let $(T, \mathcal{T}, \nu)$ be a finite measure space with $\mathcal{T} = 2^T$ and suppose that E is an order continuous Banach lattice with the PIP. Let Z be the set of all norm bounded Pettis integrable functions from T into E, endowed with the pointwise ordering induced from the ordering of E; that is, if f, $g \in Z$ then $f \geq g$ if and only if $f(t) \geq g(t)$ for all $t \in T$. (Functions which agree almost everywhere are* not *identified.) Let $\theta\colon Z \to E$ be the operator defined by setting $\theta(z) = \int z\, d\nu$ for $z \in Z$. Then*

(a) *Z is a σ-Dedekind complete Riesz space.*

(b) *θ is a positive linear operator which is almost interval preserving and has the property that if $z_n \uparrow z$ in Z then $\theta(z_n) \to \theta(z)$ (in the norm of E).*

($z_n \uparrow z$ means the sequence (z_n) is increasing with $z = \sup\{z_n\colon n = 1, 2, \ldots\}$.)

Proof. (a) Let $\widetilde{Z}$ be the set of all norm bounded functions from T into E, endowed with the pointwise ordering induced from the ordering of E. Since E is a σ-Dedekind complete Banach lattice, it is clear that $\widetilde{Z}$ is a σ-Dedekind complete Riesz space. Since, by hypothesis, $\mathcal{T} = 2^T$ and E has the PIP, every element of $\widetilde{Z}$ is Pettis integrable. Thus $\widetilde{Z} = Z$, i.e. Z is a σ-Dedekind complete Riesz space.

(b) Obviously the operator θ is linear and positive. Suppose $z_n \uparrow z$ in Z. Then, by definition of the ordering of Z, we have $z_n(t) \uparrow z(t)$ in E for each $t \in T$. Hence $z_n(t) \to z(t)$ in the norm of E for each $t \in T$ since E is order continuous. In particular, for every $p \in E_+^*$, $\langle p, z_n(t)\rangle \uparrow \langle p, z(t)\rangle$ for each $t \in T$. By the monotone convergence theorem, it follows that for each $p \in E_+^*$,

$$\int \langle p, z_n(t)\rangle\, d\nu(t) \uparrow \int \langle p, z(t)\rangle\, d\nu(t).$$

That is, $\langle p, \theta(z_n)\rangle \uparrow \langle p, \theta(z)\rangle$ for each $p \in E_+^*$.

It is clear that the sequence $(\theta(z_n))$ in E is increasing and that $\theta(z)$ is an upper bound of the set $\{\theta(z_n)\colon n = 1, 2, \ldots\}$. Consider an arbitrary upper bound of this set, say x. Then for each $p \in E_+^*$, $\langle p, \theta(z_n)\rangle \leq \langle p, x\rangle$ for all n. Hence $\langle p, \theta(z)\rangle \leq \langle p, x\rangle$ for each $p \in E_+^*$, since $\langle p, \theta(z_n)\rangle \uparrow \langle p, \theta(z)\rangle$ for such p. Consequently we must have $\theta(z) \leq x$, whence $\theta(z) = \sup\{\theta(z_n)\colon n = 1, 2, \ldots\}$. Thus $\theta(z_n) \uparrow \theta(z)$ and hence $\theta(z_n) \to \theta(z)$ in the norm of E since E is order continuous.

Finally to see that θ is almost interval preserving, pick any $z \in Z_+$ and set $A = \theta([0, z])$. Clearly $c\ell\, A \subset [0, \theta(z)]$ since θ is positive. For the reverse inclusion, consider any $x \in E_+$ with $x \notin c\ell\, A$. Note that the set A is convex and hence so is $c\ell A$. Using the Hahn-Banach theorem, select a $p \in E^*$ such that $\langle p, x\rangle < \inf\langle p, A\rangle$.

Since E is order continuous, order intervals in E are weakly compact. Thus for each $t \in T$, $\inf\langle p, [0, z(t)]\rangle$ is attained at some point in $[0, z(t)]$. That is (since any norm bounded function from T into E belongs to Z) there is a $u \in Z$ such that for each $t \in T$, $u(t) \in [0, z(t)]$ and $\langle p, u(t)\rangle = \inf\langle p, [0, z(t)]\rangle$. In particular,

$$\int \langle p, u(t)\rangle \, d\nu(t) = \inf\langle p, A\rangle$$

and

$$-\langle p, u(t)\rangle = \langle p^-, z(t)\rangle \text{ for each } t \in T.$$

Using these observations, we conclude that

$$\begin{aligned} \langle p^-, x\rangle \geq -\langle p, x\rangle &> -\inf\langle p, A\rangle \\ &= \int -\langle p, u(t)\rangle \, d\nu(t) \\ &= \langle p^-, \theta(z)\rangle. \end{aligned}$$

Thus $x \notin [0, \theta(z)]$, whence $[0, \theta(z)] \subset c\ell A$. This completes the proof of the lemma. □

Lemma 13. *Let Z be a σ-Dedekind complete Riesz space, let E be a Banach lattice, and let $\theta\colon Z \to E$ be a positive linear operator that is almost interval preserving and such that if $z_n \uparrow z$ in Z then $\theta(z_n) \to \theta(z)$ (in the norm of E). Then θ is interval preserving.*

Proof. Pick any $v \in Z_+$, with $v \neq 0$, and let $b \in [0, \theta(v)]$. We have to show that $b = \theta(z)$ for some $z \in [0, v]$. We first establish the following:

Claim: Given $u \in [0, v]$ with $\theta(u) \geq b$ and given $\epsilon > 0$ there is a $z \in [0, u]$ such that $\theta(z) \geq b$ and $\|\theta(z) - b\| \leq \epsilon$.

Let $u \in [0, v]$ with $\theta(u) \geq b$ and $\epsilon > 0$ be given. Since θ is almost interval preserving, we can find a $z_0 \in [0, u]$ such that $\|\theta(z_0) - b\| < \epsilon$. If $\theta(z_0) \geq b$ we are done. If not, consider $\theta(z_0) \vee b$. Since $\theta(z_0) \vee b \in [\theta(z_0), \theta(u)]$ and θ is almost interval preserving, given any $\epsilon' > 0$ there is a $z_1 \in [z_0, u]$ such that $\|\theta(z_1) - (\theta(z_0) \vee b)\| < \epsilon'$. We have

$$\|\theta(z_1) - b\| \leq \|\theta(z_1) - (\theta(z_0) \vee b)\| + \|(\theta(z_0) \vee b) - b\|$$

and

$$\|(\theta(z_0) \vee b) - b\| = \|(\theta(z_0) - b)^+\| \leq \|\theta(z_0) - b\| < \epsilon.$$

Hence, since ϵ' can be as small as we like, we can choose z_1 in such a way that both $\|\theta(z_1) - b\| < \epsilon$ and $\|\theta(z_1) - (\theta(z_0) \vee b)\| < 1$. If $\theta(z_1) \geq b$ we are done. If not, we repeat the construction in the following way. Consider $\theta(z_1) \vee b$. Since

$\theta(z_1) \vee b \in [\theta(z_1), \theta(u)]$ and θ is almost interval preserving, given $\epsilon' > 0$ there is a $z_2 \in [z_1, u]$ such that $\|\theta(z_2) - (\theta(z_1) \vee b)\| < \epsilon'$. We have

$$\|\theta(z_2) - b\| \leq \|\theta(z_2) - (\theta(z_1) \vee b)\| + \|(\theta(z_1) \vee b) - b\|$$

and

$$\|(\theta(z_1) \vee b) - b\| = \|(\theta(z_1) - b)^+\| \leq \|\theta(z_1) - b\| < \epsilon.$$

Hence, since ϵ' can be as small as we like, we can choose z_2 in such a way that both $\|\theta(z_2) - b\| < \epsilon$ and $\|\theta(z_2) - (\theta(z_1) \vee b)\| < 1/2$. If $\theta(z_2) \geq b$ we are done. If not, we can proceed in this manner to obtain either after a finite number of steps an element $z_n \in [0, u]$ which does the job, or an increasing sequence (z_n) in $[0, u]$ such that for all $n > 0$,

$$\|\theta(z_n) - b\| < \epsilon$$

and

$$\|\theta(z_n) - (\theta(z_{n-1}) \vee b)\| < 1/n.$$

In this latter case, since Z is σ-Dedekind complete we must have $z_n \uparrow z$ for some $z \in [0, u]$. Thus, by the hypothesized properties of θ, $\theta(z_n) \to \theta(z)$ in the norm of E. Consequently $\|\theta(z) - b\| \leq \epsilon$ and $\|\theta(z) - (\theta(z) \vee b)\| = 0$. Evidently the latter equality implies $\theta(z) \geq b$. This establishes the claim.

Using the claim, we can find a decreasing sequence (z_n) in $[0, v]$ such that $\|\theta(z_n) - b\| < 1/n$ for all $n > 0$. Since Z is σ-Dedekind complete, we have $z_n \downarrow z$ for some $z \in [0, v]$.[21] By the hypothesized properties of θ, $\theta(z_n) \to \theta(z)$ in the norm of E (since $z_n \downarrow z$ is equivalent to $-z_n \uparrow -z$). It follows that $\theta(z) = b$. This completes the proof of the lemma. □

4.5.2 Proof of Theorem 6

Let $\mathcal{E}$ be an atomless economy with commodity space E satisfying assumptions (A1) to (A5), (A9) and (M2). Clearly $\mathcal{W}(\mathcal{E}) \subset C(\mathcal{E})$. To prove the reverse inclusion, let $f \in C(\mathcal{E})$.

There is no loss of generality in assuming that the endowment mapping $t \mapsto e(t)$ is strongly measurable. Indeed, by hypothesis the commodity space E is an order continuous Banach lattice with E_+ containing quasi-interior points. Thus, by the remarks at the beginning of Section 4.5.1, E is weakly compactly generated and therefore measure-compact. Thus there is a strongly measurable function $e' \colon T \to E$ which is weakly equivalent to $t \mapsto e(t)$. In particular, e' is Pettis integrable with $\int_S e'(t)\, d\nu(t) = \int_S e(t)\, d\nu(t)$ for each $S \in \mathcal{T}$. Hence f is also a core allocation for the economy $\mathcal{E}'$ that results if the endowment mapping

[21] $z_n \downarrow z$ means the sequence (z_n) is decreasing with $z = \inf\{z_n \colon n = 1, 2, \ldots\}$.

of the economy $\mathcal{E}$ under consideration is replaced by e'. Moreover, since for any $p \in E^*$, $\langle p, e'(t)\rangle = \langle p, e(t)\rangle$ for almost all $t \in T$ (by definition of "weakly equivalent"), if (p, f) is a Walrasian equilibrium for $\mathcal{E}'$ then (p, f) is also a Walrasian equilibrium for the original economy $\mathcal{E}$. (Note also that by Lemma 2, $e'(t) \geq 0$ for almost all $t \in T$ since $e(t) \geq 0$ for all $t \in T$.) Thus we may as well assume that the endowment mapping of $\mathcal{E}$ is strongly measurable.

Let α and β be strictly positive elements of E^*, chosen according to Assumption (A9); in particular, $\alpha \leq \beta$. Denote by Γ the cone

$$\Gamma = \{x \in E \colon \alpha(x^+) > \beta(x^-)\}.$$

Note the following facts about Γ. First, $0 \notin \Gamma$ and Γ contains $E_+ \setminus \{0\}$, obviously. Second, Γ is (norm) open by virtue of the continuity of the lattice operations. Finally, Γ is convex. To see this, note that if $x, y \in E$ then for some $b \in E_+$, $(x+y)^+ = x^+ + y^+ - b$ as well as $(x+y)^- = x^- + y^- - b$. Thus whenever $x, y \in \Gamma$ then $\alpha((x+y)^+) > \beta((x+y)^-)$, because $\alpha \leq \beta$ and hence $\alpha(b) \leq \beta(b)$ for $b \geq 0$.

Next, let $\varphi \colon T \to 2^E$ be the correspondence given by

$$\varphi(t) = \{x \in E_+ \colon x \succ_t f(t)\} \cup \{e(t)\},\ t \in T.$$

The following part of the proof covers the case where every $S \in \mathcal{T}$ with $\nu(S) > 0$ has a non-measurable subset. The other case is dealt with below. (As noted in Section 4.5.1, it is consistent with ZFC that every non-trivial atomless measure space has a non-measurable subset. Clearly, the non-existence of a non-trivial atomless measure on the power set of any set implies that given any finite atomless measure space, every measurable set with measure >0 has a non-measurable subset.)

Let A be the set of all *strongly measurable* Pettis integrable selections of the correspondence φ and let

$$B = \left\{z \in E \colon z = \int g \text{ for some } g \in A.\right\}$$

Note that B is non-empty—e.g. $\int e(t)\, d\nu(t)$ belongs to this set (because the mapping $t \mapsto e(t)$ is assumed to be strongly measurable).

We claim that

$$\left(B - \left\{\int e(t)\, d\nu(t)\right\}\right) \cap -\Gamma = \varnothing.$$

Suppose, if possible, otherwise. Then, since $0 \notin \Gamma$, and by virtue of the measurability assumption M(2), there is a strongly measurable allocation $g \colon T \to E_+$ and an $S \in \mathcal{T}$, with $\nu(S) > 0$, such that $g(t) \succ_t f(t)$ for almost all $t \in S$ and $\int_S g(t)\, d\nu(t) - \int_S e(t)\, d\nu(t) = -y$ for some $y \in \Gamma$. Suppose $y \geq 0$. Then $\tilde{g} \colon T \to E_+$, defined by $\tilde{g}(t) = g(t) + (1/(\nu(S)))y$ for all $t \in T$, is an allocation with $\int_S \tilde{g}(t)\,d\nu(t) = \int_S e(t)\,d\nu(t)$. Moreover, for all $t \in S$, $\tilde{g}(t) \succ_t g(t)$ by strict monotonicity of preferences since $y \neq 0$, whence $\tilde{g}(t) \succ_t f(t)$ for almost all $t \in S$ by transitivity of preferences. We thus have a contradiction to the property of f being a core allocation. Consequently $y \geq 0$ cannot hold.

Suppose $y^- \neq 0$. Observe that we must have $y^- \leq \int_S g(t)\,d\nu(t)$ because $\int_S g(t)\,d\nu(t)$ and $\int_S e(t)\,d\nu(t)$ are positive elements (and because $-y = y^- - y^+$ and $y^- \wedge y^+ = 0$). Now since Γ is open and g is strongly measurable, an appeal to Lemma 14 below and the fact that the indefinite Pettis integral $\int_{(\cdot)} e(t)\,d\nu(t)$ is ν-continuous shows that we can assume g to be actually a simple function. Then the Riesz decomposition theorem can be used to find a measurable simple function $s\colon T \to E_+$ with $\int_S s(t)\,d\nu(t) = y^-$ and $s(t) \leq g(t)$ for all $t \in S$.

For each $t \in S$, set

$$v(t) = \frac{\beta(s(t))}{\beta(y^-)} y^+.$$

This is well defined because y^- is supposed to be $\neq 0$ and β is strictly positive; in particular, $v(t) \geq 0$ for each $t \in S$. Moreover, for all $t \in S$,

$$\alpha(v(t)) = \frac{\beta(s(t))}{\beta(y^-)} \alpha(y^+) \geq \beta(s(t))$$

with strict inequality in case $s(t) \neq 0$ since $\alpha(y^+) > \beta(y^-)$ by definition of Γ. Hence, by choice of α and β, and because $g(t) - s(t) \geq 0$ and $v(t) \geq 0$ for all $t \in S$, we have for almost all $t \in S$,

$$g(t) - s(t) + v(t) \succ_t g(t) \succ_t f(t)$$

in case $s(t) \neq 0$ and

$$g(t) - s(t) + v(t) = g(t) \succ_t f(t)$$

otherwise. Consequently, if we define $\tilde{g}\colon T \to E$ by

$$\tilde{g}(t) = \begin{cases} g(t) - s(t) + v(t) & \text{if } t \in S \\ 0 & \text{if } t \in T \setminus S, \end{cases}$$

then $\tilde{g}$ is an allocation with $\tilde{g}(t) \succ_t f(t)$ for almost all $t \in S$ by transitivity of preferences, and we have

$$\begin{aligned} \int_S \tilde{g}(t)\,d\nu(t) &= \int_S g(t)\,d\nu(t) - \int_S s(t)\,d\nu(t) + \int_S v(t)\,d\nu(t) \\ &= \int_S g(t)\,d\nu(t) - y^- + \frac{1}{\beta(y^-)} y^+ \int_S \langle \beta, s(t) \rangle\,d\nu(t) \\ &= \int_S g(t)\,d\nu(t) - y^- + y^+ \\ &= \int_S e(t)\,d\nu(t) \end{aligned}$$

thus again getting a contradiction to the property of f being a core allocation. Consequently, $(B - \{\int e(t)\, d\nu(t)\}) \cap -\Gamma = \emptyset$ as claimed above.

Since Γ is open, we must in fact have $(\mathrm{c}\ell\, B - \{\int e(t)\, d\nu(t)\}) \cap -\Gamma = \emptyset$. By Lemma 5, $\mathrm{c}\ell\, B$ is convex and hence so is $\mathrm{c}\ell\, B - \{\int e(t)\, d\nu(t)\}$. Since, as noted above, the cone Γ is convex, and since Γ and B are non-empty, it now follows from the separation theorem that there is a $p \in E^*$, with $p \neq 0$, such that

$$\inf \left\langle p, \mathrm{c}\ell\, B - \left\{ \int_T e(t)\, d\nu(t) \right\} \right\rangle \geq \sup\langle p, -\Gamma\rangle.$$

Since Γ is a cone, this implies

$$\inf\langle p, B\rangle \geq \left\langle p, \int_T e(t)\, d\nu(t) \right\rangle \equiv \int \langle p, e(t)\rangle\, d\nu(t)\,. \tag{4}$$

Note also that p must be strictly positive because Γ is open and $E_+ \setminus \{0\} \subset \Gamma$.

We claim:

(5) For any $x \in E_+$, $\{t \in T\colon x \succ_t f(t) \text{ and } p(x) < p(e(t))\}$ is a null set in T.

Indeed, pick any $x \in E_+$ and let $g\colon T \to E_+$ be given by

$$g(t) = \begin{cases} x & \text{if } x \succ_t f(t) \text{ and } p(x) < p(e(t)) \\ e(t) & \text{otherwise.} \end{cases}$$

From (M2), the set $\{t \in T\colon x \succ_t f(t)\}$ belongs to $\mathcal{T}$, and because the mapping $t \mapsto e(t)$ is weakly measurable, so does the set $\{t \in T\colon p(x) < p(e(t))\}$. Hence, g is Pettis integrable. Moreover, from the assumption (made at the beginning of this proof) that the mapping $t \mapsto e(t)$ is in fact strongly measurable, it follows that g is strongly measurable. From the definition of B, then, $\int_T g(t)\, d\nu(t) \in B$ and hence from (4), $\int \langle p, g(t)\rangle\, d\nu(t) \geq \int \langle p, e(t)\rangle\, d\nu(t)$. Thus (5) must hold.

Let

$$\widetilde{S} = \{t \in T\colon \text{ there is an } x \in E_+ \text{ with } x \succ_t f(t) \text{ and } p(x) < p(e(t))\}.$$

We are going to show that $\widetilde{S}$ is a null set. Proceeding by contradiction, suppose $\widetilde{S}$ is a non-null set and let $g\colon \widetilde{S} \to E_+$ be a function with $g(t) \succ_t f(t)$ and $p(g(t)) < p(e(t))$ for each $t \in \widetilde{S}$.

Appealing to Lemma 9—which applies since E is order continuous and E_+ contains quasi-interior points—select a family $(x_i, p_i)_{i \in I}$ of elements of $E \times E^*$ such that:

(i) $\langle p_i, x_j\rangle \neq 0$ if and only if $i = j$.

(ii) The set $\{p_i\colon i \in I\}$ is a total subset of E^*.

(iii) Let Q denote the set of all (finite) linear combinations of the x_i such that the coefficients are rational. Then $Q \cap E_+$ is dense in E_+.

Then by continuity of preferences, we may assume that $g(t) \in Q$ for each $t \in \widetilde{S}$. We claim that there are an $\overline{S} \subset \widetilde{S}$, with $\nu^*(\overline{S}) > 0$,[22] and an $a \in E$ such that for each $i \in I$, $\langle p_i, g(t)\rangle = \langle p_i, a\rangle$ for almost all $t \in \overline{S}$.

To see this, first note that since every $g(t)$ is a linear combination of the x_i, (i) implies that for every $t \in \widetilde{S}$, $\{i \in I\colon \langle p_i, g(t)\rangle \neq 0\}$ is finite. By the fact that a countable union of null sets is a null set, this means we can find an integer k and a set $S_1 \subset \widetilde{S}$, with $\nu^*(S_1) > 0$, such that $|\{i \in I\colon \langle p_i, g(t)\rangle \neq 0\}| = k$ for all $t \in S_1$, where $|\cdot|$ stands for "cardinality."

Consider the following condition on pairs (S, F) where $S \subset \widetilde{S}$ and $F \subset I$:

(∗) $S \subset S_1$, $\nu^*(S) > 0$, and for each $i \in F$, $\langle p_i, g(t)\rangle \neq 0$ for all $t \in S$.

By choice of S_1, if (S, F) satisfies (∗) then F is a finite set with $|F| \leq k$. Let

$$L = \{\ell \in \mathbb{N}\colon \ell = |F| \text{ for some } (S, F) \text{ that satisfies } (*)\}.$$

Clearly, $(S_1, \emptyset)$ satisfies (∗). Thus $0 \in L$. Set $\overline{\ell} = \max L$. If $\overline{\ell} = 0$, the claim holds for $\overline{S} = S_1$ together with $a = 0$. If $\overline{\ell} \geq 1$, choose $S_2 \subset S_1$ and $F \subset I$ such that (S_2, F) satisfies (∗) and $|F| = \overline{\ell}$. Then $\nu^*(S_2) > 0$ and, from the definition of $\overline{\ell}$, for each $i \in I \setminus F$, $\langle p_i, g(t)\rangle = 0$ for almost all $t \in S_2$. Now since every $g(t)$ is a linear combination of the x_i such that all coefficients are rational, it follows from (i) above that for each $i \in I$ and every $t \in \widetilde{S}$, $\langle p_i, g(t)\rangle = r_t(i)\langle p_i, x_i\rangle$ for some rational number $r_t(i)$. But this fact combined with the facts that F is finite and $\nu^*(S_2) > 0$ implies that there are an $S_3 \subset S_2$, with $\nu^*(S_3) > 0$, and rational numbers $r(i)$, $i \in F$, such that $\langle p_i, g(t)\rangle = r(i)\langle p_i, x_i\rangle$ for all $t \in S_3$ and each $i \in F$ (because the set of all functions from a finite set into the set of rational numbers is countable, and because the union of countably many null sets is a null set). Set $a = \sum_{i \in F} r(i) x_i$. Another appeal to (i) above shows that $\langle p_i, a\rangle = r(i)\langle p_i, x_i\rangle$ for $i \in F$, and that $\langle p_i, a\rangle = 0$ for $i \in I \setminus F$. Finally, since $S_3 \subset S_2$, for each $i \in I \setminus F$ we have $\langle p_i, g(t)\rangle = 0$ for almost all $t \in S_3$. Thus the claim holds for $\overline{S} = S_3$ together with a as just defined.

Choose and fix objects $\overline{S}$ and a as described in the claim. Recall that E is weakly compactly generated. Hence by Lemma 3, (ii) above implies that in fact for *each* $q \in E^*$, we have $\langle q, g(t)\rangle = \langle q, a\rangle$ for almost all $t \in \overline{S}$ (applying Lemma 3 to $\overline{g}\colon \overline{S} \to E$ given by $\overline{g}(t) = g(t) - a$). In particular, then, for each positive $q \in E^*$ we have $\langle q, a\rangle = \langle q, g(t)\rangle$ for almost all $t \in \overline{S}$ and hence, by the Hahn Banach theorem, we must have $a \geq 0$ because $g(t) \geq 0$ for all $t \in \overline{S}$ and $\overline{S}$ is a non-null set.

Now, since $\overline{S}$ is a non-null set, $\overline{S}$ has a non-measurable subset, say S' (according to what has been hypothesized for this part of the proof; of course, it is

[22] As above, if A is any subset of T, then $\nu^*(A)$ denotes the outer measure of A.

possible that already $\overline{S}$ itself is non-measurable). Let $h\colon T \to E$ be the function defined by setting

$$h(t) = \begin{cases} a & \text{if } t \in T \setminus S' \\ g(t) & \text{if } t \in S'. \end{cases}$$

Then $h(t) \geq 0$ for all $t \in T$. Moreover, h is Pettis integrable, because for each $q \in E^*$, $\langle q, h(t)\rangle = \langle q, a\rangle$ for almost all $t \in T$. Thus h is an allocation. Set

$$S_1 = \{t \in T\colon h(t) \succ_t f(t)\}$$

and

$$S_2 = \{t \in T\colon h(t) \succ_t f(t) \text{ and } \langle p, h(t)\rangle < \langle p, e(t)\rangle\}.$$

Then by Assumption (M2), S_1 belongs to $\mathcal{T}$, and hence so does S_2. Set

$$S_a = \{t \in T\colon a \succ_t f(t) \text{ and } \langle p, a\rangle < \langle p, e(t)\rangle\}.$$

Evidently $S_2 = S' \cup S_a$. Since S_2 is a measurable set but S' is not, this shows that S_a cannot be a null set. This contradicts (5) and proves that $\widetilde{S}$ is a null set.

Since preferences are continuous and strictly monotone, and p is strictly positive, the usual arguments now apply to show that in fact

$$\{t \in T\colon \text{ there is an } x \in E_+ \text{ with } x \succ_t f(t) \text{ and } p(x) \leq p(e(t))\}$$

is a null set, and that $\langle p, f(t)\rangle = \langle p, e(t)\rangle$ must hold for almost all $t \in T$. Thus the allocation f is Walrasian.

We show now how to proceed when it is not necessarily true that every $S \in \mathcal{T}$ with $\nu(S) > 0$ has a non-measurable subset. We first consider the pure case where in fact $\mathcal{T} = 2^T$.

Let A' be the set of *all* Pettis integrable selections of the correspondence φ given by

$$\varphi(t) = \{x \in E_+\colon x \succ_t f(t)\} \cup \{e(t)\},\ t \in T,$$

and set

$$B' = \left\{z \in E\colon z = \int g \text{ for some } g \in A'\right\}.$$

Then B' is non-empty—e.g. it contains $\int e(t)\,d\nu(t)$. As noted at the beginning of this proof, E is measure-compact, and thus by Lemma 6, $c\ell\, B'$ is convex and hence so is $c\ell\, B' - \{\int e(t)\,d\nu(t)\}$. We claim that $\left(B' - \{\int e(t)\,d\nu(t)\}\right) \cap -\Gamma = \emptyset$. Suppose, if possible, otherwise. Then, because $0 \notin \Gamma$, there is an allocation $g\colon T \to E_+$ and an $S \in \mathcal{T}$, with $\nu(S) > 0$, such that $g(t) \succ_t f(t)$ for almost all $t \in S$ and $\int_S g(t)\,d\nu(t) - \int_S e(t)\,d\nu(t) = -y$ for some $y \in \Gamma$.

As above we see that $y \geq 0$ is impossible. Thus suppose $y^- \neq 0$ and note that we must have $y^- \leq \int_S g(t)\,d\nu(t)$. Since Γ is open it follows from Lemma 10 and the fact that the indefinite Pettis integral $\int_{(\cdot)} e(t)\,d\nu(t)$ is ν-continuous that we can assume that $1_S g$ is norm bounded. Now since E is measure-compact, E has the PIP (see the beginning of Section 4.5.1). Hence by Lemma 11, the fact that $1_S g$ is norm bounded implies that there is a Pettis integrable function $h\colon T \to E_+$ with $\int_S h(t)\,d\nu(t) = y^-$ and $h(t) \le g(t)$ for all $t \in S$. We can now proceed as above to get (with h in place of s) a contradiction to the property of f being a core allocation. Thus $(B' - \{\int e(t)\,d\nu(t)\}) \cap -\Gamma = \emptyset$, as predicted. Since Γ is open, it follows that also $(c\ell\, B' - \{\int e(t)\,d\nu(t)\}) \cap -\Gamma = \emptyset$.

The separation theorem now applies to provide a $p \in E^* \setminus \{0\}$ such that

$$\inf\langle p, B'\rangle \ge \left\langle p, \int_T e(t)\,d\nu(t)\right\rangle . \tag{6}$$

As above, p must be strictly positive. Again let

$$\widetilde{S} = \{t \in T\colon \text{ there is an } x \in E_+ \text{ with } x \succ_t f(t) \text{ and } p(x) < p(e(t))\}.$$

We have $\widetilde{S} \in \mathcal{T}$ because $\mathcal{T} = 2^T$. Suppose $\nu(\widetilde{S}) > 0$ and let $\widetilde{g}\colon \widetilde{S} \to E_+$ be a function such that $\widetilde{g}(t) \succ_t f(t)$ and $\langle p, \widetilde{g}(t)\rangle < \langle p, e(t)\rangle$ for each $t \in \widetilde{S}$. For every integer $n > 0$ let $S_n = \{t \in \widetilde{S}\colon \|\widetilde{g}(t)\| \le n\}$. Again since $\mathcal{T} = 2^T$, $S_n \in \mathcal{T}$. For some n, $\nu(S_n) > 0$, since the countable union of null sets is a null set. Choose and fix such an n. Since E has the PIP and $\mathcal{T} = 2^T$, the function $1_{S_n}\widetilde{g}$ is Pettis integrable. Hence so is the function $\overline{g}\colon T \to E_+$ given by

$$\overline{g}(t) = \begin{cases} \widetilde{g}(t) & \text{if } t \in S_n \\ e(t) & \text{if } t \in T \setminus S_n. \end{cases}$$

By definition of B', $\int_T \overline{g}(t)\,d\nu(t) \in B'$. On the other hand, since $\nu(S_n) > 0$,

$$\left\langle p, \int_T \overline{g}(t)\,d\nu(t)\right\rangle \equiv \int_T \langle p, \overline{g}(t)\rangle\,d\nu(t) < \int_T \langle p, e(t)\rangle\,d\nu(t) \equiv \left\langle p, \int_T e(t)\,d\nu(t)\right\rangle.$$

Thus we have a contradiction to (6). Consequently $\widetilde{S}$ must be a null set. By continuity and strict monotonicity of preferences, together with the fact that p is strictly positive, it follows that the pair (p, f) is a Walrasian equilibrium.

Finally, we will address the "mixed situation" where

$$\overline{r} := \sup\{r \in \mathbb{R}\colon \text{there is an } S \in \mathcal{T} \text{ with } \nu(S) \ge r \text{ and } 2^S \subset \mathcal{T}\}$$

is > 0 but $< \nu(T)$. Suppose this situation occurs. Then for each integer $n > 0$ there is a set $S_n \in \mathcal{T}$ with $2^{S_n} \subset \mathcal{T}$ and $\nu(S_n) > \overline{r} - (1/n)$. Set $T^1 = \bigcup_{n>0} S_n$. Then $T^1 \in \mathcal{T}$ and $2^{T^1} \subset \mathcal{T}$; in particular, $\nu(T^1) = \overline{r}$, by the construction of T^1 and by definition of $\overline{r}$. Set $T^2 = T \setminus T^1$. By construction, every subset S of T^2 with $S \in \mathcal{T}$ and $\nu(S) > 0$ has a non-measurable subset.

Let A'' be the set of all Pettis integrable selections g of the correspondence φ such that $1_{T^2} g$ is strongly measurable. Set

$$B'' = \{z \in E\colon z = \int g \text{ for some } g \in A''\}.$$

Then B'' is non-empty—e.g. $\int_T e(t)\,d\nu(t) \in B''$ (recall: $t \mapsto e(t)$ is strongly measurable). According to Lemma 7, $c\ell\, B''$ is convex, and so is $c\ell\, B'' - \{\int e(t)\,d\nu(t)\}$. We claim

$$\left(B'' - \left\{\int e(t)\,d\nu(t)\right\}\right) \cap -\Gamma = \emptyset.$$

For suppose otherwise. Then, since $0 \notin \Gamma$, and because of Assumption M(2), there is a Pettis integrable function $g\colon T \to E_+$, with $1_{T^2}g$ is strongly measurable, and some $S \in \mathcal{T}$ with $\nu(S) > 0$ such that $g(t) \succ_t f(t)$ for almost all $t \in S$ and $\int_S g(t)\,d\nu(t) - \int_S e(t)\,d\nu(t) = -\gamma$ for some $\gamma \in \Gamma$. As earlier, we see that $\gamma \geq 0$ is impossible, so suppose $\gamma^- \neq 0$. We have

$$\gamma^- \leq \int_S g(t)\,d\nu(t) = \int_{S\cap T^1} g(t)\,d\nu(t) + \int_{S\cap T^2} g(t)\,d\nu(t)\,.$$

The Riesz decomposition theorem asserts the existence of elements $b_1, b_2 \in E_+$ with $\gamma^- = b_1 + b_2$ and $b_1 \leq \int_{S\cap T^1} g$ and $b_2 \leq \int_{S\cap T^2} g$.

Since Γ is open, we can assume both that $1_{T^2}g$ is a simple function and that $1_{S\cap T^1}g$ is norm bounded, appealing to Lemma 14, Lemma 10, and the fact that the vector measure $\int_{(\cdot)} e(t)\,d\nu(t)$ is ν-continuous. Then another appeal to the Riesz decomposition theorem ensures that there is a measurable simple function $s\colon T \to E_+$ with $\int_{S\cap T^2} s = b_2$ and $s(t) \leq g(t)$ for all $t \in S \cap T^2$, and Lemma 11 ensures that there is a Pettis integrable function $h\colon T \to E_+$ such that $\int_{S\cap T^1} h = b_1$ and $h(t) \leq g(t)$ for all $t \in S \cap T^1$.

Set $\overline{h} = 1_{T^1}h + 1_{T^2}s$. Then $\overline{h}$ is Pettis integrable and we have $\int_S \overline{h} = \gamma^-$ and $0 \leq \overline{h}(t) \leq g(t)$ for all $t \in S$. Arguing as in the first part of this proof (with $\overline{h}$ in place of s) we get a contradiction to the property of f being in the core. Thus $(B'' - \{\int e(t)\,d\nu(t)\}) \cap -\Gamma = \emptyset$ must be true, which implies that we also have $(c\ell\, B'' - \{\int e(t)\,d\nu(t)\}) \cap -\Gamma = \emptyset$ because Γ is open.

Invoking the separation theorem again, we can now find a $p \in E^* \setminus \{0\}$ such that

$$\inf\langle p, B''\rangle \geq \left\langle p, \int_T e(t)\,d\nu(t)\right\rangle.$$

As earlier, p is strictly positive. Also, just as in the first part of the proof it follows (with B'' in place of B) that (5) holds for p. Once again, let

$$\tilde{S} = \{t \in T\colon \text{ there is an } x \in E_+ \text{ with } x \succ_t f(t) \text{ and } p(x) < p(e(t))\}.$$

As before, $\tilde{S}$ must be a null set. Indeed, suppose that $\nu^*(\tilde{S} \cap T^2) > 0$. Then, since every $S \subset T^2$ with $S \in \mathcal{T}$ and $\nu(S) > 0$ has a non-measurable subset, we can proceed as in the first part of the proof—but starting with $\tilde{S} \cap T^2$ in place of $\tilde{S}$— to get a contradiction to (5).

Suppose that $\nu^*(\tilde{S} \cap T^1) > 0$. Then since $2^{T^1} \subset \mathcal{T}$, we can proceed as in the second part of the proof—but starting with $\tilde{S} \cap T^1$ in place of $\tilde{S}$—to get a direct contradiction to the fact that $\inf\langle p, B''\rangle \geq \langle p, \int_T e(t)\,d\nu(t)\rangle$. (Note that if we construct $\overline{g}$ as in this second part of the proof, modulo that we start with $\widetilde{S} \cap T^1$ in place of $\widetilde{S}$, we have $\overline{g}(t = e(t)$ for all $t \in T^2$. Thus $1_{T^2}\overline{g}$ is strongly measurable because $t \mapsto e(t)$ is assumed to be so, and consequently $\int_T \overline{g}$ must belong to B''.)

Thus $\widetilde{S}$ is a null set, and we conclude that the pair (p, f) is a Walrasian equilibrium. To complete the proof of the theorem, the following lemma, which twice was invoked above, must be established.

Lemma 14. *Let E be a Banach lattice and let $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succ_t, e(t))_{t\in T}]$ be an economy with commodity space E satisfying assumptions* (A2), (A4) *and* (M2). *Let f and g be allocations for $\mathcal{E}$, let $S \in \mathcal{T}$ with $\nu(S) > 0$, and suppose that $g(t) \succ_t f(t)$ for almost all $t \in S$. Assume that g is strongly measurable. Then given any real number $\epsilon > 0$ there is a measurable simple function $h\colon T \to E_+$ and an $S' \in \mathcal{T}$ with $S' \subset S$ such that $\|\int_{S'} h - \int_S g\| < \epsilon$ and $h(t) \succ_t f(t)$ for almost all $t \in S'$. Moreover, given $\delta > 0$, S' can be chosen so that $\nu(S \setminus S') < \delta$.*

Proof. Let ϵ, $\delta > 0$ be given. According to Lemma 8, there is an $\overline{S} \in \mathcal{T}$ with $\overline{S} \subset S$, $\nu(\overline{S}) > 0$, $\nu(S \setminus \overline{S}) < \delta/2$, and such that $1_{\overline{S}}g$ is Bochner integrable and

$$\left\|\int_{\overline{S}} g - \int_S g\right\| < \epsilon/2.$$

Set $\overline{g} = 1_{\overline{S}}g$. By definition of Bochner integrability, select a sequence (s_n) of simple functions from T into E such that $\int_T \|\overline{g}(t) - s_n(t)\|\, d\nu(t) \to 0$ and, passing to a subsequence if necessary, such that $s_n(t) \to \overline{g}(t)$ in the norm $\|\cdot\|$ of E for almost all $t \in T$. For each n let $h_n\colon T \to E_+$ be given by $h_n(t) = s_n(t) \vee 0$, $t \in T$. Then each h_n is also a simple function, and by virtue of the continuity of the lattice operations we have $h_n(t) \to \overline{g}(t)$ for almost all $t \in T$. Moreover, $\|h_n(t)\| \le \|s_n(t)\|$ for all n and t (since $\|\cdot\|$ is a lattice norm). For each n set

$$S_n = \{t \in \overline{S} \colon h_m(t) \succ_t f(t) \text{ for all } m \ge n\}$$

and note that $S_n \in \mathcal{T}$ by Assumption (M2). Evidently $S_n \subset S_{n+1}$ for all n, and by continuity of preferences, for some null set N in $\overline{S}$ we have $\overline{S} \setminus N = \bigcup_{n=1}^{\infty} S_n$. Consequently $(1_{S_n} h_n)(t) \to (1_{\overline{S}}\, \overline{g})(t)$ for almost all $t \in T$, and since

$$\|(1_{S_n} h_n)(t)\| \le \|(1_{\overline{S}}\, h_n)(t)\| \le \|s_n(t)\|$$

for all $t \in T$ (and $\int_T \|\overline{g}(t) - s_n(t)\|\, d\nu(t) \to 0$), an appeal to Vitali's convergence theorem shows that $\int_{S_n} h_n(t)\, d\nu(t) \to \int_{\overline{S}} \overline{g}(t)\, d\nu(t)$.

Since

$$\left\|\int_{S_n} h_n - \int_S g\right\| \le \left\|\int_{S_n} h_n - \int_{\overline{S}} \overline{g}\right\| + \left\|\int_{\overline{S}} \overline{g} - \int_S g\right\|$$

and $\nu(S_n) \to \nu(\overline{S})$, the lemma is proved. □

References

Aumann, R. J. (1964): "Markets with a Continuum of Traders," *Econometrica*, 32, 39–50.

Bewley, T. F. (1973): "The Equality of the Core and the Set of Equilibria in Economies with Infinitely Many Commodities and a Continuum of Agents," *International Economic Review*, 14, 383–394.

Diestel, J. and J. J. Uhl, Jr. (1977): *Vector Measures*, vol. 15 of *Mathematical Surveys and Monographs*, Providence, Rhode Island: American Mathematical Society.

Edgar, G. A. (1977): "Measurability in a Banach Space," *Indiana University Mathematics Journal*, 26, 663–677.

——— (1979): "Measurability in a Banach Space, II," *Indiana University Mathematics Journal*, 28, 559–579.

Gabszewicz, J. (1968): *Coeurs et Allocations Concurrentielles dans des Economies d'Echange avec un Continu de Biens*, Louvain: Librairie Universitair.

Jin, R. and H. J. Keisler (2000): "Maharam Spectra of Loeb Spaces," *The Journal of Symbolic Logic*, 65, 550–566.

Juhász, I. and Z. Szentmiklóssy (1992): "Convergent Free Sequences in Compact Spaces," *Proceedings of the American Mathematical Society*, 116, 1153–1160.

Kelley, J. L. and I. Namioka (1976): *Linear Topological Spaces*, Graduate Texts in Mathematics, New York: Springer-Verlag, second printing.

Khan, M. A. and Y. Sun (1997): "The Capital-Asset-Pricing Model and Arbitrage Pricing Theory: A Unification," *Proc. Natl. Acad. Sci. USA*, 94, 4229–4232.

Lindenstrauss, J. and L. Tzafriri (1979): *Classical Banach Spaces II*, New York: Springer-Verlag.

Mas-Colell, A. (1975): "A Model of Equilibrium with Differentiated Commodities," *Journal of Mathematical Economics*, 2, 263–295.

Mertens, J.-F. (1970): "An Equivalence Theorem for the Core of an Economy with Commodity Space $L_\infty - \tau(L_\infty, L_1)$," in *Equilibrium Theory in Infinite Dimensional Spaces*, ed. by Khan, M. Ali and N. C. Yannelis, New York, 1991: Springer-Verlag, reprint of CORE DP 7028 (1970).

Negrepontis, S. (1984): “Banach Spaces and Topology,” in *Handbook of Set-Theoretic Topology*, ed. by K. Kunen and J. E. Vaughan, Amsterdam: North-Holland, chap. 23, 1045–1142.

Ostroy, J. M. and W. R. Zame (1994): “Nonatomic Economies and the Boundaries of Perfect Competition,” *Econometrica*, 62, 593–633.

Podczeck, K. (2002): “Note on the Core-Walras Equivalence Problem when the Commodity Space is a Banach Lattice,” Tech. rep., Institut für Wirtschaftswissenschaften, Universität Wien.

——— (2003): “Core and Walrasian Equilibria when Agents’ Characteristics are Extremely Dispersed,” *Economic Theory*, 22, 699–725.

Rustichini, A. and N. C. Yannelis (1991): “Edgeworth’s Conjecture in Economies with a Continuum of Agents and Commodities,” *Journal of Mathematical Economics*, 20, 307–326.

Sun, Y. (1996): “Hyperfinite Law of Large Numbers,” *The Bulletin of Symbolic Logic*, 2, 189–198.

Tourky, R. and N. C. Yannelis (2001): “Markets with Many More Agents than Commodities: Aumann’s “Hidden” Assumption,” *Journal of Economic Theory*, 101, 189–221.

Yannelis, N. C. (1991): “Integration of Banach-Valued Correspondences,” in *Equilibrium Theory in Infinite Dimensional Spaces*, ed. by Khan, M. Ali and N. C. Yannelis, New York: Springer-Verlag, 2–35.

Zame, W. R. (1986): “Markets with a Continuum of Traders and Infinitely Many Commodities,” Working paper, SUNY at Buffalo.