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Note on the Core-Walras Equivalence Problem when the Commodity Space is a Banach Lattice

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Note on the Core-Walras Equivalence Problem when the Commodity Space is a Banach Lattice[*]

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Abstract

The core-Walras equivalence problem for an atomless economy is considered in the commodity space setting of Banach lattices. In particular, necessary and sufficient conditions on the commodity space in order for core-Walras equivalence to hold are established. In general, these conditions can be regarded as implying that an economy with a continuum of agents has indeed "many more agents than commodities." However, it turns out that there are special commodity spaces in which core-Walras equivalence holds for every atomless economy satisfying certain standard assumptions, but in which an atomless economy does not have the meaning of there being "many more agents than commodities."

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1 Introduction

In his now classical paper "Markets with a Continuum of Traders," Aumann (1964) argued that the appropriate model for perfect competition is one in which the set of agents is specified by an atomless measure space. Aumann showed that, in such a model, a certain test of perfect competition is indeed satisfied: the core of an economy coincides with the set of Walrasian equilibrium allocations. Aumann's model is formulated in terms of finitely many commodities. However, if one follows, say, Chamberlin's view of commodity differentiation, then a continuum of agents means that infinitely many commodities should be

taken into account. The question naturally arises then as to whether Aumann's core-Walras equivalence theorem carries over to infinite dimensional commodity spaces. In fact, this question has been taken up several times in the literature.

In this note we address the core-Walras equivalence problem in the abstract setting of commodity spaces that are Banach lattices. In particular, necessary and sufficient conditions on the commodity space in order for core-Walras equivalence to obtain are established.

Recently, Tourky and Yannelis (2001) showed that given a non-separable *Hilbert* space E, and given any atomless measure space $(T, \mathcal{T}, \nu)$, there is an economy with E as commodity space and $(T, \mathcal{T}, \nu)$ as space of traders such that—feasibility being formalized by the Bochner integral—there is a core allocation not supportable as a Walrasian equilibrium although the following strong conditions, listed in Tourky and Yannelis (2001) under the term "desirable assumptions," all hold: a) for some vector ordering on E, E is an ordered Hilbert space so that E_+, the positive cone of E, has a non-empty interior[1]; (b) endowments belong to the interior of E_+; (c) consumption sets are equal to E_+; (d) preferences are complete preorderings and are continuous, convex and strictly monotone; (e) the preference mapping is measurable in the sense of Aumann (1964)[2]. Subsequently, responding to a question raised by Tourky and Yannelis (2001), it was shown in Podczeck (2001) that a core-Walras non-equivalence result like that of Tourky and Yannelis (2001) actually holds in any non-separable *Banach* space. Combining this latter fact with the core-Walras equivalence result for separable Banach spaces by Rustichini and Yannelis (1991), it thus turns out that, under the "desirable assumptions," the class $\{E\}$ of Banach spaces with the property that any atomless economy with commodity space E exhibits core-Walras equivalence is exactly the class of Banach spaces that are separable[3].

The interpretation of these results of Tourky and Yannelis (2001) and Podczeck (2001) is that a large number of agents does not guarantee perfect competition unless there are in fact "many more agents than commodities;" if this latter condition does not hold, then a large number of agents means that agents' characteristics may be extremely dispersed, so that the standard theory of perfect competition fails.[4]

However, many interesting commodity spaces have the property that the

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[1]"Ordered Hilbert space" means that the positive cone is closed.

[2]That is, given any two allocations the set of agents preferring what they get in the first allocation to what they get in the second is measurable, allocations being defined as Bochner integrable (hence strongly measurable) functions.

[3]Actually, the assumptions employed by Rustichini and Yannelis (1991, Theorem 4.1) are weaker than the "desirable assumptions;" in particular, preferences are not assumed to be convex or complete.

[4]We refer to Tourky and Yannelis (2001) and Podczeck (2001) for a more detailed discussion of this point.

positive cone has empty interior and are thus not covered in the framework of the "desirable assumptions." Moreover, note the phrase "for some ordering…" in the listening of the "desirable assumptions." In fact, the core-Walras non-equivalence results in Tourky and Yannelis (2001) and Podczeck (2001) are not established for a Banach space with a certain given ordering; rather, the ordering is constructed in the proofs. On the other hand, in most economic models the ordering of the commodity space is taken as a priori given. Indeed, the typical examples of infinite dimensional commodity spaces appearing in the literature are vector lattices, and the ordering of the commodity space is taken to be the given lattice ordering and is not an object of construction.

With these facts as motivation—and as economic justification—we investigate in this note the question as to whether the analysis of Tourky and Yannelis (2001) and Podczeck (2001) carries over to the context of Banach lattices, in particular addressing the general case in which the positive cone may have an empty interior.

It should be noted that, in fact, the core-Walras non-equivalence results of Tourky and Yannelis (2001) and Podczeck (2001) do not apply to any Banach lattice (with the given lattice ordering). Indeed, in the proofs given by these authors, the ordering of the commodity space is chosen in such a way that not only the commodity space itself has positive cone with non-empty interior, but also its dual space.[5] However, no infinite dimensional Banach lattice has the property that its own positive cone and that of its dual both have non-empty interior.[6]

Let us remark also that non-separable Banach lattices indeed appear as commodity spaces in economic models. An example are models of commodity differentiation where the commodity space is $M(\Omega)$, the space of all regular bounded Borel measures on a compact Hausdorff space Ω; under its usual norm and ordering, $M(\Omega)$ is a Banach lattice which is non-separable if Ω is uncountable. The theory of thick and thin markets developed by Ostroy and Zame (1994) uses this

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[5] In fact, the positive cone of the commodity space X is chosen to be the cone generated by $\{v\} + B_X$ where B_X is the closed unit ball of the Banach space X and v some point of X with $v \notin B_X$.

[6] A further point concerns the use of the Bochner integral to formalize aggregation of commodity bundles. Since, by definition, a Bochner integrable function is strongly measurable, hence essentially separably valued, it can be argued that formalizing aggregation in terms of the Bochner integral makes the core "large" in some sense, thus implying a bias in favor of core-Walras non-equivalence when the commodity space is non-separable. The analysis of the core-Walras equivalence problem when aggregation is formalized in terms of a weaker notion of integrability will be the topic of a future paper. In the present note the question is to what extent the results of Tourky and Yannelis (2001) and Podczeck (2001) carry over to Banach lattices. We remark, however, that all the results of this note would continue to be valid in a framework in which allocations were defined to be functions that are Pettis integrable but still strongly measurable.

framework of commodity differentiation. As shown by these authors, in order to have examples of thin markets, preferences must not be weak* continuous (as frequently assumed in models of commodity differentiation), but just norm continuous, so that (norm) non-separability of $M(\Omega)$ actually matters.

Other examples of commodity spaces that are non-separable Banach lattices can be found in models of uncertainty. Frequently, in these models the commodity space is specified to be an $L_p(\mu)$ space for $1 \le p < \infty$. Such a space is separable if and only if the underlying measure space is separable. Since there is no a priori economic reason why this measure space should be separable, non-separability must be taken into account in the framework of commodity spaces that are $L_p(\mu)$ spaces. In fact, some authors (e.g. Khan and Sun, 1997) advertize Loeb measure spaces as appropriate for models of uncertainty, and $L_p(\mu)$ spaces on such measure spaces *are* non-separable.

The results of our paper can be summarized as follows. In Section 3.2.1 it is shown, concerning the question of conditions on the commodity space in order for core-Walras equivalence to hold, that it indeed matters whether the positive cone of the commodity space is taken to be a priori given or, as in Tourky and Yannelis (2001) and Podczeck (2001), taken to be an object of construction. In this section, the commodity space is a $C(\Omega)$ space[7]—in other words, a Banach lattice whose positive cone has non-empty interior. In the framework of commodity spaces that are Banach lattices, this setting is the special case that is most directly comparable with the discussion in Tourky and Yannelis (2001) or Podczeck (2001). In particular, complications arising in the general Banach lattice framework through emptiness of the interior of the positive cone are ruled out. It will turn out that the pivotal property for a given $C(\Omega)$ space, say E, in order that core-Walras equivalence holds for every atomless economy with commodity space E satisfying the "desirable assumptions"—modulo the fact that the ordering of E is now a priori given—is not separability, but rather that for each *positive* linear functional on E there be some countable subset of E separating this functional from the other *positive* linear functionals on E. (See Section 3 for precise formulation.) This latter condition is of course satisfied by any separable $C(\Omega)$ space (in fact, by any separable Banach lattice), but is also satisfied by some $C(\Omega)$ spaces that are non-separable (and for which the "desirable assumptions" can indeed be met). Thus, relative to the context of commodity spaces that are $C(\Omega)$ spaces, the correlation between separability of the commodity space and core-Walras equivalence is less strict than in the setting used by Tourky and Yannelis (2001) and Podczeck (2001), where the ordering of the commodity space is not taken to be a priori given; in partic-

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[7]By "$C(\Omega)$ space" we mean a Banach lattice that is isomorphic as a Banach lattice to a space of all continuous real valued functions defined on some compact Hausdorff space Ω.

ular, there are $C(\Omega)$ spaces in which core-Walras equivalence holds for every atomless economy satisfying the "desirable assumptions"—modulo that the ordering of the space is the given lattice ordering—but in which the property of an economy being atomless does not mean that there are "many more agents than commodities."

In Section 3.2.2 we turn to the general case where the commodity space is a Banach lattice whose positive cone may have an empty interior. Concerning this case, it is well known that if an economy with an infinite dimensional commodity space has consumption sets with empty interior, then—regardless of whether or not the commodity space is separable—one way in which core-Walras equivalence can fail is through preferences displaying marginal rates of substitution that are not properly bounded; cf. the example of a failure of core-Walras equivalence described in Rustichini and Yannelis (1991), an example where the commodity space is actually separable. This sort of failure of core-Walras equivalence reflects the general fact that if consumption sets in an infinite dimensional commodity space have empty interior, then continuity of preferences by itself does not provide the appropriate bounds on marginal rates of substitution in order for preferred sets to admit supporting price systems.

Under assumptions imposing bounds on marginal rates of substitution, positive results on core-Walras equivalence were established in Rustichini and Yannelis (1991) and Zame (1986) for the case where the commodity space is a separable Banach lattice. The assumption on marginal rates of substitution we use in the present paper is taken from Zame (1986). It may be found in Section 3 under the label (A9).

A sufficient and necessary condition on the commodity space in order for core-Walras equivalence to hold when this latter hypothesis on preferences is in force in addition to the "desirable assumptions"—modulo the fact that the positive cone of the commodity space is a priori given and now may have empty interior—is described in Theorem 2 in Section 3. We will here mention only that this condition can actually be regarded as implying that in an atomless economy there are "many more agents than commodities." This is in accordance with the analysis in Tourky and Yannelis (2001) and Podczeck (2001). However, to have "many more agents than commodities," it is no longer necessary that the commodity space be separable. The reasons for this are the following. First, the assumption on marginal rates of substitution implies strong continuity properties for preferences, so that it is not the norm topology of the commodity space that is relevant, but some weaker topology. Second, because all (feasible) trading activities in an economy have to take place in the (closure of the) order ideal generated by the aggregate endowment, topological properties of order ideals, rather than topological properties of the entire commodity space, are

important.[8]

In Section 3.2.3, finally, the analysis of Section 3.2.2 will be specialized to the commodity space setting of σ-Dedekind complete Banach lattices whose positive cones contain quasi-interior points and whose duals contain strictly positive elements. This class of Banach lattices includes many of those spaces which came to prominence in the modeling of situations with infinitely many commodities.[9] It will turn out that for a commodity space belonging to this class the property of being separable *is* the decisive condition for core-Walras equivalence to hold. Moreover, separability of the commodity space and the notion of "many more agents than commodities" amount to the same condition in the context of an atomless economy.

2 Notation and Terminology

(1) Let F be a vector lattice.

(a) As usual, the order of F is denoted by $\geq$, and F_+ denotes the positive cone of F, i.e. $F_+ = \{x \in F : x \geq 0\}$. For $x, y \in F$ the expressions x^+, x^-, $|x|$, $x \vee y$, $x \wedge y$, and $x \perp y$ have the usual lattice theoretical meaning.

(b) Let $x, y \in F$. Then:

– $[x, y]$ denotes the order interval $\{z \in F : x \leq z \leq y\}$.

– A_x denotes the order ideal in F generated by x. Thus, if $x \in F_+$ then

$$A_x = \bigcup_{n=1}^{\infty} [-nx, nx] = \{z \in F : |z| \leq nx \text{ for some } n \in \mathbb{N}\}.$$

(2)(a) $C(\Omega)$ stands for the space of all continuous real valued functions on some compact Hausdorff space Ω, endowed with the supremum norm and the usual pointwise ordering; thus $C(\Omega)$ is a Banach lattice.

(b) By a "$C(\Omega)$ space" we mean a Banach lattice that is isomorphic as a Banach lattice to a concrete space $C(\Omega)$.

(3) Let E be any Banach lattice.

(a) E^* denotes the dual space of E, i.e. the space of all continuous linear functions from E into $\mathbb{R}$. If $x \in E$ and $p \in E^*$, the value $p(x)$ of p at x will often be denoted by $\langle p, x \rangle$ for notational convenience. E^* will always be regarded as endowed with the dual norm and the dual ordering. Then E^* is also a Banach lattice; in particular:

– $E_+^* = \{q \in E^* : q(x) \geq 0 \text{ for all } x \in E_+\}$.

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[8] This is not in contradiction to our result for $C(\Omega)$ spaces. For in that context, the aggregate endowment of an economy belongs, under the "desirable assumptions," to the interior of the positive cone and therefore generates an order ideal equal to the entire space (by definition of a $C(\Omega)$ space).

[9] E.g., all the spaces $L_p(\mu)$, $1 \leq p \leq \infty$, the measure μ σ-finite, belong to this class.

(b) Let $x \in E_+$.

– The point x is said to be a *quasi-interior point* of E_+ if A_x is dense in E. Recall that this can be equivalently expressed by saying that x is a quasi-interior point of E_+ if $q(x) > 0$ whenever $q \in E_+^* \setminus \{0\}$.

(c) Let $q \in E^*$.

– $\ker q$ denotes the kernel of q, i.e. $\ker q = \{x \in E : q(x) = 0\}$.

– q is said to be *strictly positive* if $q(x) > 0$ whenever $x \in E_+ \setminus \{0\}$.

(d) Let A be a subset of E. Then:

– $\operatorname{int} A$ denotes the (norm) interior of A;

– $\operatorname{c\ell} A$ or $\bar{A}$ denote the (norm) closure of A;

– $\langle p, A \rangle$, $p \in E^*$, denotes the set $\{p(x) : x \in A\}$.

(e) $\sigma(E, E^*)$ denotes the weak topology of E and $\sigma(E^*, E)$ the weak* topology of E^*. Further, for a strictly positive $q \in E_+^*$:

– $\sigma(E, A_q)$ denotes the weak topology of E with respect to the order ideal A_q;

– $|\sigma|(E, A_q)$ denotes the absolute weak topology of E with respect to A_q.

(Note that when q is strictly positive, A_q separates the points of E.)

(f) Let V be a subset of E or E^*, and let τ be some topology on (the set underlying) E or E^*, respectively. Then

– (V, τ) means V regarded as endowed with the (relativized) topology τ (instead of the norm topology). E.g., for $q \in E_+^*$:

– $(E, \sigma(E, A_q))$ means E regarded as endowed with the topology $\sigma(E, A_q)$.

(g) Let $(T, \mathcal{T}, \nu)$ be a measure space. Then:

– Given a mapping $f : T \to E$, by "f is integrable" we always mean f is Bochner integrable.

– Given a correspondence $\varphi : T \to 2^E$, $\int_T \varphi(t) \, d\nu(t)$ means the set

$$\left\{ z \in E : z = \int_T f(t) \, d\nu(t) \text{ for some (Bochner) integrable function} \atop f : T \to E \text{ with } f(t) \in \varphi(t) \text{ for almost all } t \in T \right\}.$$

3 The Model and the Results

3.1 The Model

Let E be a Banach lattice. An *economy* $\mathcal{E}$ with commodity space E is a pair $[(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ where

- $(T, \mathcal{T}, \nu)$ is a complete positive finite measure space of agents;

- $X(t) \subset E$ is the consumption set of agent t;

- $\succcurlyeq_t \subset X(t) \times X(t)$ is the preference/indifference relation of agent t;

- $e(t) \in E$ is the initial endowment of agent t;

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and where the endowment mapping $e\colon T \to E$, given by $t \mapsto e(t)$, is assumed to be integrable[10].

The economy $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ is said to be *atomless* if the measure space $(T, \mathcal{T}, \nu)$ is atomless.

An *allocation* for the economy $\mathcal{E}$ is an integrable function $f\colon T \to E$ such that $f(t) \in X(t)$ for almost all $t \in T$. An allocation f is said to be *feasible* if

$$\int_T f(t)\, d\nu(t) = \int_T e(t)\, d\nu(t).$$

A *Walrasian equilibrium* for the economy $\mathcal{E}$ is a pair (p, f) where f is a feasible allocation and $p \in E^* \setminus \{0\}$ is a price system such that for almost all $t \in T$:

(i) $\langle p, f(t) \rangle \le \langle p, e(t) \rangle$ and

(ii) if $x \in X(t)$ satisfies $x \succ_t f(t)$ then $\langle p, x \rangle > \langle p, e(t) \rangle$.[11]

A feasible allocation f is said to be a *Walrasian allocation* if there is a $p \in E^* \setminus \{0\}$ such that (p, f) is a Walrasian equilibrium. An allocation f is a *core allocation* if it is feasible and if there does not exist a coalition $S \in \mathcal{T}$ with $\nu(S) > 0$ and an integrable function $g\colon T \to E$ such that

(i) $\int_S g(t)\, d\nu(t) = \int_S e(t)\, d\nu(t)$, i.e. g is feasible for S, and

(ii) $g(t) \succ_t f(t)$ for almost all $t \in S$.

We denote by $C(\mathcal{E})$ the set of all core allocations of the economy $\mathcal{E}$, and by $\mathcal{W}(\mathcal{E})$ the set of Walrasian allocations.

We shall take the following standard assumptions into consideration:

(A1) $X(t) = E_+$ for every $t \in T$.

(A2) $\succcurlyeq_t$ is reflexive, transitive, and complete for every $t \in T$.

(A3) For every $t \in T$, $\succcurlyeq_t$ is continuous, i.e. for each $x \in E_+$ the sets
$\{y \in X(t)\colon y \succcurlyeq_t x\}$ and $\{y \in X(t)\colon x \succcurlyeq_t y\}$ are (norm) closed in E_+.

(A4) For every $t \in T$, $\succcurlyeq_t$ is strictly monotone, i.e. whenever $x, x' \in E_+$ with $x \ge x'$ and $x \ne x'$ then $x \succ_t x'$.

(A5) For every $t \in T$, $\succcurlyeq_t$ is convex, i.e. for each $x \in E_+$ the set
$\{y \in E_+\colon y \succcurlyeq_t x\}$ is convex.

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[10]As said in the previous section, throughout this paper "integrable" means "Bochner integrable." We do not discuss the implications of other notions of integrability for the core-Walras equivalence problem.

[11]As usual, $x \succ_t y$ means "$x \succcurlyeq_t y$ and $y \not\succcurlyeq_t x$" i.e. that x is valued better than y by t.

(A6) If f and g are any two allocations then $\{t \in T : f(t) \succ_t g(t)\}$ is a measurable set, i.e. it belongs to $\mathcal{T}$. ("Aumann measurability" of the profile of agents' preferences.)

Remark 1. Requiring assumptions (A1) to (A5) to be *simultaneously* satisfied may amount to making an assumption on the commodity space E. For example, if E is actually a $C(\Omega)$ space then these assumptions taken together imply that E^* must contain strictly positive elements; or, to say it the other way round, these assumptions can hold simultaneously only when strictly positive linear functionals on E do exist. (Indeed, when these assumptions hold and $\mathrm{int}\, E_+ \neq \varnothing$, then, given any $t \in T$ and $x \in E_+$, the set of all $y \in E_+$ preferred to x by t is supported by a positive $p \in E^* \smallsetminus \{0\}$, and when x actually belongs to $\mathrm{int}\, E_+$ then p must in fact be strictly positive, by the usual argument.) On the other hand, when strictly positive linear functionals exist on a given Banach lattice E, then, of course, assumptions (A1) to (A5) can be satisfied at the same time. (To see this, suppose q is a strictly positive linear functional on E and look at the preference relation $\succcurlyeq$ on E_+ defined by setting $x \succcurlyeq y$ if and only if $q(x) \geq q(y)$, $x, y \in E_+$.) Examples of Banach lattices whose duals contain strictly positive linear functionals will be presented below at the appropriate places. Here let us recall just that there are strictly positive linear functionals on any *separable* Banach lattice. (See e.g. the first paragraph of the proof of Proposition 1.b.15 in Lindenstrauss and Tzafriri, 1979, p. 25).

3.2 Results

3.2.1 $C(\Omega)$ spaces

In this section the core-Walras equivalence problem is treated in the setting where the commodity space is a $C(\Omega)$ space, or, in other words, a Banach lattice whose positive cone has a non-empty interior. As noted in the introduction, in the general framework of commodity spaces that are Banach lattices, this setting is the special case that is most closely related to the analysis in Tourky and Yannelis (2001) and Podczeck (2001). The difference is that here we take the ordering of the commodity space to be the given lattice ordering. Concerning conditions on the commodity space in order for core-Walras equivalence to hold, we shall show in particular that this difference indeed matters.

Let $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ be an economy with commodity space E where E is a $C(\Omega)$ space. In addition to the conditions listed in the previous section, we take the following one into consideration in this case.

(A7) $e(t) \in \mathrm{int}\, E_+$ for every $t \in T$.

Recall from the introduction that (A1) to (A7) together yield what is called in Tourky and Yannelis (2001) the "desirable assumptions."

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We come now to the main point regarding the setting where the commodity space is a $C(\Omega)$ space. As advertized in the introduction, in this setting the condition on the commodity space crucial to obtain core-Walras equivalence for an atomless economy satisfying the "desirable assumptions" is not separability, but rather that all positive linear functionals be countably determined in some sense.

Precisely, this latter condition is as follows. Given a Banach lattice E let us call an element q of E_+^* *countably determined* if there is a countable subset D of E such that whenever $q' \in E_+^*$ and $q'(d) = q(d)$ for all $d \in D$ then $q' = q$. (Note that this definition applies only to *positive* elements $q, q' \in E^*$. Obviously, if $\operatorname{int} E_+ \neq \varnothing$ then the zero element of E^* is countably determined.)

We say that a Banach lattice E has *property CD* if every $q \in E_+^*$ is countably determined. (The sets D from the definition of "countably determined" may vary with $q \in E_+^*$, of course.)

Clearly, every separable Banach lattice has property CD. However, for a non-separable Banach lattice, property CD may or may not be satisfied:

Example 1. *A non-separable $C(\Omega)$ space with property CD.* Let Ω be the compact Hausdorff space known as the "split interval" (or "double arrow space")[12] and recall that this space is separable but not second countable. Let $E = C(\Omega)$. Then, because Ω is not second countable, E is non-separable.

We identify $E^* \equiv C(\Omega)^*$ with the space $M(\Omega)$ of all finite regular Borel measures on Ω. In particular, E_+^* is identified with $M(\Omega)_+$.

Let $P \subset M(\Omega)_+$ be the set of all regular probability measures on Ω. Then, Ω being the "split interval," according to Pol (1982, 8.4)[13], (P, weak^*) is first countable. Evidently this means that each $q \in P$ is countably determined (by the definition of the weak* topology), which in turn implies that the same is true for each $q \in M(\Omega)_+ (\equiv E_+^*)$. Thus E satisfies property CD.

Note also for later reference that $E^* \equiv M(\Omega)$ has strictly positive elements. (Indeed, as Ω is separable, $M(\Omega)_+$ contains measures with support equal to Ω, and such measures are strictly positive linear functionals on $E \equiv C(\Omega)$.)

Example 2. *$C(\Omega)$ spaces without property CD.* Examples are provided by any infinite dimensional space $L_\infty(\mu)$ and, in particular, by ℓ_∞. As is well known, these spaces are $C(\Omega)$ spaces.

To see that they do not have property CD, note first that if a Banach lattice E satisfies property CD, then there is an injection from E_+^* into the cartesian product of the set of all sequences in E and the set of all sequences of real

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[12]Recall that this is just the subset X of $\mathbb{R}^2$ defined by $X = (0, 1] \times \{0\} \cup [0, 1) \times \{1\}$, endowed with the lexicographical order topology; cf., e.g., Engelking (1989, 3.10.C, p. 212).

[13]Note that what we have called "countably determined" corresponds to what is call in Pol (1982) "*strongly* countably determined."

numbers.[14] Thus, recalling that

$$\operatorname{card}(\ell_\infty^*)_+ = \operatorname{card} \ell_\infty^* > \operatorname{card} \ell_\infty = \mathfrak{c} = \mathfrak{c}^{\aleph_0} \cdot \mathfrak{c}^{\aleph_0},$$

where "card" stands for "cardinality" and $\mathfrak{c}$ is the cardinality of the continuum, it is clear that ℓ_∞ does not have property CD.

Next note that if E is any Banach lattice with property CD and F is a Banach lattice such that there is a positive linear operator from E onto F, then F must have property CD, too. Thus, property CD fails for any Banach lattice that admits a positive linear operator onto ℓ_∞. In particular, any infinite dimensional space $L_\infty(\mu)$ fails property CD.

We are ready to formulate our first result. The hypothesis in the following theorem that E^* contain strictly positive elements is commented in Remark 3 below (see also there for corresponding examples).

Theorem 1. *Let E be a $C(\Omega)$ space with E^* containing strictly positive elements. Assume the continuum hypothesis. Then the following are equivalent.*

(i) *E has property CD.*

(ii) *$C(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ holds for every atomless economy $\mathcal{E}$ with commodity space E satisfying assumptions (A1) to (A7).*

(See Section 4.2 for the proof. The continuum hypothesis is needed only for implication (ii)⇒(i). Let us remark here that the continuum hypothesis is also assumed in the Tourky and Yannelis (2001) result on core-Walras non-equivalence in non-separable Hilbert spaces.)

Theorem 1 yields two simple corollaries. Combining Theorem 1(ii)⇒(i) with the fact that there are $C(\Omega)$ spaces E which fail property CD, but such that E^* has strictly positive elements—see Example 2 and Remark 3(d) below—we obtain (recalling that a Banach lattice which fails property CD must be non-separable):

Corollary 1. *Assume the continuum hypothesis. Then there exist non-separable $C(\Omega)$ spaces E such that $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$ holds for some atomless economy $\mathcal{E}$ with commodity space E satisfying assumptions (A1) to (A7).*

(For the formulation of this corollary note that "$C(\mathcal{E}) \neq \mathcal{W}(\mathcal{E})$" is equivalent to "$C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$" because a Walrasian allocation must be a core allocation.)

On the other hand, however, as shown in Example 1, there are non-separable $C(\Omega)$ spaces E for which E^* contains strictly positive elements and property CD is satisfied. (That the dual of the space of Example 1 contains strictly positive

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[14]Take any mapping that assigns to each $q \in E_+^*$ a sequence $(x_i)_{i=0}^\infty$ of elements of E and a sequence $(r_i)_{i=0}^\infty$ of real numbers such that the set $\{x_i : i = 0, 1, \ldots\}$ determines q according to property CD and such that $q(x_i) = r_i$ for each i.

elements was noted at the end of that example.) Combining this fact with Theorem 1(i)⇒(ii) yields:

Corollary 2. *There are non-separable $C(\Omega)$ spaces E, with E^* containing strictly positive elements, such that $C(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ holds for every atomless economy $\mathcal{E}$ with commodity space E satisfying assumptions (A1) to (A7).*

(For this second corollary note that, as already remarked, implication (i)⇒(ii) of Theorem 1 holds without the continuum hypothesis. Recall also from Remark 1 that if E^* contains strictly positive elements, it is guaranteed that economies satisfying assumptions (A1) to (A5) do exist.)

Corollary 1 confirms the analysis of Tourky and Yannelis (2001) and Podczeck (2001): Even if the commodity space is a $C(\Omega)$ space (and the ordering considered is the given lattice ordering), core-Walras equivalence may fail when this space is non-separable. Corollary 2, however, shows that the analysis of Tourky and Yannelis (2001) and Podczeck (2001) does not totally carry over to the class of $C(\Omega)$ spaces. In particular, Corollary 2 shows, concerning criteria for core-Walras equivalence, that it matters whether the ordering of the commodity space is taken to be a priori given or, as in Tourky and Yannelis (2001) and Podczeck (2001), taken to be an object of construction. Moreover, this corollary implies that the pivotal condition on a $C(\Omega)$ space in order for core-Walras equivalence to hold cannot in general be interpreted by saying that under this condition an atomless economy has "many more agents than commodities."

We finish the treatment of $C(\Omega)$ spaces with some remarks.

Remark 2. As was already noted, separable Banach lattices have property CD, and their duals possess strictly positive elements. Thus implication (i)⇒(ii) of Theorem 1 entails a core-Walras equivalence result for separable $C(\Omega)$ spaces. For this case, in accordance with other known core-Walras equivalence results, (ii) in Theorem 1 remains valid when the convexity assumption (A5) is dropped from its statement. We do not know whether this is so in case of a $C(\Omega)$ space that has property CD but is non-separable. Since the focus in this paper is not on establishing a core-Walras equivalence result in largest generality, but rather on the properties of the commodity space that play a role for the core-Walras equivalence problem, we leave this as an open question. Finally, let us mention that in the statement of (ii) in Theorem 1, Assumption (A7) can be replaced by the weaker assumption

(A7') $e(t) \in E_+$ for every $t \in T$ and $\int_T e(t)\, d\nu(t) \in \operatorname{int} E_+$

without making this theorem false. On the other hand, implication (ii)⇒(i) yields a stronger result with (A7) instead of (A7'), and with the convexity assumption (A5) rather than without it.

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Remark 3. (a) The hypothesis in Theorem 1 that E^* contain strictly positive elements (i.e. that there be price systems for which every non-zero positive commodity bundle has value > 0) is natural from an economic viewpoint. Moreover, as pointed out in Remark 1 above, if the commodity space is a $C(\Omega)$ space, i.e. if the positive cone has non-empty interior, then the standard assumptions (A1) to (A5) can be simultaneously satisfied only when strictly positive linear functionals do exist. In particular, therefore, implication (ii)$\Rightarrow$(i) of Theorem 1 does not hold without this hypothesis. (Indeed, fix any uncountable set Γ and consider the $C(\Omega)$ space $\ell_\infty(\Gamma)$. According to point (e) below, its dual has no strictly positive elements. But therefore, (ii) in Theorem 1 is valid with $\ell_\infty(\Gamma)$ substituted for E—for the trivial reason that economies satisfying (A1) to (A5) do not exist. On the other hand, $\ell_\infty(\Gamma)$ does not have property CD—see the last paragraph of Example 2.) Thus, the hypothesis that E^* contain strictly positive elements is necessary for Theorem 1 to be true.[15]

(b) If $E = C(\Omega)$, then in order for E^* to have strictly positive elements (i.e. for Ω to carry finite positive regular Borel measures with support equal to Ω), it is sufficient (however, not necessary) that Ω be separable. Note on the other hand that separability of Ω does not imply separability of $C(\Omega)$; see e.g. Example 1 above.

(c) $C(\Omega)$ is separable if and only if Ω is metrizable. Thus if $C(\Omega)$ is separable, then so is Ω. Therefore, by what has been remarked under the previous point, *separable* $C(\Omega)$ spaces have duals with strictly positive elements. (As was noted in Remark 1, strictly positive linear functionals do in fact exist on any separable Banach lattice.)

(d) Examples of *non-separable* $C(\Omega)$ spaces whose duals have strictly positive elements are provided by $L_\infty(\mu)$ if it is infinite dimensional and μ is σ-finite, and, in particular, by ℓ_∞.

(e) An example of a $C(\Omega)$ space whose dual does not contain strictly positive elements is provided by $\ell_\infty(\Gamma)$ for Γ an uncountable set. To see this, recall that $\ell_\infty(\Gamma)$ may be identified with $C(\beta\Gamma)$ via a positive linear operator—denoting by $\beta\Gamma$ the Stone-Čech compactification of the discrete set Γ. Evidently, when Γ is uncountable, there is no finite Borel measure on $\beta\Gamma$ with support equal to the whole space.

Remark 4. The notion of property CD makes sense in any ordered Banach space, so one may ask whether or not in the core-Walras non-equivalence results of Tourky and Yannelis (2001) and Podczeck (2001) the commodity spaces satisfy

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[15]The hypothesis that E^* contain strictly positive elements could be dropped from Theorem 1 would we replace the assumption that preferences be *strictly* monotone by the assumption that they be monotone and locally non-satiated. However, this would lead out of the context of the "desirable assumptions."

property CD. Now in these latter results, the ordering of the commodity space is constructed in such a way that not only the commodity space itself, but also, for the dual ordering, its dual space has a positive cone with non-emptyinterior (so that, in particular, the ordering constructed cannot be a lattice ordering, unless the space in question is finite dimensional). It may easily be seen that if Y is an ordered Banach space such that the positive cone of the dual space has non-empty interior, then Y has property CD if and only if Y is separable. Consequently, the commodity spaces in the core-Walras non-equivalence results of Tourky and Yannelis (2001) and Podczeck (2001) do not have propertyCD.

On the other hand, in an ordered Banach space satisfying property CD, core-Walras equivalence holds under the "desirable assumptions." In fact, the proof of (i)⇒(ii) in Theorem 1 makes no use of any lattice properties.

3.2.2 General Banach lattices

In this section we turn to the general case where the commodity space is a Banach lattice E whose positive cone may have an empty interior. Concerning individual endowments in an economy with commodity space E, we will now consider the simple assumption:

(A8) For every $t \in T$, $e(t) \in E_+ \setminus \{0\}$.

As was pointed out in the introduction, if consumption sets in an infinite dimensional commodity space may have empty interior, then (even when the commodity space is separable) core-Walras equivalence may fail through lack of proper boundedness of marginal rates of substitution. To eliminate the possibility of this kind of failure of core-Walras equivalence, we will make use of a condition on marginal rates of substitution introduced by Zame (1986) (see also Ostroy and Zame, 1994). Given an economy with commodity space E, this condition is as follows:

(A9) There are strictly positive linear functionals $\alpha, \beta \in E^*$ with $\alpha \leq \beta$ such that for every $t \in T$, whenever $x, u, v \in E_+$ satisfy $u \leq x$ and $\alpha(v) > \beta(u)$ then $x - u + v \succ_t x$.

Note that this is a requirement on preferences that is uniform over agents as well as over the consumption set E_+. We refer to Zame (1986) for a discussion of this condition as well as for corresponding examples. (It may be seen that (A9), together with the convexity assumption (A5), is equivalent to the following statement: "There are strictly positive elements α, β in E^*, with $\alpha \leq \beta$, such that given any $t \in T$ and $x \in E_+$ there is a p in the order interval $[\alpha, \beta]$ such that $p(x) \leq p(y)$ for all $y \in E_+$ with $y \succcurlyeq_t x$." Thus, since supporting price systems are measures of marginal rates of substitution, (A9) is indeed a condition putting bounds on these rates.)

Of course, given a Banach lattice E as commodity space, Assumption (A9) can be satisfied only when strictly positive linear functionals on E do actually exist. As already noted, strictly positive linear functionals exist on any Banach lattice that is separable. Some non-separable Banach lattices whose duals have strictly positive elements were presented above in the discussion of $C(\Omega)$ spaces. The following example lists some non-separable Banach lattices whose duals have strictly positive elements but whose positive cones have empty interior.

Example 3. *Non-separable Banach lattices E such that E^* has strictly positive elements and such that* $\operatorname{int} E_+ = \varnothing$. Straightforward examples are provided by the Lebesgue spaces $L_p(\mu)$, $1 \leq p < \infty$, if the underlying measure space is non-separable but σ-finite, by the space $\ell_1(\Gamma)$ for Γ an uncountable set, and, for a compact uncountable Hausdorff space Ω, by $M(\Omega)$, the space of all bounded regular Borel measures on Ω; all these spaces understood as being endowed with their usual norms and orderings. (To see that $\ell_1(\Gamma)^*$ and $M(\Omega)^*$ possess strictly positive elements, recall simply that $\ell_1(\Gamma)^* = \ell_\infty(\Gamma)$, and that $M(\Omega)^*$ includes $C(\Omega)$.)

Now to give an intuition for the condition on the commodity space E crucial for core-Walras equivalence when A(8) and A(9) are in force in addition to the general assumptions (A1) to (A6),[16] we remark first that (A9) implies, together with the transitivity part of (A2), that preferences are uniformly proper, with a properness cone including E_+ and open actually for the absolute weak topology $|\sigma|(E, A_\beta)$, where β is the strictly positive element of E^* from Assumption A(9). (For details, see paragraphs 2–4 of the proof of Theorem 2.) Together with assumptions (A1) to (A4), this property in turn implies that preferences are $|\sigma|(E, A_\beta)$-lower semicontinuous, as may readily be seen. Thus, recalling standard core-Walras equivalence proofs, it should be sufficient for core-Walras equivalence to hold for an atomless economy satisfying all these assumptions that the commodity space E be $|\sigma|(E, A_\beta)$-separable, or, equivalently (by the Hahn-Banach theorem), $\sigma(E, A_\beta)$-separable.

In fact, it should be sufficient that the order ideal generated by the aggregate endowment be $\sigma(E, A_\beta)$-separable, because, under (A1) and (A8), all feasible trading activities in an economy have to take place in the norm closure of this order ideal. (Cf. Lemma 5 in Section 4.1. Uniform properness should provide the appropriate reservation values for commodity bundles not lying in this ideal.)

Thus, given a Banach lattice E, in order to have core-Walras equivalence for any atomless economy with commodity space E satisfying assumptions (A1) to

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[16]For the following recall from Section 2 that given $e \in E_+$ and $q \in E_+^*$, A_e denotes the order ideal in E generated by e, and A_q the order ideal in E^* generated by q; recall also that $\sigma(E, A_q)$ denotes the weak topology of E with respect to A_q, and $|\sigma|(E, A_q)$ the absolute weak topology of E with respect to A_q.

(A6) as well as (A8) and (A9), it should be sufficient that for any $e \in E_+$ and any strictly positive $q \in E^*$ the restriction of the topology $\sigma(E, A_q)$ to A_e be separable. The following theorem states that this intuition is indeed correct. Moreover, this theorem states that this latter condition on a Banach lattice E is also necessary for core-Walras equivalence to hold for every atomless economy with commodity space E satisfying the just listed assumptions. In particular, a characterization of Banach lattices in regard to the core-Walras equivalence problem in the context of economies for which these assumptions hold is given by this theorem.

Theorem 2. *Let E be a Banach lattice such that E^* has strictly positive elements. Assume the continuum hypothesis. Then the following are equivalent.*

(i) *For every $e \in E_+$ and every strictly positive $q \in E^*$, the relativization of the topology $\sigma(E, A_q)$ to A_e is separable.*

(ii) *$C(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ holds for every atomless economy $\mathcal{E}$ with commodity space E satisfying assumptions (A1) to (A6), (A8) and (A9).*

(See Section 4.3 for the proof.)

Some comments regarding this theorem are in order, and in particular some comments and examples regarding condition (i) and its relationship to the property of a commodity space being (norm) separable, to the notion of "many more agents than commodities," and to property CD defined above.

First, as already noted in the introduction, the fact that properties of order ideals, rather than properties of the entire commodity space, play a role in Theorem 2 is not in contradiction with our results for $C(\Omega)$ spaces. For in this latter context, the aggregate endowment of an economy was supposed to belong to the interior of the positive cone of the commodity space, and hence to generate an order ideal that coincides with the entire space.

Second, concerning the hypothesis that E^* contain strictly positive elements, this hypothesis can of course be removed from the statement of Theorem 2. For if E^* does not possess strictly positive elements, then (i) in Theorem 2 holds vacuously, and the same is true of (ii) since in this case economies satisfying Assumption (A9) cannot exist. Since this triviality is of no meaning, the hypothesis of the existence of strictly positive linear functionals has been incorporated in the statement of Theorem 2. On the other hand, when strictly positive linear functionals exist on a Banach lattice E, then, for E taken as commodity space, the existence of economies satisfying (A9) (as well as the other assumptions listed in (ii) of Theorem 2) is of course guaranteed.

Now as for condition (i) in the statement of Theorem 2, if this condition holds for a given Banach lattice E, then in particular the norm closure of the order ideal generated by the aggregate endowment of an economy with commodity

space E is $\sigma(E, A_q)$-separable for any strictly positive $q \in E^*$, therefore also $|\sigma|(E, A_q)$-separable.[17] Hence, under this condition, any atomless economy with commodity space E satisfying the assumptions listed in (ii) of Theorem 2 can be regarded as having "many more agents than commodities;" for as noted above, under (A1) and (A8) all feasible trading activities have to take place in the norm closure of the order ideal generated by the aggregate endowment, and under (A1) to (A4) and (A9), preferences are $|\sigma|(E, A_q)$-lower semicontinuous for some strictly positive $q \in E^*$. Consequently, Theorem 2 can be thought of as saying that, under assumptions (A1) to (A6) and (A8) and (A9), having "many more agents than commodities" is pivotal in order for core-Walras equivalence to be guaranteed to hold.

Thus, if one accepts this interpretation, Theorem 2 confirms the analysis of Tourky and Yannelis (2001) and Podczeck (2001) for the commodity space setting of Banach lattices, provided that economies are supposed to satisfy the above listed assumptions, and in particular (A9).

Of course, (i) of the statement of Theorem 2 holds for any separable Banach lattice E, because, given $e \in E_+$ and $q \in E_+^*$, the topology induced on A_e by $\sigma(E, A_q)$ is weaker than the (relative) norm topology of A_e. Thus Theorem 2 implies that in the framework of economies satisfying assumptions (A1) to (A6), (A8) and (A9), core-Walras equivalence holds whenever the commodity space is a Banach lattice that is separable.

However, (i) of Theorem 2 also holds for some non-separable Banach lattices; in particular, in order that an economy satisfying the assumptions listed in (ii) of Theorem 2 can be viewed to have "many more agents than commodities," it is not required that the commodity space be separable. Indeed, for the same reason for which (i) of Theorem 2 holds for a separable Banach lattice, this latter condition is in fact satisfied by any Banach lattice E with the property that, for each element e of E_+, the order ideal A_e is (norm) separable[18]—and there are Banach lattices with this property which are themselves non-separable (and for which the dual has strictly positive elements, so that, for them taken as commodity spaces, it is guaranteed that economies satisfying the assumptions listed in (ii) of Theorem 2 do actually exist):

Example 4. *Non-separable Banach lattices E for which E^* has strictly positive elements and* (i) *of Theorem 2 holds.* (a) Let $E = \ell_1(\Gamma)$ where Γ is an uncountable set. As noted in Example 3, this space is non-separable, and its dual contains

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[17]To see this, observe that if Z is a linear subspace of E and q is a strictly positive element of E^*, then the norm closure of Z is contained in the $\sigma(E, A_q)$-closure of Z (because the topology $\sigma(E, A_q)$ is weaker than the norm topology of E), and the $\sigma(E, A_q)$-closure of Z coincides with the $|\sigma|(E, A_q)$-closure of Z (by the Hahn-Banach theorem).

[18]The converse statement is false: there are Banach lattices E for which (i) of Theorem 2 is true but such that for some $e \in E_+$, A_e is not (norm) separable; see e.g. Example 5 below.

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strictly positive elements. But since for any $e \in \ell_1(\Gamma)$ the set of all $\gamma \in \Gamma$ for which $e(\gamma) \neq 0$ is countable, it is clear that for any $e \in \ell_1(\Gamma)_+$ the order ideal A_e is (norm) separable and hence, given any strictly positive $q \in E^*$, separable in the topology inherited from $\sigma(E, A_q)$. Thus $\ell_1(\Gamma)$ satisfies (i) of Theorem 2.

(b) Let $E = M(\Omega)$, where Ω is a compact *metric* uncountable space. By what was noted in Example 3, this Banach lattice also is non-separable and the dual has strictly positive elements. Now Ω, being a compact metric space, is second countable and hence its Borel σ-algebra is countably generated. Therefore, for any $\mu \in M(\Omega)_+$, the subspace $L_1(\mu)$ of $M(\Omega)$ is (norm) separable whence the order ideal A_μ is (norm) separable as well; thus (i) of Theorem 2 holds. Let us remark that there are also compact Hausdorff spaces Ω that are non-metrizable but such that for every $\mu \in M(\Omega)_+$, $L_1(\mu)$ is separable and consequently (i) of Theorem 2 is satisfied by $M(\Omega)$; see for instance the space Ω of Example 5 below.

The spaces of Example 4 have, in particular, the property that the positive cone has empty interior.[19] Thus, non-separable Banach lattices for which, taken as commodity spaces, core-Walras equivalence holds for any given atomless economy satisfying certain standard assumptions (and such that the existence of such economies is guaranteed) can also be found outside the context of $C(\Omega)$ spaces.

For examples of Banach lattices that do not satisfy condition (i) of Theorem 2, see Example 6(b) in Section 3.2.3, together with Lemma 1 in that section.

Now to pay attention to the relationship between (i) of the statement of Theorem 2 and property CD, it is stated in the corollary following the proof of Lemma 4 in Section 4.1 that whenever a Banach lattice E has property CD, it also satisfies (i) of Theorem 2. In particular, therefore, the $C(\Omega)$ space exhibited in Example 1 provides another non-separable Banach lattice for which the dual has strictly positive elements and (i) of Theorem 2 holds.

On the other hand, there are Banach lattices having these properties but not property CD. For instance, consider the space $\ell_1(\Gamma)$. As noted in Example 4, this space satisfies (i) of Theorem 2, and its dual has strictly positive elements, but when Γ is actually uncountable, it obviously fails property CD. In fact, (i) of Theorem 2 and the hypothesis of there being strictly positive linear functionals do not even imply property CD for $C(\Omega)$ spaces. This is shown in the next example which, in particular, exhibits a further non-separable Banach lattice for which (i) of Theorem 2 holds.

That the condition on the commodity space is weaker in Theorem 2 than in Theorem 1 is not surprising; for in Theorem 2, Assumption (A9) is required to hold for economies, while in Theorem 1 this is not the case.

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[19]In fact, if E is any non-separable Banach lattice such that for each $e \in E_+$ the order ideal A_e is separable then, of course, E cannot have a strong unit, i.e. the interior of E_+ must be empty.

Example 5. *A $C(\Omega)$ space E for which E^* contains strictly positive elements and* (i) *of Theorem 2 holds, but for which property CD fails to hold.* Let $E = C(\Omega)$ where Ω is a compact Hausdorff space which is scattered, separable, but not second countable; according to Semadeni (1971, 8.5.10(G), p. 150) such an Ω does exist.[20] Then, as Ω is separable, E^* has strictly positive elements (see Remark 3(b) above).

Because Ω is scattered, all regular Borel measures on Ω are purely atomic (see Semadeni, 1971, Theorem 19.7.6, p. 338). Therefore the dual of $E \equiv C(\Omega)$ may be identified with $\ell_1(\Omega)$. It is a well known fact (and is easy to see) that order intervals in $\ell_1(\Omega)$ are compact in the norm topology of $\ell_1(\Omega)$. Since the norm topology of $\ell_1(\Omega)$ is stronger than the weak* topology $\sigma(\ell_1(\Omega), C(\Omega))$, it follows that order intervals in $\ell_1(\Omega)$ are weak* metrizable (i.e. metrizable in the weak* topology $\sigma(\ell_1(\Omega), C(\Omega))$). By using Lemma 4 of Section 4.1, it follows from this that $E \equiv C(\Omega)$ satisfies (i) of Theorem 2.

As for property CD, consider the set $V = \{\delta_\omega : \omega \in \Omega\} \subset E_+^*$, where δ_ω is the Dirac measure at ω, and note first that Ω and (V, weak^*) are homeomorphic via the mapping $\omega \mapsto \delta_\omega$; in particular, (V, weak^*) is compact. Now since the compact Hausdorff space Ω is scattered but not second countable, it follows from a theorem of Mazurkiewicz and Sierpiński (see Semadeni, 1971, Theorem 8.6.10, p. 155) that there must be some point in Ω, say $\overline{\omega}$, at which Ω is not first countable. Accordingly, (V, weak^*) is not first countable at $\delta_{\overline{\omega}}$. By the compactness of (V, weak^*) this means that no countable subset of E can separate the point $\delta_{\overline{\omega}}$ from the other points of V (by definition of the weak* topology). Consequently, E does not satisfy property CD.

3.2.3 σ-Dedekind complete Banach lattices with positive cones having quasi-interior points

In this section the analysis of 3.2.2 is specialized to Banach lattices that are σ-Dedekind complete and whose positive cones contain quasi-interior points. This class of Banach lattices includes many of the spaces that have been used in economic models with infinitely many commodities. (See Example 6 below for some spaces belonging to this class.) It turns out that for a commodity space E in this class with E^* possessing strictly positive elements (so that, in particular, the existence of economies satisfying the assumptions listed in Theorem 2 is guaranteed), the condition that E be separable is actually crucial for core-Walras

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[20]Recall that a topological space Z is called *scattered* if every non-empty subset of Z has an isolated point (in the subspace topology). Actually, Semadeni (1971, 8.5.10(G), p. 150) does not speak of a separable space but of a space with countably many isolated points. But in a scattered space this latter property is equivalent to separability, because in such a space the set of isolated points is dense.

equivalence to hold. In fact, for such a space E, condition (i) in the statement of Theorem 2 is equivalent to separability:

Lemma 1. *Let E be a σ-Dedekind complete Banach lattice such that E_+ contains quasi-interior points and such that E^* contains strictly positive elements. Then the following are equivalent.*

(a) *E is separable.*

(b) (i) *in the statement of Theorem 2 holds for E.*

(That (a) implies (b) was mentioned already. The proof of (b)⇒(a) can be found in Section 4.4. It is only for this latter implication to be valid that the hypotheses about E stated in the preamble of the lemma have to be assumed; see Remark 9.)

Thus, combining Lemma 1 and Theorem 2, we have the following result.

Theorem 3. *Let E be a σ-Dedekind complete Banach lattice such that E_+ contains quasi-interior points and such that E^* contains strictly positive elements. Assume the continuum hypothesis. Then the following are equivalent.*

(i) *E is separable.*

(ii) *$C(\mathcal{E}) = \mathcal{W}(\mathcal{E})$ holds for every atomless economy $\mathcal{E}$ with commodity space E satisfying assumptions (A1) to (A6), (A8) and (A9).*

In view of Lemma 1 and the discussion of (i) of Theorem 2 given in the previous section, Theorem 3 can in particular be interpreted as saying that, in the class of spaces covered by this latter theorem, the property of a commodity space being separable and the notion of "many more agents than commodities" amount to the same criterion for core-Walras equivalence in the context of atomless economies for which assumptions (A1) to (A6) as well as (A8) and (A9) hold.

Some Banach lattices that satisfy the general hypotheses of Theorem 3, and some which do not, are listed in the final example.

Example 6. *Quasi-interior points, strictly positive linear functionals, and σ-Dedekind completeness.* All the concrete spaces that will be mentioned are understood as being endowed with their usual norms and orderings. With the exception of the space mentioned under (c) below, they are all σ-Dedekind complete, in fact Dedekind complete.

(a) If E is any *separable* Banach lattice then E_+ contains quasi-interior points and E^* contains strictly positive elements. (Indeed, if $\{x_n : n = 1, 2, \ldots\}$ is dense in $E_+ \setminus \{0\}$ then $\sum_{n=1}^{\infty} 2^{-n} \|x_n\|^{-n} x_n$ is a quasi-interior point of E_+ as may readily be verified. That the second property is satisfied by any separable Banach lattice was already noted in Remark 1 above.) Examples of separable Banach lattices are provided by the sequence spaces c_0 and ℓ_p, $1 \le p < \infty$. Other examples are

the Lebesgue spaces $L_p(\mu)$, $1 \leq p < \infty$, if the underlying measure space is σ-finite and separable. (That these spaces have the property that the positive cone contains quasi-interior points and the dual contains strictly positive elements can of course be seen directly.)

(b) Examples of *non-separable* Banach lattices E with E_+ containing quasi-interior points and E^* containing strictly positive elements are provided (i) by the Lebesgue spaces $L_p(\mu)$, $1 \leq p < \infty$, if the underlying measure space is non-separable and σ-finite; and (ii) by $L_\infty(\mu)$ if it is infinite dimensional and μ is σ-finite, and, as a special case, by ℓ_∞. (For the space $L_\infty(\mu)$, the positive cone does in fact have interior points, so the set of its quasi-interior points coincides with the set of its interior points.)

(c) An example of a Banach lattice that is not σ-Dedekind complete is provided by $C([0,1])$, the space of continuous functions on the unit interval. Note that this space is actually separable.

(d) Let $E = \ell_1(\Gamma)$. Then $E^* = \ell_\infty(\Gamma)$, so E^* contains strictly positive elements. However, E_+ does not possess quasi-interior points if Γ is uncountable.

(e) Let $E = \ell_\infty(\Gamma)$. Then the interior of E_+ is non-empty and is the same as the set of all quasi-interior points of E_+. However, see Example 2, E^* does not possess strictly positive elements if Γ is uncountable.

(e) Let $E = c_0(\Gamma)$. Then, if Γ is uncountable, E_+ does not contain quasi-interior points and $E^* \equiv \ell_1(\Gamma)$ does not contain strictly positive elements.

We close this section with a few remarks.

Remark 5. As with Theorem 1, the core-Walras equivalence part of Theorems 2 and 3 (i.e. implication (i)⇒(ii) of these theorems) holds without the continuum hypothesis, of course. This can be seen from the proof of Theorem 2.

Remark 6. Regarding Theorem 3, the convexity assumption (A5) may be removed from the statement of (ii) in that theorem. Indeed, as may be seen in the proof of (i)⇒(ii) in Theorem 2, this assumption is needed only when the positive cone of the commodity space does not possess quasi-interior points. On the other hand, implication (ii)⇒(i) of Theorems 2 and 3 yields a stronger result with Assumption (A5).

Remark 7. The core-Walras non-equivalence part of Theorems 2 and 3 has nothing to do with an occurrence of minimum wealth problems. In fact, under the assumptions listed in (ii) of those theorems, a non-trivial quasi-equilibrium of an economy must be a Walrasian equilibrium. (A quasi-equilibrium is *non-trivial* if there is a non-negligible set of consumers which can dispose some income.) Thus, under these assumptions, if a core allocation is not Walrasian then it is not the allocation of a non-trivial quasi-equilibrium either.

Remark 8. It may be seen that for a Banach lattice E whose positive cone has quasi-interior points, condition (i) in Theorem 2 is equivalent to the more easily readable condition "$(E, \sigma(E, A_q))$ is separable for every strictly positive $q \in E^*$" (regardless of whether or not E is σ-Dedekind complete).

Remark 9. For Lemma 1(b)$\Rightarrow$(a) to be valid, none of the assumptions about E made in the preamble of this lemma can be removed. Indeed, let (†) stand for "E_+ contains quasi-interior points," (‡) for "E is σ-Dedekind complete," and ($\star$) for "E^* contains strictly positive elements." (I) Let $E = \ell_1(\Gamma)$ where Γ is uncountable. Then ($\star$) and (‡) hold. (In fact, $\ell_1(\Gamma)$ is Dedekind complete.) Moreover, (b) holds (see Example 4). However, (a) is false. (II) Let $E = \ell_\infty(\Gamma)$ where Γ is uncountable. Then (†) and (‡) hold. (In fact, $\ell_\infty(\Gamma)$ is Dedekind complete and int $\ell_\infty(\Gamma)_+ \neq \varnothing$.) Moreover, (b) holds (vacuously, since $\ell_\infty(\Gamma)^*$ has no strictly positive elements; see Remark 3(e)). But of course, (a) is false. (III) Let E be the $C(\Omega)$ space of Example 5. Then (†), ($\star$), and (b) hold but (a) is false.

4 Proofs[21]

4.1 Preliminaries

The principal mathematical tool to prove the core-Walras non-equivalence part of Theorems 1 and 2 is provided by the following lemma.

Lemma 2. *Let X be a Banach space, let $V \subset X^*$, and let $q \in V$. Suppose that (V, weak^*) is compact but not first countable at q. Then, assuming the continuum hypothesis, and denoting by ω_1 the first uncountable ordinal number, there is a transfinite sequence $(q_\alpha)_{\alpha < \omega_1}$ of elements of V such that*

(a) *$q_\alpha \neq q$ for each ordinal $\alpha \in [0, \omega_1)$; but*

(b) *given any separable subset S of X, there is an ordinal $\alpha_S < \omega_1$ such that for each $\alpha \in [\alpha_S, \omega_1)$, $q_\alpha(x) = q(x)$ for all $x \in S$.[22]*

Proof. Since (V, weak^*) is compact but not first-countable at $q \in V$, and because the continuum hypothesis has been assumed to hold, it follows from Juhász and Szentmiklóssy (1992, Corollary 2.1 and the lines before the statement of that result) that there exists a transfinite sequence $(q_\alpha)_{\alpha < \omega_1}$ in V that converges to q with respect to the weak* topology, but such that $q_\alpha \neq q$ for each $\alpha \in [0, \omega_1)$. In particular, (a) of the proposition holds for such a sequence. To see that (b) holds as well, simply note that "$q_\alpha \to q$ in (V, weak^*) as $\alpha \uparrow \omega_1$" means that given

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[21]For the general facts about Banach lattices used in the following proofs see Aliprantis and Burkinshaw (1985) and Meyer-Nieberg (1991).

[22]If α, β are ordinal numbers then $[\alpha, \beta)$ denotes the ordinal interval $\{\gamma : \alpha \leq \gamma < \beta\}$.

any $x \in X$ and $n \in \mathbb{N}$ there is a $\beta < \omega_1$ such that $|q_\alpha(x) - q(x)| < 1/n$ for all $\alpha < \omega_1$ with $\alpha > \beta$, and recall that if A is any countable set of ordinals $\gamma < \omega_1$ then there is an ordinal $\overline{\gamma} < \omega_1$ such that for all $\gamma \in A$, $\gamma < \overline{\gamma}$. $\qquad\square$

For the next lemma recall that a Riesz dual system $\langle E, F \rangle$ is a linear pairing where E is a Riesz space, F is an ideal of the order dual of E separating the points of E, and such that the duality function $\langle \cdot, \cdot \rangle$ is the natural one, i.e. $\langle x, q \rangle = q(x)$ for all $x \in E$ and $q \in F$.

Lemma 3. *Let $\langle E, F \rangle$ be a Riesz dual system such that F has a strong unit, i.e. $F = A_\beta$ for some $\beta \in F_+$, and such that F is Dedekind complete. Then given any $e \in E_+$, there is a $q \in F_+$ such that $(A_e, \sigma(E, F))^* = A_q$ (i.e. such that the topological dual of A_e, when A_e is endowed with the (relativized) topology $\sigma(E, F)$, may be identified with A_q, the order ideal in F generated by q).*

Proof. Pick any $e \in E$ and set

$$Z = \{p \in F : p(x) = 0 \text{ for all } x \in A_e\}.$$

Then Z is a band in F, and since F is Dedekind complete by hypothesis, Z is in fact a projection band.[23] Thus we may write

$$F = Z \oplus Z^d$$

where Z^d is the disjoint complement of Z in F, i.e.

$$Z^d = \{p' \in F : p' \perp p \text{ for all } p \in Z\}.$$

Let π_1 denote the band projection of F onto Z and π_2 the band projection of F onto Z^d. Then for any $x \in A_e$ and any $p \in F$,

$$p(x) = \pi_1(p)(x) + \pi_2(p)(x) = \pi_2(p)(x),$$

showing that the relativization of the weak topology $\sigma(E, F)$ to A_e coincides with the weak topology $\sigma(A_e, Z^d)$. But A_e separates the points of Z^d by construction, and therefore we do in fact have $(A_e, \sigma(E, F))^* \equiv (A_e, \sigma(A_e, Z^d))^* = Z^d$.

Thus it remains to show that $Z^d = A_q$ for some $q \in F_+$. By hypothesis, we can select a $\beta \in F_+$ so that $F = A_\beta$. Then $\pi_2(\beta) \geq 0$ and $\pi_2(F) \equiv \pi_2(A_\beta) \subset A_{\pi_2(\beta)}$, because π_2 is linear and positive. Consequently $Z^d \subset A_{\pi_2(\beta)}$. On the other hand, because $\pi_2(\beta) \in Z^d$ and Z^d is an ideal in F, we must also have $A_{\pi_2(\beta)} \subset Z^d$. We conclude that $Z^d = A_{\pi_2(\beta)}$. This completes the proof of the lemma. $\qquad\square$

The following lemma was invoked in Example 5 of Section 3.2.2 and will also be used in the proof of Theorem 2.

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[23]See Aliprantis and Burkinshaw (1985, pp. 170–171, and Theorem 3.8, p. 33).

Lemma 4. *Let E be a Banach lattice which does not satisfy* (i) *of the statement of Theorem 2. Then there are an $e \in E_+$ and a $q \in E_+^*$ such that both* (a) *$([-q, q], \text{weak}^*)$ is not first countable at 0 and* (b) *A_e separates the points of A_q.*

Proof. By hypothesis there are an e in E_+ and a strictly positive β in E^* such that the space $(A_e, \sigma(E, A_\beta))$ is non-separable. Observe that β being strictly positive means that A_β separates the points of E and recall that for a Banach lattice the (topological) dual and the order dual coincide. Thus $\langle E, A_\beta \rangle$ is a Riesz dual system. We may therefore apply the previous lemma to find a positive $q \in A_\beta$ so that $(A_e, \sigma(E, A_\beta))^*$ may be identified with A_q. (Note that since A_β is an ideal in E^*, A_q regarded as an ideal in A_β is the same object as A_q regarded as an ideal in E^*; in particular, $[-q, q]$ regarded as an order interval in A_β is the same object as $[-q, q]$ regarded as an order interval in E^*.)

Now since $(A_e, \sigma(E, A_\beta))^* = A_q$, A_e separates the points of A_q. However, because $(A_e, \sigma(E, A_\beta))$ is non-separable, there is no *countable* subset of A_e that separates the points of A_q, and in particular, there is no countable subset of A_e that separates the point 0 from the other points of the order interval $[-q, q]$. Thus $[-q, q]$ is not first countable at 0 for the topology $\sigma(A_q, A_e)$.

Note that the topology $\sigma(A_q, A_e)$ is Hausdorff since A_e separates the points of A_q. Moreover, on A_q, this topology is weaker than the topology $\sigma(E^*, E)$. Recall also that order intervals in the dual of a Banach lattice are weak* compact. Thus $[-q, q]$ is $\sigma(E^*, E)$ compact and it follows that the topologies $\sigma(A_q, A_e)$ and $\sigma(E^*, E)$ agree on $[-q, q]$. Consequently $[-q, q]$ is not first countable at 0 for the weak* topology $\sigma(E^*, E)$, as required. The proof of the lemma is thus complete. $\qquad\square$

Corollary. *Let E be a Banach lattice with property CD. Then* (i) *of Theorem 2 holds for E.*

Proof. We will prove the contrapositive. Thus suppose (i) of Theorem 2 does not hold. Then by Lemma 4 we can select a $q \in E_+^*$ such that $([-q, q], \text{weak}^*)$ is not first countable at 0. But because order intervals in the dual of a Banach lattice are weak* compact, the fact that $([-q, q], \text{weak}^*)$ is not first countable at 0 implies that no countable subset of E can separate 0 from the other points of the order interval $[-q, q]$ (by the definition of the weak* topology). Consequently, no countable subset of E can separate q from the other points of the order interval $[0, 2q]$, whence E does not satisfy property CD. $\qquad\square$

The final two lemmata will be used in the proof of the core-Walras equivalence part of Theorem 2. They are needed because in that theorem it is not assumed that the positive cone of the commodity space possesses quasi-interior points.

Lemma 5. *Let E be a Banach lattice, let $(T, \mathcal{T}, \nu)$ be a finite positive measure space, and let $g \colon T \to E$ be a (Bochner) integrable function such that $g(t) \in E_+$ for almost all $t \in T$. Then, setting $\overline{x} = \int_T g(t)\, d\nu(t)$, we have $g(t) \in \mathrm{c}\ell\, A_{\overline{x}}$ for almost all $t \in T$ (i.e., for almost all $t \in T$, $g(t)$ belongs to the (norm) closure of the order ideal in E generated by $\overline{x}$).*

Proof. By hypothesis, for any $q \in E_+^*$ we have $\langle q, g(t) \rangle \geq 0$ for almost all $t \in T$. Hence, for any $q \in E_+^*$ and each $S \in \mathcal{T}$,

$$0 \leq \int_S \langle q, g(t) \rangle\, d\nu(t) \leq \int_T \langle q, g(t) \rangle\, d\nu(t)$$

i.e.

$$0 \leq \left\langle q, \int_S g(t)\, d\nu(t) \right\rangle \leq \left\langle q, \int_T g(t)\, d\nu(t) \right\rangle.$$

Consequently, $0 \leq \int_S g(t)\, d\nu(t) \leq \int_T g(t)\, d\nu(t)$. Thus $\int_S g(t)\, d\nu(t)$ belongs to $A_{\overline{x}}$ for each $S \in \mathcal{T}$. Set $F = \mathrm{c}\ell\, A_{\overline{x}}$.

Consider the quotient E/F, endowed with the quotient norm; as F is closed in E, E/F is a Banach space. Let $\pi \colon E \to E/F$ denote the projection. π is a bounded linear operator, so the composition $\pi \circ g$ is Bochner integrable since g is. In particular, $\int_S \pi(g(t))\, d\nu(t) = \pi \left(\int_S g(t)\, d\nu(t) \right)$ for each $S \in \mathcal{T}$. Since $\int_S g(t)\, d\nu(t) \in F$ for each $S \in \mathcal{T}$, it follows that $\int_S \pi(g(t))\, d\nu(t) = 0$ for each $S \in \mathcal{T}$. According to a standard fact, this means that $\pi(g(t)) = 0$ for almost all $t \in T$, whence $g(t) \in F$ for almost all $t \in T$. $\qquad\square$

Lemma 6. *Let E be a (real) Banach space and let U and V be convex subsets of E with V open and such that $U \cap V \neq \emptyset$. Let $z \in U \cap \mathrm{c}\ell\, V$, let $q \in E^*$ and suppose $qz \leq qz'$ for each $z' \in U \cap V$. Then there exist elements q_1 and q_2 of E^* such that $q_1 z \leq q_1 u$ for all $u \in U$, $q_2 z \leq q_2 v$ for all $v \in V$, and $q_1 + q_2 = q$.*

For a proof see Podczeck (1996, Lemma 2).

4.2 Proof of Theorem 1

(i)$\Rightarrow$(ii): Let $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ be an atomless economy with commodity space E such that assumptions (A1) to (A7) are satisfied.[24] Recall that according to (A1), consumption sets are equal to E_+. In what follows, this fact will be used without explicit invocation.

Clearly, $\mathcal{W}(\mathcal{E}) \subset C(\mathcal{E})$. To verify the reverse inclusion, suppose $f \in C(\mathcal{E})$. Since $\mathrm{int}\, E_+ \neq \emptyset$ by the general hypotheses about E, and since $\mathcal{E}$ is atomless, a standard argument using (A1), (A2), (A4) and (A6) provides a $p \in E_+^* \setminus \{0\}$ such that:

(1) For any $x \in E_+$, $\{t \in T \colon x \succ_t f(t) \text{ and } p(x) < p(e(t))\}$ is a null set in T.

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[24]Throughout this proof, whenever necessary for the sake of an argument, it is assumed that E is non-zero dimensional, i.e. that $E \neq \{0\}$.

(Cf. the proof of Theorem 1 in Rustichini and Yannelis (1991). The argument is presented below for sake of completeness and for later reference.)

Now f, being integrable, is almost separably valued, and according to (i) of the theorem, p is countably determined. Thus we can select a closed separable subspace F of E such that for almost all $t \in T$, $f(t) \in F$ and such that whenever $q \in E_+^*$ and $q(x) = p(x)$ for all $x \in F$, then $q = p$. Observe that

$$(2) \qquad\qquad F \cap \operatorname{int} E_+ \neq \varnothing.$$

Indeed, as $f(t) \in F$ for almost all $t \in T$ and f is a feasible allocation—i.e. $\int_T f(t)\, dv(t) = \int_T e(t)\, dv(t)$—we have $\int_T e(t)\, dv(t) \in F$. On the other hand, according to (A7), $e(t) \in \operatorname{int} E_+$ for all $t \in T$, and so $\int_T e(t)\, dv(t) \in \operatorname{int} E_+$, too. (To see that "$e(t) \in \operatorname{int} E_+$ for all $t \in T$" implies "$\int_T e(t)\, dv(t) \in \operatorname{int} E_+$," note that a point $z \in E$ belongs to $\operatorname{int} E_+$ if and only if $q(z) > 0$ for all $q \in E_+^* \setminus \{0\}$, by the separation theorem, and that $\langle q, \int_T e(t)\, dv(t)\rangle = \int_T \langle q, e(t)\rangle\, dv(t)$ for all $q \in E^*$. To see that "$f(t) \in F$ for almost all $t \in T$" implies "$\int_T f(t)\, dv(t) \in F$," note that if a point $z \in E$ does not belong to the closed linear subspace F of E, then for some $q \in E^*$, $q(z) > 0 = q(y)$ for all $y \in F$ whence $z \neq \int_T f(t)\, dv(t)$ because $\langle q, \int_T f(t)\, dv(t)\rangle = \int_T \langle q, f(t)\rangle\, dv(t)$.)

Since F is separable, so is $F_+ \equiv E_+ \cap F$; let D be a countable dense subset of F_+. From (1) it can be seen that, for almost all $t \in T$, if $d \in D$ satisfies $d \succ_t f(t)$ then $p(d) \geq p(e(t))$. By the continuity of preferences this implies:

$$(3) \qquad \text{For almost all } t \in T, \text{ if } x \in F_+ \text{ satisfies } x \succ_t f(t) \text{ then } p(x) \geq p(e(t)).$$

By choice of F (and Assumption (A1)), for almost all $t \in T$ we have $f(t) \in F_+$ and thus, given any $v \in F_+$, $f(t) + v \in F_+$ as well. Hence, by strict monotonicity of preferences and (3), $\langle p, f(t)\rangle \geq \langle p, e(t)\rangle$ for almost all $t \in T$ (note: $F_+ \neq \{0\}$ because of (2)), whence, by feasibility of f,

$$(4) \qquad\qquad \langle p, f(t)\rangle = \langle p, e(t)\rangle \text{ for almost all } t \in T.$$

Summing up, for some $T_1 \in \mathcal{T}$ with $v(T \setminus T_1) = 0$, the following holds:

$$(5) \qquad \begin{aligned} &\text{For every } t \in T_1, \text{ (a) } f(t) \in F, \text{ (b) } \langle p, f(t)\rangle = \langle p, e(t)\rangle, \\ &\qquad\qquad \text{(c) if } x \in F_+ \text{ satisfies } x \succ_t f(t) \text{ then } p(x) \geq p(e(t)). \end{aligned}$$

Pick any $t \in T_1$, set $B = \{x \in E_+ : x \succ_t f(t)\}$, and let Γ be the cone generated by $B - \{f(t)\}$. By assumptions (A5) and (A2), B is convex and hence so is Γ. Furthermore, by strict monotonicity of preferences, $\{f(t)\} + (E_+ \setminus \{0\}) \subset B$ whence $E_+ \subset \Gamma$. In particular, $\operatorname{int} \Gamma \neq \varnothing$, and in fact, from (2), $(\operatorname{int} \Gamma) \cap F \neq \varnothing$. Finally, from (5), if $y \in \Gamma \cap F$ then $p(y) \geq 0$. Using the Krein-Rutman theorem, it follows that there is a $\tilde{p} \in E^*$ such that (i) $\tilde{p}(y) \geq 0$ for all $y \in \Gamma$ but (ii) $\tilde{p}(x) = p(x)$ for all $x \in F$. From (i), $\tilde{p}$ actually belongs to E_+^* since $E_+ \subset \Gamma$,

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whence, from (ii) and the choice of F, $\tilde{p} = p$. Hence from (i) again, $p(y) \geq 0$ for all $y \in \Gamma$, and by the choice of Γ and (b) of (5) we conclude that for *any* $x \in E_+$, if $x \succ_t f(t)$ then $p(x) \geq p(e(t))$. As $t \in T_1$ was arbitrary (the cone Γ depending on t of course), and since $\nu(T \setminus T_1) = 0$, we have shown:

(6)
> For almost every $t \in T$,
> if $x \in E_+$ satisfies $x \succ_t f(t)$ then $\langle p, x \rangle \geq \langle p, e(t) \rangle$.

But since $p \in E_+^* \setminus \{0\}$, and since for all $t \in T$, $e(t) \in \text{int } E_+$ (Assumption (A7)) and $\succ_t$ is continuous (Assumption (A3)), (6) implies (by a standard argument):

(7)
> For almost every $t \in T$,
> if $x \in E_+$ satisfies $x \succ_t f(t)$ then $\langle p, x \rangle > \langle p, e(t) \rangle$.

In view of (4) and (7) we may conclude that the pair (p, f) is a Walrasian equilibrium, as was to be shown.

Finally, we will give the argument leading to a $p \in E_+^* \setminus \{0\}$ such that (1) holds. In this regard, let $\varphi: T \to 2^E$ be the correspondence given by

$$\varphi(t) = \{x \in E_+ : x \succ_t f(t)\} \cup \{e(t)\}, \ t \in T.$$

Then $\int_T \varphi(t) \, d\nu(t)$ is non-empty—e.g. $\int_T e(t) \, d\nu(t)$ belongs to this set—and $\text{c}\ell \int_T \varphi(t) \, d\nu(t)$ is convex because the measure space $(T, \mathcal{T}, \nu)$ is atomless (see e.g. Yannelis, 1991, Theorem 6.2, p. 22). Moreover, because $f \in C(\mathcal{E})$,

(8)
$$\left(\text{c}\ell \int_T \varphi(t) \, d\nu(t) - \left\{ \int_T e(t) \, d\nu(t) \right\} \right) \cap \text{int}(-E_+) = \varnothing.$$

Indeed, suppose the contrary. Then, since $\text{int}(-E_+)$ is an open set, there are a $\nu \in \text{int}(-E_+)$—in particular $\nu \neq 0$—and an integrable function $g: T \to E_+$ such that $g(t) \in \varphi(t)$ for almost all $t \in T$ and

$$\int_T g(t) \, d\nu(t) = \int_T e(t) \, d\nu(t) + \nu.$$

Set $S = \{t \in T : g(t) \succ_t f(t)\}$. By Assumption (A6), $S \in \mathcal{T}$. By definition of φ we have $g(t) = e(t)$ for almost all $t \in T \setminus S$ and hence

$$\int_S g(t) \, d\nu(t) = \int_S e(t) \, d\nu(t) + \nu.$$

In particular, therefore, we must have $\nu(S) > 0$ since $\nu \neq 0$. Note that $-\nu \geq 0$. Let $\tilde{g}: T \to E_+$ be given by

$$\tilde{g}(t) = g(t) - \frac{1}{\nu(S)} \nu.$$

Then $\int_S \tilde{g}(t) \, d\nu(t) = \int_S e(t) \, d\nu(t)$; that is, the allocation $\tilde{g}$ is feasible for the coalition S. Moreover, for almost all $t \in S$, $\tilde{g}(t) \succ_t g(t)$ because preferences are

strictly monotone, whence $\tilde{g}(t) \succ_t f(t)$ by transitivity. Thus f is not in $C(\mathcal{E})$, a contradiction establishing (8).

Since $\text{int}(-E_+) \neq \varnothing$ by hypothesis (and since $-E_+$ is convex), it follows now from the separation theorem that there is a $p \in E_+^* \setminus \{0\}$ such that

$$\inf \left\langle p, c\ell \int_T \varphi(t)\, dv(t) - \left\{\int_T e(t)\, dv(t)\right\} \right\rangle \geq 0$$

i.e. such that

$$(9) \qquad \inf \left\langle p, \int_T \varphi(t)\, dv(t) \right\rangle \geq \int_T \langle p, e(t) \rangle\, dv(t).$$

But (9) implies that (1) holds for p. Indeed, pick any $x \in E_+$ and let $g \colon T \to E_+$ be given by

$$g(t) = \begin{cases} x & \text{if } x \succ_t f(t) \text{ and } p(x) < p(e(t)) \\ e(t) & \text{otherwise.} \end{cases}$$

From (A6), the set $\{t \in T \colon x \succ_t f(t)\}$ is in $\mathcal{T}$, and because the mapping $t \mapsto e(t)$, being integrable, is weakly measurable, so is the set $\{t \in T \colon p(x) < p(e(t))\}$. It follows that g is integrable and that $\int_T g(t)\, dv(t) \in \int_T \varphi(t)\, dv(t)$. Thus (9) implies (1). This completes the proof of (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(i):[25] We will prove the contrapositive. Thus suppose (i) is not true and select a $\hat{q} \in E_+^*$ that is not countably determined. (Observe that $\hat{q} \neq 0$ since $\text{int}\, E_+ \neq \varnothing$ by hypothesis.) Fix any interior point $\bar{e}$ of E_+ and let

$$V = \{q \in E_+^* \colon q(\bar{e}) = \hat{q}(\bar{e})\}.$$

Evidently, $\hat{q}$ being not countably determined implies that (V, weak^*) is not first countable at $\hat{q}$, and since $\bar{e} \in \text{int}\, E_+$, (V, weak^*) is compact. Therefore, by Lemma 2 in Section 4.1, and since the continuum hypothesis is supposed to be in force, we can select a transfinite sequence $(q_\alpha)_{\alpha < \omega_1}$ of elements of V (denoting by ω_1 the first uncountable ordinal number) such that $q_\alpha \neq \hat{q}$ for each ordinal $\alpha \in [0, \omega_1)$ but such that

$$(10) \qquad \begin{aligned} &\text{if } S \text{ is any separable subset of } E \text{ then there is an ordinal } \alpha_S < \omega_1 \\ &\text{such that for each } \alpha \in [\alpha_S, \omega_1), q_\alpha(x) = \hat{q}(x) \text{ for all } x \in S. \end{aligned}$$

Let $(T, \mathcal{T}, v)$ be any complete finite positive atomless measure space. Using the family $(q_\alpha)_{\alpha < \omega_1}$ just obtained, we will now construct an economy $\mathcal{E}$ with $(T, \mathcal{T}, v)$ as measure space of agents and E as commodity space such that $C(\mathcal{E}) \not\subseteq \mathcal{W}(\mathcal{E})$ but such that all the assumptions listed in (ii) of the statement of Theorem 1 hold. Concerning consumption sets and endowments, for each

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[25]This part of the proof has some overlap with the proof of Theorem 1 of Podczeck (2001). However, for sake of completeness we give the whole argument.

agent t in T we let the endowment $e(t)$ be equal to $\bar{e}$ ($\bar{e}$ being the point of int E_+ chosen above) and the consumption set $X(t)$ be equal to E_+. Then assumptions (A1) and (A7) are met. Further, since the measure ν is finite, the endowment mapping $t \mapsto \bar{e}$ is integrable, as required in our definition of an economy.

Since the measure space $(T, \mathcal{T}, \nu)$ is atomless, and since the continuum hypothesis holds, we may use Proposition 5.2 of Tourky and Yannelis (2001) to write $T = \bigcup_{\alpha < \omega_1} N_\alpha$ where $(N_\alpha)_{\alpha < \omega_1}$ is a family of pairwise disjoint null sets in T. Denote by $\phi \colon T \to [0, \omega_1)$ the mapping that takes a $t \in T$ to that ordinal number α for which $t \in N_\alpha$.

For each $t \in T$ set $q_t = q_{\phi(t)}$. Then

(11) $$q_t \neq \hat{q} \text{ for each } t \in T$$

since $q_\alpha \neq \hat{q}$ for each $\alpha < \omega_1$, and from (10):

(12)   For any given separable subset S of E the set
$\{t \in T \colon q_t(s) \neq \hat{q}(s) \text{ for some } s \in S\}$ is a null set in T,

because for each ordinal number $\alpha < \omega_1$ we have $\phi^{-1}([0, \alpha)) = \bigcup_{\alpha' < \alpha} N_{\alpha'}$, each $N_{\alpha'}$ is a null set, and for each $\alpha < \omega_1$ the set $[0, \alpha)$ is countable.

Finally, recall that E^* contains strictly positive elements by hypothesis. We may therefore assume that

(13) $$q_t \text{ is strictly positive for each } t \in T$$

by adding, if necessary, a common strictly positive element of E^* to each q_t *and* to $\hat{q}$ (recalling that each q_t is positive by construction and noting that the sum of two elements of E^*, one of them being strictly positive and the other positive, is strictly positive).

Now, for each $t \in T$, let a utility function $u_t \colon E_+ \to \mathbb{R}$ be defined by

$$u_t(x) = q_t(x), \ x \in E_+.$$

The family of preferences so defined satisfies all the assumptions from (A2) to (A6). Indeed, this is clear for (A2), (A3), and (A5), and because of (13), this is equally clear for (A4). As for (A6), since allocations are Bochner integrable by definition, they are in particular essentially separably valued. Thus from (12):

(14)   If $h \colon T \to E_+$ is an allocation then
for almost all $t \in T$, $u_t(h(t)) = \langle \hat{q}, h(t) \rangle$.

Consequently, since a Bochner integrable function is weakly measurable, and since the measure space $(T, \mathcal{T}, \nu)$ is complete, given any allocation $h \colon T \to E_+$ the mapping $t \mapsto u_t(h(t))$ is measurable. It is plain that this implies (A6).

We have thus constructed an atomless economy $\mathcal{E}$ with commodity space E such that the assumptions listed in (ii) of the statement of Theorem 1 all hold.

We claim that the initial allocation $t \mapsto \bar{e}$ belongs to $C(\mathcal{E})$. To see this, fix any coalition $C \in \mathcal{T}$ with $\nu(C) > 0$ and let $f \colon T \to E_+$ be an allocation which is feasible for C. Thus, $\int_C f(t)\,d\nu(t) = \nu(C)\bar{e}$. In particular, $\langle \hat{q}, \int_C f(t)\,d\nu(t) \rangle = \langle \hat{q}, \nu(C)\bar{e} \rangle$ whence $\int_C \langle \hat{q}, f(t) \rangle\,d\nu(t) = \nu(C)\langle \hat{q}, \bar{e} \rangle$. But therefore, in view of (14), $u_t(f(t)) > u_t(\bar{e})$ cannot hold for almost all $t \in C$, and it follows that the initial allocation $t \mapsto \bar{e}$ indeed belongs to $C(\mathcal{E})$ as claimed.

However, the initial allocation $t \mapsto \bar{e}$ is not Walrasian. To see this, assume to the contrary that there is a $p \in E^*$ such that the pair $(p, t \mapsto \bar{e})$ is a Walrasian equilibrium. Now $\bar{e}$ is in the interior of the consumption set E_+, so the equilibrium conditions for the pair $(p, t \mapsto \bar{e})$ imply that for almost all $t \in T$, $\ker p \subset \ker q_t$; that is, by a standard fact from linear algebra,

(15)   for almost every $t \in T$, $q_t = \lambda_t p$ for some real number λ_t.

According to (12), however,

(16)   for any $z \in E$, $q_t(z) = \hat{q}(z)$ for almost all $t \in T$,

so (15) means in fact that $q_t = \lambda p$ for some real number λ and almost all $t \in T$. Using (16) once more, it follows that $q_t = \hat{q}$ for almost all $t \in T$. But this is impossible because of (11), and we conclude that the initial allocation $t \mapsto \bar{e}$ is not Walrasian. Thus "not (ii)" has been established and the proof of Theorem 1 is complete.

4.3   Proof of Theorem 2

(i)$\Rightarrow$(ii): Let $\mathcal{E} = [(T, \mathcal{T}, \nu), (X(t), \succcurlyeq_t, e(t))_{t \in T}]$ be an atomless economy with commodity space E such that assumptions (A1) to (A6) as well as (A8) and (A9) are satisfied. Clearly, $\mathcal{W}(\mathcal{E}) \subset C(\mathcal{E})$. To prove the reverse inclusion, note first that according to Assumption (A1), the consumption sets are all equal to E_+; in particular, if $h \colon T \to E$ is an allocation then $h(t) \in E_+$ for each $t \in T$. Further, according to Assumption (A8), the endowment $e(t)$ belongs to E_+ for each $t \in T$. In the sequel these facts will be used frequently without explicit reference.

Now let α and β be strictly positive elements of E^*, chosen according to Assumption (A9); in particular, $\alpha \le \beta$. Denote by Γ the cone

$$\Gamma = \{x \in E \colon \alpha(x^+) > \beta(x^-)\}.$$

Note the following facts about Γ. First, by choice of α and β:

(17)   For every $t \in T$, if $z \in E_+$ and $y \in \Gamma$ satisfy $z + y \ge 0$ then $z + y \succ_t z$

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(because for $z \in E_+$, $z + y \geq 0$ implies $y^- \leq z$). Second, Γ contains $E_+ \setminus \{0\}$, obviously. Furthermore, Γ is (norm) open by virtue of the continuity of the lattice operations. Finally, Γ is convex. To see this, note that if $x, y \in E$ then for some $b \in E_+$, $(x + y)^+ = x^+ + y^+ - b$ as well as $(x + y)^- = x^- + y^- - b$. Thus whenever $x, y \in \Gamma$ then $\alpha((x + y)^+) > \beta((x + y)^-)$, because $\alpha \leq \beta$ and hence $\alpha(b) \leq \beta(b)$ for $b \geq 0$.

Consider A_β, the order ideal in E^* generated by β. Since for a Banach lattice the topological dual coincides with the order dual, besides the norm topology on E we can also consider the absolute weak topology $|\sigma|(E, A_\beta)$.[26] Observe that β being strictly positive means that A_β separates the points of E. Hence, according to a standard fact, the topological dual of $(E, |\sigma|(E, A_\beta))$ is A_β.[27]

According to another standard fact, the lattice operations on E are continuous for the topology $|\sigma|(E, A_\beta)$. In particular, therefore, the cone Γ is also $|\sigma|(E, A_\beta)$-open, because $\alpha \leq \beta$ and thus α (as well as β) belongs to A_β, the topological dual of $(E, |\sigma|(E, A_\beta))$.

Now suppose f is a core allocation for $\mathcal{E}$. We claim:

(∗)   There is no pair (S, s) where $S \in \mathcal{T}$ with $\nu(S) > 0$ and $s: T \to E_+$ is a ($\mathcal{T}$-measurable) simple function such that $s(t) \succ_t f(t)$ for almost all $t \in S$ but $\int_S s(t)\, d\nu(t) - \int_S e(t)\, d\nu(t) \in -\Gamma$.

To see this,[28] suppose to the contrary that there are points $x_1, \ldots, x_n$ in E_+ and elements $S_1, \ldots, S_n$ of $\mathcal{T}$, with $S_i \cap S_j = \varnothing$ for $i \neq j$, such that for all $i = 1, \ldots, n$, $\nu(S_i) > 0$ and $x_i \succ_t f(t)$ for almost all $t \in S_i$, and such that

$$\sum_{i=1}^{n} \nu(S_i) x_i - \int_S e(t)\, d\nu(t) = -y$$

where $y \in \Gamma$ and $S = \bigcup_{i=1}^{n} S_i$. Note that $y \neq 0$ by the definition of Γ.

Suppose first that $y \geq 0$. For each $t \in S_i$ set $y(t) = x_i + \frac{1}{\nu(S)} y$, $i = 1, \ldots, n$. Then $y(t) \succ_t x_i$ for each $t \in S_i$ since preferences are strictly monotone (Assumption (A4)), whence $y(t) \succ_t f(t)$ for almost all $t \in S_i$, $i = 1, \ldots, n$, by transitivity of preferences. On the other hand,

$$\int_S y(t)\, d\nu(t) = \sum_{i=1}^{n} \nu(S_i) x_i + y = \int_S e(t)\, d\nu(t),$$

and we have thus got a contradiction to the property of f being a core allocation; thus $y \geq 0$ cannot hold.

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[26]See Aliprantis and Burkinshaw (1985, p. 166) for the notion of the absolute weak topology as well as for the facts about this topology which are invoked in the following.

[27]Recall from Section 2 that (E, τ) means E with the topology τ instead of the norm topology.

[28]The argument given in the sequel to establish this claim is taken from Zame (1986, p. 10-15).

Thus suppose $\gamma^- \neq 0$. Set $\lambda_i = \nu(S_i)$, $i = 1, \ldots, n$. (Thus $\lambda_i > 0$ for all i.) Since $\sum_{i=1}^{n} \lambda_i x_i$ and $\int_S e(t) \, d\nu(t)$ are positive elements of E (and since $-\gamma = \gamma^- - \gamma^+$ and $\gamma^- \wedge \gamma^+ = 0$) we must have $\gamma^- \leq \sum_{i=1}^{n} \lambda_i x_i$, so the Riesz decomposition theorem asserts the existence of elements $u_1, \ldots, u_n$ in E_+ with $\sum_{i=1}^{n} \lambda_i u_i = \gamma^-$ and $u_i \leq x_i$, $i = 1, \ldots, n$. Set

$$v_i = \frac{\beta(u_i)}{\beta(\gamma^-)} \gamma^+, \ i = 1, \ldots, n.$$

This is well defined because γ^- is supposed to be $\neq 0$ and β is strictly positive; in particular, $v_i \geq 0$ for each i. Moreover, since $\alpha(\gamma^+) > \beta(\gamma^-)$ by definition of Γ,

$$\alpha(v_i) = \frac{\beta(u_i)}{\beta(\gamma^-)} \alpha(\gamma^+) \geq \beta(u_i), \ i = 1, \ldots, n,$$

with strict inequality if $u_i \neq 0$. Hence, by choice of α and β, and because $u_i \leq x_i$ and $v_i \geq 0$, we have $x_i - u_i + v_i \succcurlyeq_t x_i$ for all t and i (in fact, $x_i - u_i + v_i \succ_t x_i$ in case $u_i \neq 0$), and therefore, by transitivity of preferences, $x_i - u_i + v_i \succ_t f(t)$ for almost all $t \in S_i$, $i = 1, \ldots, n$. Also

$$\sum_{i=1}^{n} \lambda_i(x_i - u_i + v_i) = \sum_{i=1}^{n} \lambda_i x_i - \gamma^- + \frac{\beta(\sum_{i=1}^{n} \lambda_i u_i)}{\beta(\gamma^-)} \gamma^+$$

$$= \sum_{i=1}^{n} \lambda_i x_i - \gamma^- + \gamma^+$$

$$= \int_S e(t) \, d\nu(t).$$

Consequently, if we set $y_t = x_i - u_i + v_i$ for $t \in S_i$, $i = 1, \ldots, n$, we have $y(t) \succ_t f(t)$ for almost all $t \in S$ ($\equiv \bigcup_{i=1}^{n} S_i$) but $\int_S y(t) \, d\nu(t) = \int_S e(t) \, d\nu(t)$, thus again getting a contradiction to the property of f being a core allocation. Thus $(*)$ holds.

We shall now deduce that the term "simple function" in the statement of $(*)$ can be replaced by "allocation" (i.e. by "integrable function"). Arguing by contradiction, suppose for some $S \in \mathcal{T}$ with $\nu(S) > 0$ and some allocation $g \colon T \to E_+$ we have $g(t) \succ_t f(t)$ for almost all $t \in S$ but $\int_S g(t) \, d\nu(t) - \int_S e(t) \, d\nu(t) \in -\Gamma$. By definition of Bochner integrability, we can select a sequence (s_n) of simple functions from T into E such that $s_n(t) \to g(t)$ in the norm $\|\cdot\|$ of E for almost all $t \in T$ and $\int_T \|g(t) - s_n(t)\| \, d\nu(t) \to 0$. For each n let $g_n \colon T \to E_+$ be given by $g_n(t) = s_n(t) \vee 0$, $t \in T$. Then each g_n is also a simple function, and by virtue of the continuity of the lattice operations we have $g_n(t) \to g(t)$ for almost all $t \in T$. Moreover, $\|g_n(t)\| \leq \|s_n(t)\|$ for all n and t (since $\|\cdot\|$ is a lattice norm). For each n set

$$S_n = \{t \in S \colon g_m(t) \succ_t f(t) \text{ for all } m \geq n\}$$

and note that $S_n \in \mathcal{T}$ by Assumption (A6). Evidently $S_n \subset S_{n+1}$ for all n, and by continuity of preferences, for some null set N in S we have $S \setminus N = \bigcup_{n=1}^{\infty} S_n$. Consequently $(1_{S_n} \cdot g_n)(t) \to (1_S \cdot g)(t)$ for almost all $t \in T$, and since

$$\| (1_{S_n} \cdot g_n)(t) \| \leq \| (1_S \cdot g_n)(t) \| \leq \| s_n(t) \|$$

for all $t \in T$ (and $\int_T \| g(t) - s_n(t) \| \, dv(t) \to 0$), an appeal to Vitali's convergence theorem shows that $\int_{S_n} g_n(t) \, dv(t) \to \int_S g(t) \, dv(t)$.

Noting that $\int_{S_n} e(t) \, dv(t) \to \int_S e(t) \, dv(t)$, and recalling from above that the cone Γ is open, we may conclude that for some n, $v(S_n) > 0$ and $g_n(t) \succ_t f(t)$ for almost all $t \in S_n$ but $\int_{S_n} g_n(t) \, dv(t) - \int_{S_n} e(t) \, dv(t) \in -\Gamma$. But g_n is a simple function and we thus get a contradiction to (∗). Thus we have shown:

> (∗∗) There is no pair (S, g) where $S \in \mathcal{T}$ with $v(S) > 0$ and $g \colon T \to E_+$ is an allocation (i.e. integrable function) such that $g(t) \succ_t f(t)$ for almost all $t \in S$ but $\int_S g(t) \, dv(t) - \int_S e(t) \, dv(t) \in -\Gamma$.

Consider now the correspondence $\varphi \colon T \to 2^E$ given by

$$\varphi(t) = \{ x \in E_+ : x \succ_t f(t) \} \cup \{ e(t) \}, \ t \in T.$$

As in the last part of (i)⇒(ii) in the proof of Theorem 1, we have that $\int_T \varphi(t) \, dv(t)$ is non-empty and that $c\ell \int_T \varphi(t) \, dv(t)$ is a convex set in E. Recall that according to Assumption (A6), given any allocation g the set $\{ t \in T : g(t) \succ_t f(t) \}$ belongs to $\mathcal{T}$. Hence from (∗∗),

$$\left(\int_T \varphi(t) \, dv(t) - \left\{ \int_T e(t) \, dv(t) \right\} \right) \cap -\Gamma = \varnothing$$

(because $0 \notin \Gamma$), and thus in fact

$$\left(c\ell \int_T \varphi(t) \, dv(t) - \left\{ \int_T e(t) \, dv(t) \right\} \right) \cap -\Gamma = \varnothing$$

because Γ is open. Since the cone Γ is convex, too, we can now appeal to the separation theorem to find an element p in $E^* \setminus \{0\}$ [29] such that $\langle p, \Gamma \rangle \geq 0$ and such that—again see the last part of (i)⇒(ii) in the proof of Theorem 1:

(18) For any $x \in E_+$, $\{ t \in T : x \succ_t f(t) \text{ and } p(x) < p(e(t)) \}$ is a null set in T.

Observe that since $p \neq 0$ and Γ is open, $\langle p, \Gamma \rangle \geq 0$ means in fact $\langle p, \Gamma \rangle > 0$ whence p is strictly positive, since Γ contains $E_+ \setminus \{0\}$ as noted above. Recall also that Γ is actually open for the topology $|\sigma|(E, A_\beta)$. Hence $\langle p, \Gamma \rangle \geq 0$ means, in particular, that p is bounded on some $|\sigma|(E, A_\beta)$-neighborhood of 0; that is, p is actually continuous for the topology $|\sigma|(E, A_\beta)$.

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[29] Note that by Assumption (A8), $E \neq \{0\}$; in particular, $\Gamma \neq \varnothing$.

Set $\bar{e} = \int_T e(t)\,d\nu(t)$ and let $F = c\ell\,A_{\bar{e}}$, i.e. F is the (norm) closure of the order ideal in E generated by the aggregate endowment $\int_T e(t)\,d\nu(t)$. Note that F, being the closure of an ideal in E, is an ideal in E as well. Set $F_+ = F \cap E_+$. Then, by Lemma 5, both $e(t)$, $f(t) \in F_+$ for almost all $t \in T$ (since for all $t \in T$, $e(t)$ and $f(t)$ are ≥ 0 and since f, being a core allocation, is feasible, i.e. $\int_T f(t)\,d\nu(t) = \bar{e} \equiv \int_T e(t)\,d\nu(t)$). In particular, $F_+ \setminus \{0\}$ is non-empty (since according to Assumption A(8), $e(t) \neq 0$ for any $t \in T$.)

We claim that the space $(F_+, |\sigma|(E, A_\beta))$ (i.e. F_+ endowed with the relativized topology $|\sigma|(E, A_\beta)$) is separable. Indeed, according to (i) of the theorem, the space $(A_{\bar{e}}, \sigma(E, A_\beta))$ is separable. But by definition of F, $A_{\bar{e}}$ is (norm) dense in F, therefore also dense in F with respect to the topology $\sigma(E, A_\beta)$ (since this latter topology is obviously weaker than the norm topology of E). Thus, the space $(F, \sigma(E, A_\beta))$ is separable. Now the spaces $(E, \sigma(E, A_\beta))$ and $(E, |\sigma|(E, A_\beta))$ have the same dual,[30] and therefore, by the Hahn-Banach theorem, $(F, \sigma(E, A_\beta))$ and $(F, |\sigma|(E, A_\beta))$ have the same dual, too. Hence since $(F, \sigma(E, A_\beta))$ is separable, $(F, |\sigma|(E, A_\beta))$ is separable as well, by the geometric form of the Hahn-Banach theorem (since $(F, |\sigma|(E, A_\beta))$ is a locally convex space). But F is an ideal in E; in particular, whenever $x \in F$ then $x^+ \in E_+ \cap F \equiv F_+$. Hence the fact that $(F, |\sigma|(E, A_\beta))$ is separable implies that $(F_+, |\sigma|(E, A_\beta))$ is separable as claimed, because—as also noted above—the lattice operations on E are continuous for the topology $|\sigma|(E, A_\beta)$.

Let D be a countable subset of F_+ that is dense in F_+ for the topology $|\sigma|(E, A_\beta)$. Since D is countable, it follows from (18) that for some $T' \subset T$, with $T \setminus T'$ a null set in T, if $t \in T'$ then $p(d) \geq p(e(t))$ whenever $d \in D$ and $d \succ_t f(t)$. We assert that in fact:

(19) For every $t \in T'$, if x is *any* element of F_+ with $x \succ_t f(t)$ then $p(x) \geq p(e(t))$.

To see this, pick any $t \in T'$ and any $x \in F_+$ with $x \succ_t f(t)$. Set

$$B = (\{x\} + \Gamma) \cap F_+.$$

As noted earlier, Γ is $|\sigma|(E, A_\beta)$-open and contains $E_+ \setminus \{0\}$. Consequently B is open in F_+ for the (relativized) topology $|\sigma|(E, A_\beta)$ and $\{x\} + F_+ \setminus \{0\} \subset B$ so x belongs to the $|\sigma|(E, A_\beta)$-closure of B (recall: $F_+ \neq \{0\}$). Thus, since D is dense in F_+ with respect to the topology $|\sigma|(E, A_\beta)$, the point x belongs to the $|\sigma|(E, A_\beta)$-closure of $B \cap D$. Now from (17), if $y \in B$ then $y \succ_t x$ and therefore, by transitivity of preferences, $y \succ_t f(t)$. Thus if $y \in B \cap D$, then $p(y) \geq p(e(t))$ since $t \in T'$. But therefore, since x belongs to the $|\sigma|(E, A_\beta)$-closure of $B \cap D$ and—as noted above—p is $|\sigma|(E, A_\beta)$-continuous, we must have $p(x) \geq p(e(t))$. Thus (19) holds.

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[30] As noted in the beginning of this proof, the dual of $(E, |\sigma|(E, A_\beta))$ can be identified with A_β, and the same is true for the dual of $(E, \sigma(E, A_\beta))$ (since E separates the points of A_β).

From (19) it follows in particular that $p(f(t)) = p(e(t))$ must be true for almost all $t \in T$. Indeed, for almost all $t \in T$, $f(t) \in F_+$ by choice of F, and thus, given any $v \in F_+$, $f(t) + v \in F_+$ as well. Hence, since $F_+ \neq \{0\}$, (19) implies that $p(f(t)) \geq p(e(t))$ for almost all $t \in T$ because preferences are strictly monotone, whence $p(f(t)) = p(e(t))$ for almost all $t \in T$ because f is feasible i.e. $\int_T f(t)\, dv(t) = \int_T e(t)\, dv(t)$. Note also that $p(e(t)) > 0$ for all $t \in T$ since p is strictly positive and since according to (A8), $e(t)$ is positive and non-zero for all $t \in T$.

Summarizing the discussion so far:

(20)
There is a subset $\tilde{T} \subset T$, with $T \setminus \tilde{T}$ a null set in T, such that for every $t \in \tilde{T}$,
(i) both $f(t), e(t) \in F_+$,
(ii) whenever $x \in F_+$ satisfies $x \succ_t f(t)$ then $p(x) \geq p(e(t))$, and
(iii) $p(f(t)) = p(e(t)) > 0$.

Observe that (20) implies (because $p(e(t)) > 0$ for $t \in \tilde{T}$) that the allocation f is Walrasian relative to the subspace F. To show now that f is Walrasian indeed for the entire commodity space E, we will establish the following:

Claim: There are a real number $k > 0$ and elements $p_t \in E_+^* \setminus \{0\}$, $t \in \tilde{T}$, such that for every $t \in \tilde{T}$: (a) $p_t \leq k\beta$, (b) $p_t(y) \leq p(y)$ for all $y \in F_+$, (c) $p_t(f(t)) = p(e(t))$, and (d) if $x \in E_+$ satisfies $x \succ_t f(t)$ then $p_t(x) \geq p(e(t))$.

Let us assume for the time being that the claim has been verified and see how to finish this proof. Thus let k and $p_t \in E_+^*$, $t \in \tilde{T}$, be chosen according to the claim. Being the dual of a Banach lattice, E^* is Dedekind complete and thus by virtue of (a), the set $\{p_t : t \in \tilde{T}\}$ has a supremum in E^*, say $\tilde{p}$. Note that from (b), $\tilde{p}(y) \leq p(y)$ for all $y \in F_+$, because F is an ideal in E by construction. Hence, from (i) of (20), $\tilde{p}(e(t)) \leq p(e(t))$ for all $t \in \tilde{T}$. Consequently, using (c) of the claim, $\tilde{p}(e(t)) \leq \tilde{p}(f(t))$ for all $t \in \tilde{T}$ (since $f(t) \geq 0$), whence, by feasibility of f and since $T \setminus \tilde{T}$ a null set,

(21) $$\tilde{p}(f(t)) = \tilde{p}(e(t)) \text{ for almost all } t \in T.$$

In particular, $\tilde{p}(e(t)) > 0$ for almost all $t \in T$, by a combination of (21) with (c) of the claim and (iii) of (20). Finally, using (d) of the claim together with the fact that $\tilde{p}(e(t)) \leq p(e(t))$ for every $t \in \tilde{T}$, we see that for each $t \in \tilde{T}$, whenever $x \in E_+$ satisfies $x \succ_t f(t)$ then $\tilde{p}(x) \geq \tilde{p}(e(t))$. But since $\tilde{p}(e(t)) > 0$ for almost all $t \in T$, and since $T \setminus \tilde{T}$ is a null set, this latter sentence implies, by the usual standard argument, that for almost all $t \in T$, whenever $x \in E_+$ satisfies $x \succ_t f(t)$ then, in fact, $\tilde{p}(x) > \tilde{p}(e(t))$. In view of this and (21), the allocation f is Walrasian as was to be shown.

To complete the proof of implication (i)$\Rightarrow$(ii) of the theorem, it remains only to show that the above claim is correct. To this end, pick any $t \in \tilde{\tilde{T}}$ and note that according to (i) of (20), $f(t) \in F_+$. Set

$$A = \{x \in E_+ : x \succ_t f(t)\} \text{ and } B = A + \Gamma$$

(where Γ is still the cone introduced at the start of this proof). Observe that A is convex by assumptions (A2) and (A5), and that $f(t) \in c\ell A$ by strict monotonicity of preferences. Recall also from above that Γ is convex and (norm) open. Consequently, B convex and open, and, as Γ is a cone, $f(t) \in c\ell B$. Recall further that Γ contains $E_+ \setminus \{0\}$; by strict monotonicity of preferences again, this implies that $B \cap F_+ \neq \varnothing$ (because $F_+ \setminus \{0\} \neq \varnothing$). Finally, since preferences are transitive, by (17) we have $x \succ_t f(t)$ whenever $x \in B \cap E_+$; thus from (ii) and (iii) of (20), $p(f(t)) \leq p(x)$ for all $x \in B \cap F_+$.

In view of these facts, and since F_+ is convex, we can now appeal to Lemma 6 to find elements p_t, r_t in E^* such that (I) $p_t(f(t)) \leq p_t(x)$ for all $x \in B$, (II) $r_t(f(t)) \leq r_t(y)$ for all $y \in F_+$, and (III) $p_t + r_t = p$.

Combining (II) with the facts that $f(t) \in F_+$ and F_+ is a cone, we see that $0 = r_t(f(t)) \leq r_t(y)$ for all $y \in F_+$, and from this combined with (III) that $p_t(y) \leq p(y)$ for all $y \in F_+$, and in particular that $p_t(f(t)) = p(f(t))$. From (iii) of (20), then, $p_t(f(t)) = p(e(t)) > 0$; in particular, $p_t \neq 0$. Now from (I), $p_t(f(t)) \leq p_t(x)$ for all $x \in A$ since Γ is a cone, whence $p(e(t)) \leq p_t(x)$ for all $x \in A$; in particular, $p_t \geq 0$ by strict monotonicity of preferences.

Summarizing, $p_t \in E_+^* \setminus \{0\}$ and (b), (c) and (d) of the claim hold. As for (a), note first that since $f(t) \in c\ell A$, (I) also implies that $0 \leq p_t(y)$ for all $y \in \Gamma$, which, since Γ is open and $p_t \neq 0$, actually means $0 < p_t(y)$ for all $y \in \Gamma$. Observe also that the definition of Γ as stated in the beginning of this proof can be equivalently written in the form

(22) $\Gamma = \{z \in E : z = a - b \text{ for some } a, b \in E_+ \text{ with } \alpha(a) > \beta(b)\}$.

(Indeed, let $a, b \in E_+$. Then $(a - b)^+ = a - a \wedge b$ and $(a - b)^- = b - a \wedge b$. Thus $\alpha((a - b)^+) - \beta((a - b)^-) = \alpha(a) - \beta(b) + (\beta - \alpha)(a \wedge b)$. Consequently, when $\alpha(a) - \beta(b) > 0$ then $\alpha((a - b)^+) - \beta((a - b)^-) > 0$ as well, because $\beta \geq \alpha$ and $a \wedge b \geq 0$.)

Fix any $y \in F_+ \setminus \{0\}$ and note that $\alpha(y) > 0$ since α is strictly positive. We claim that $p_t \leq p_t(y)[\alpha(y)]^{-1}\beta$. To see this, suppose to the contrary that there is an $x \in E_+ \setminus \{0\}$ for which $p_t(x) > p_t(y)[\alpha(y)]^{-1}\beta(x)$. Note that $p_t(x) > 0$ since both p_t and β are ≥ 0 and $\alpha(y) > 0$. Set $a = y$ and $b = p_t(y)[p_t(x)]^{-1}x$. Then $a, b \in E_+$ and $\alpha(a) > \beta(b)$, so $a - b \in \Gamma$ according to (22). On the other hand, $p_t(a - b) = 0$ and thus we have a contradiction to "$0 < p_t(y)$ for all $y \in \Gamma$." Consequently $p_t \leq p_t(y)[\alpha(y)]^{-1}\beta$ as predicted. Now from above, since

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$y \in F_+$, $p_t(y) \le p(y)$ and it follows that $p_t \le p(y)[\alpha(y)]^{-1}\beta$. But the term $p(y)[\alpha(y)]^{-1}\beta$ is independent of the particular t just under consideration, and thus (a) of the claim holds, too. This completes the proof of implication (i)⇏(ii).

(ii)⇒(i): We will prove the contrapositive. Thus suppose (i) is false. We may then apply Lemma 4 to find a $\overline{q}$ in E_+^* together with an $\overline{e}$ in E_+ such that $([-\overline{q}, \overline{q}], \text{weak}^*)$ is not first countable at 0 and such that $A_{\overline{e}}$ separates the points of $A_{\overline{q}}$. (In particular, then, $\overline{e} \ne 0$.) Let $(T, \mathcal{T}, \nu)$ be any complete finite positive atomless measure space. Using the element $\overline{e}$ and the order interval $[-\overline{q}, \overline{q}]$, we shall construct an economy $\mathcal{E}$ with $(T, \mathcal{T}, \nu)$ as measure space of agents and E as commodity space such that all the assumptions listed in (ii) of the statement of Theorem 3 hold but $C(\mathcal{E}) \not\subset \mathcal{W}(\mathcal{E})$. To begin with, for each individual t in the set T of agents, we let the endowment $e(t)$ be equal to $\overline{e}$ and the consumption set $X(t)$ be equal to E_+. Then assumptions (A1) and (A8) hold. Further, since the measure ν is finite, the endowment mapping $t \mapsto \overline{e}$ is integrable, as required in our definition of an economy.

By hypothesis, E^* has strictly positive elements. Fix any such element, say $\tilde{q}$, set $\hat{q} = \tilde{q} + \overline{q}$, and consider the order interval $[\tilde{q}, \tilde{q} + 2\overline{q}]$. Evidently $\hat{q} \in [\tilde{q}, \tilde{q} + 2\overline{q}]$ and $([\tilde{q}, \tilde{q} + 2\overline{q}], \text{weak}^*)$ is not first countable at $\hat{q}$ (because $([-\overline{q}, \overline{q}], \text{weak}^*)$ is not first countable at 0 and the mapping $q \mapsto q + \hat{q}$ is a homeomorphism between $([-\overline{q}, \overline{q}], \text{weak}^*)$ and $([\tilde{q}, \tilde{q} + 2\overline{q}], \text{weak}^*)$). Note also that $([\tilde{q}, \tilde{q} + 2\overline{q}], \text{weak}^*)$ is compact. We can therefore apply the constructions and arguments from (ii)⇏(i) in the proof of Theorem 1 (first, third and fourth paragraph, with $[\tilde{q}, \tilde{q} + 2\overline{q}]$ in place of V and $\hat{q}$ now specified as $\hat{q} = \tilde{q} + \overline{q}$) to find a family $(q_t)_{t \in T}$ of elements of $[\tilde{q}, \tilde{q} + 2\overline{q}]$ such that $q_t \ne \hat{q}$ for each $t \in T$ but such that:

(23)   For any given separable subset S of E the set
$\{t \in T : q_t(s) \ne \hat{q}(s) \text{ for some } s \in S\}$ is a null set in T.

Recall from above that the order ideal $A_{\overline{e}}$ separates the points of $A_{\overline{q}}$. Hence since $q_t - \hat{q} \in [-\overline{q}, \overline{q}]$, the statement "$q_t \ne \hat{q}$ for each $t \in T$" actually means:

(24)   For each $t \in T$ there is a $z \in A_{\overline{e}}$ such that $q_t(z) \ne \hat{q}(z)$

(the element z possibly depending on t, of course). Finally, note that every element of the order interval $[\tilde{q}, \tilde{q} + 2\overline{q}]$ is strictly positive (since $\tilde{q}$ is); in particular, $\hat{q}$ and each q_t are strictly positive.

Now for each $t \in T$ define a utility function $u_t : E_+ \to \mathbb{R}$ by

$$u_t(x) = q_t(x), \ x \in E_+.$$

Clearly assumptions (A2) to (A5) hold for this specification of preferences. (For (A4), recall that each q_t is strictly positive). Also—see the proof of Theorem 1,

implication (ii)⇒(i)—(A6) is satisfied, and in particular we have:

(25)   If $h: T \to E_+$ is any allocation then
           for almost all $t \in T$, $u_t(h(t)) = \langle \hat{q}, h(t) \rangle$.

Finally, regarding (A9), set $\alpha = \tilde{q}$ and $\beta = \tilde{q} + 2\overline{q}$. Then, as required in (A9), both α and β are strictly positive elements of E^* and $\beta \geq \alpha$. Pick any $q \in [\tilde{q}, \tilde{q} + 2\overline{q}]$ and any points u, v in E_+ with $\alpha(u) > \beta(v)$. Then

$$q(u) - q(v) \geq \tilde{q}(u) - (\tilde{q} + 2\overline{q})(v) \equiv \alpha(u) - \beta(v) > 0;$$

hence, for any $t \in T$ and $x \in E_+$, if $x - v + u \geq 0$ then

$$u_t(x - v + u) \equiv q_t(x - v + u) > q_t(x) \equiv u_t(x)$$

since $q_t \in [\tilde{q}, \tilde{q} + 2\overline{q}]$. Thus (A9) holds.

Summing up, an atomless economy $\mathcal{E}$ with commodity space E has been constructed so that the assumptions listed in (ii) of the statement of Theorem 3 all hold. Using (25) it may be seen that the initial allocation $t \mapsto \overline{e}$ is in $C(\mathcal{E})$—cf. the corresponding part of (ii)⇒(i) in the proof of Theorem 1—so that it remains only to show that this allocation is not Walrasian. Arguing by contradiction, assume there is a $p \in E^*$ such that the pair $(p, t \mapsto \overline{e})$ is a Walrasian equilibrium. Consider $A_{\overline{e}}$, the order ideal generated by $\overline{e}$. Given any $x \in A_{\overline{e}}$, for some real number $\lambda > 0$ we have $\overline{e} + \lambda x \geq 0$, and therefore the equilibrium conditions with respect to $(p, t \mapsto \overline{e})$ imply:

(26)   For almost all $t \in T$, $A_{\overline{e}} \cap \ker p \subset \ker q_t$.

But this leads to the desired contradiction, similar as in the last paragraph of the proof of Theorem 1. Indeed, let $q_t | A_{\overline{e}}$ denote the restriction of q_t to $A_{\overline{e}}$, $t \in T$, and let $\hat{q} | A_{\overline{e}}$ and $p | A_{\overline{e}}$ denote the restrictions to $A_{\overline{e}}$ of $\hat{q}$ and p, respectively. In terms of $q_t | A_{\overline{e}}$ and $p | A_{\overline{e}}$, (26) may be rephrased to say that

(27)   for almost every $t \in T$, $q_t | A_{\overline{e}} = \lambda_t p | A_{\overline{e}}$ for some real number λ_t.

From (23), however,

(28)   for any $z \in A_{\overline{e}}$, $q_t | A_{\overline{e}}(z) = \hat{q} | A_{\overline{e}}(z)$ for almost all $t \in T$,

and hence (27) means in fact that, for some real number λ, and almost all $t \in T$, $q_t | A_{\overline{e}} = \lambda p | A_{\overline{e}}$, whence, by (28) again, $q_t | A_{\overline{e}} = \hat{q} | A_{\overline{e}}$ for almost all $t \in T$. But this contradicts (24) and we conclude that the allocation $t \mapsto \overline{e}$ is not Walrasian. Thus "not (ii)" has been established and the proof of Theorem 3 is complete.

4.4 Proof of Lemma 1

Only implication (b)⇒(a) needs proof. Thus assume (b) to be true. We first claim:

(29)   Given any strictly positive $\beta \in E^*$ there is a countable subset of E that separates the points of A_β.

Indeed, let β be a strictly positive element of E^*. Pick any quasi-interior point of E_+, say e; this is possible by hypothesis. According to (b), the space $(A_e, \sigma(E, A_\beta))$ is separable, and according to the definition of a quasi-interior point, A_e is dense in E for the norm topology, therefore also for the topology $\sigma(E, A_\beta)$. Thus, the space $(E, \sigma(E, A_\beta))$ is separable. But since E separates the points of A_β, the topological dual of $(E, \sigma(E, A_\beta))$ can be identified with A_β, and it follows that some countable subset of E separates the points of A_β. Thus (29) holds.

We proceed by showing that the hypotheses imply that E must be order continuous, and it will follow from this that E is separable.

Suppose E is not order continuous. Then, since E is σ-Dedekind complete by hypothesis, E contains a closed sublattice that is isomorphic as a Banach lattice to ℓ_∞ (this latter space endowed with its usual norm).[31] But ℓ_∞ is an injective Banach lattice, so this means there is a positive (in particular, continuous) linear operator from E onto ℓ_∞, say T.[32] Noting that ℓ_∞ can be identified with $C(\beta\mathbb{N})$ via a positive operator, we may view T as a positive linear operator from E onto $C(\beta\mathbb{N})$. (As usual, $\beta\mathbb{N}$ denotes the Stone-Čech compactification of $\mathbb{N}$.)

According to arguments of Rosenthal (1969, Proposition 1.4 and the proof of Proposition 3.4) there is a regular finite positive Borel measure μ on $\beta\mathbb{N}$ such that—denoting by $\mathcal{B}_{\beta\mathbb{N}}$ the Borel σ-algebra of $\beta\mathbb{N}$—the subspace $L_1(\beta\mathbb{N}, \mathcal{B}_{\beta\mathbb{N}}, \mu)$ of $C(\beta\mathbb{N})^*$ is non-separable in the usual L_1-norm. Note that this norm agrees with the one induced from $C(\beta\mathbb{N})^*$. Consider the order interval $[-\mu, \mu]$. Since $L_1(\beta\mathbb{N}, \mathcal{B}_{\beta\mathbb{N}}, \mu)$ is non-separable in the norm topology of $C(\beta\mathbb{N})^*$, so is $[-\mu, \mu]$ because the linear span of $[-\mu, \mu]$ is norm dense in $L_1(\beta\mathbb{N}, \mathcal{B}_{\beta\mathbb{N}}, \mu)$. An appeal to the Hahn-Banach theorem shows that $[-\mu, \mu]$ is also non-separable in the weak topology of $C(\beta\mathbb{N})^*$ (i.e. the topology $\sigma(C(\beta\mathbb{N})^*, C(\beta\mathbb{N})^{**})$). Finally, note that the order interval $[-\mu, \mu]$ is compact in the weak topology of $C(\beta\mathbb{N})^*$, because $C(\beta\mathbb{N})^*$ is an L-space.

Set $V = T^*([-\mu, \mu])$ where $T^* \colon C(\beta\mathbb{N})^* \to E^*$ is the adjoint operator of the continuous linear operator T. Note that since T is positive, so is T^* and thus $T^*(\mu) \in E_+^*$ and $V \subset [-T^*(\mu), T^*(\mu)]$. Now by definition of an adjoint operator, T^* is continuous for the weak* topologies of E^* and $C(\beta\mathbb{N})^*$, hence continuous for the weak* topology of E^* and the weak topology of $C(\beta\mathbb{N})^*$, too. Moreover, since T is onto, T^* is one to one. Thus, since $([-\mu, \mu], \text{weak})$ is compact, T^* is

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[31]See Aliprantis and Burkinshaw (1985, Theorem 4.14, p. 220).

[32]See Meyer-Nieberg (1991, Definition 3.2.3 and Theorem 3.2.4, p. 170).

a homeomorphism between $([-\mu, \mu], \mathrm{weak})$ and (V, weak^*), and consequently (V, weak^*) is non-separable since $([-\mu, \mu], \mathrm{weak})$ is.

Pick any strictly positive element β' in E^*; this is possible by the hypotheses about E. Set $\beta = \beta' + T^*(\mu)$. Then β is strictly positive (since β' is and $T^*(\mu)$ is positive.) Moreover, we have $V \subset [-\beta, \beta]$.

Note that the order interval $[-\beta, \beta]$ in E^* is weak* compact. However, $[-\beta, \beta]$ cannot be weak* metrizable since $V \subset [-\beta, \beta]$ and V is not weak* separable. But $([-\beta, \beta], \mathrm{weak}^*)$ being compact but not metrizable means that no countable subset of E separates the points of $[-\beta, \beta]$, which in turn implies that no countable subset of E separates the points of A_β. However, this contradicts (29), thus proving that E is order continuous as predicted.

Now to see that E is in fact separable, again pick any strictly positive element in E^*, say β. Note that β being strictly positive means that A_β separates the points of E. In other words, A_β is weak* dense in E^*. Observe next that, for any integer $n > 0$, the order interval $[-n\beta, n\beta]$ in E^*, being weak* compact, is also weak* metrizable, because of (29). Thus, for each n, $[-n\beta, n\beta]$ is weak* separable. Consequently, $A_\beta \equiv \bigcup_{n=1}^{\infty}[-n\beta, n\beta]$ is weak* separable, and hence so is E^* because A_β is weak* dense in E^*. Now pick a quasi-interior point of E_+, say e, and consider the order interval $[-e, e]$. Since E is order continuous, $[-e, e]$ is weakly compact, i.e. compact for the topology $\sigma(E, E^*)$. But therefore, since E^* is weak* separable, $[-e, e]$ is weakly metrizable as well. In particular, then, $[-e, e]$ is weakly separable, hence norm separable (by the Hahn-Banach theorem). But e is a quasi-interior point of E_+, i.e. the order ideal generated by e is dense in E. It follows that E is separable, as was to be shown. The proof of the lemma is thus complete.

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