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Rationalizable Foresight Dynamics:
Evolution and Rationalizability

September 2002

Working Paper No: 0302

DEPARTMENT OF ECONOMICS

UNIVERSITY OF VIENNA

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Rationalizable Foresight Dynamics: Evolution and Rationalizability*

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February 26, 2002; revised September 5, 2002

*We are grateful to seminar participants at Hitotsubashi University, Keio University, Kyoto University, Nagoya University, Tohoku University, and University of Tokyo, and participants at the 2002 Annual Meeting of the Japanese Economic Association in Otaru and the Fifth Spanish Meeting on Game Theory in Sevilla, as well as Atsushi Kajii, Michihiro Kandori, Roger Lagunoff, and Rafael Rob for helpful conversations and comments.

Abstract

This paper considers a dynamic adjustment process in a society with a continuum of agents. Each agent takes an action upon entry and commits to it until he is replaced by his successor at a stochastic point in time. In this society, rationality is common knowledge, but beliefs may not be coordinated with each other. A *rationalizable foresight path* is a feasible path of action distribution along which every agent takes an action that maximizes his expected discounted payoff against another path which is in turn a rationalizable foresight path. An action distribution is *accessible* from another distribution *under rationalizable foresight* if there exists a rationalizable foresight path from the latter to the former. An action distribution is said to be a *stable state under rationalizable foresight* if no rationalizable foresight path departs from the distribution. A set of action distributions is said to be a *stable set under rationalizable foresight* if it is closed under accessibility and any two elements of the set are mutually accessible. Stable sets under rationalizable foresight always exist. These concepts are compared with the corresponding concepts under perfect foresight. Every stable state under rationalizable foresight is shown to be stable under perfect foresight. But the converse is not true. An example is provided to illustrate that the stability under rationalizable foresight gives a sharper prediction than that under perfect foresight.

KEYWORDS: Nash equilibrium, evolution, rationalizability, rationalizable foresight, perfect foresight, stability under rationalizable foresight (RF-stability), stability under perfect foresight (PF-stability).

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1 Introduction

For a long time game theory has centered around the concept of equilibrium introduced by Nash (1951). In spite of its dominance as a theory of rational agents, the equilibrium theory leaves one question to be answered: how do agents coordinate their beliefs? When an agent makes a decision in a strategic situation, he does not know other agents' strategies, and therefore, often makes some conjecture about them. Equilibrium requires that this conjecture be correct. In many cases, the coordination of beliefs requires that agents be not only rational but also omniscient. In particular, this problem becomes acute if there are multiple equilibria as it entails the problem of equilibrium selection.

In response to the claim that requiring the coordination of beliefs among agents goes beyond their rational decision making abilities, two theories have been proposed: one is the theory of rationalizability, and the other is the evolutionary game theory.

The concept of rationalizability was proposed by Bernheim (1984) and Pearce (1984) to cope with a one-shot situation in which, while rationality is common knowledge, agents do not know which strategies the opponents are going to take. The conjectures formed by the agents do not have to be correct, but consistent with common knowledge of rationality, and therefore, the actual decisions need not be best responses to each other. Although the theory of rationalizability provides us with new insights, it has not served useful tool to analyze economic problems since it gives less sharper predictions than the equilibrium theory.[1]

The evolutionary game theory, on the other hand, has gained game theorists' attention as a theory of unsophisticated agents since the seminal work of Maynard Smith and Price (1973).[2] This theory typically considers a dynamic adjustment process in a large and anonymous population, examining the stability of behavior pattern. Agents therein are not sophisticated enough to make rational decisions due to informational constraints and/or the lack of ability. Instead, their behavior is gradually adjusted according to some rule. Again, their failure to take best responses to each other is a natural consequence of the environment, unless the process is at some equilibrium state. The evolutionary game theory, however, has been criticized as it ignores the rationality of human decision makers. That is to say, it has abandoned not only the coordination of beliefs, but also the rationality of human beings and/or their knowledge on the structure of the society.

In the present paper, we synthesize the ideas of rationalizability and evolution in a dynamic environment in which rational agents, who are neither

[1] Pearce (1984) showed that rationalizability in extensive form games may refine the set of equilibrium outcomes. See also Battigalli (1997).

[2] See, e.g., Fudenberg and Levine (1998), Hofbauer and Sigmund (1998), Vega-Redondo (1996), Weibull (1997), and Young (1998).

genes nor omniscient beings, use their reasoning abilities in determining their actions. We then demonstrate that rationalizability combined with evolution can have more predictive power than equilibrium.

We study infinite horizon societal games in which a continuum of agents are randomly matched to play a normal form game repeatedly. Each agent stays in the society for a stochastic time period to be replaced by a new entrant. Each entrant inherits certain knowledge about the current action distribution, takes an action to maximize his expected discounted payoff, and commits to it through the rest of his life.[3] While the structure of the society as well as the rationality of the agents is common knowledge, beliefs about the future path of action distribution are not necessarily coordinated among them. We postulate that each agent forms his belief in a *rationalizable* manner. We call the process induced by such agents the *rationalizable foresight dynamics*.

In bold strokes, the main concept of rationalizable foresight path is inductively defined in the following way. First, given the set of (physically) feasible paths of action distribution, Φ^0, define its subset Φ^1 as the set of paths along which each agent takes an optimal action against *some* path in Φ^0. Here, we allow different agents to optimize against different paths. Then given this newly defined set Φ^1, define its subset Φ^2 of paths along which each agent takes an optimal action against some path in Φ^1. We repeat this procedure inductively and take the limit of these sets: this limit is called the set of *rationalizable foresight paths*. Indeed, along a rationalizable foresight path each agent optimizes against another—typically different—rationalizable foresight path.

We use the rationalizable foresight paths to define stability concepts as in Gilboa and Matsui (1991). An action distribution is *accessible under rationalizable foresight* from another distribution if there exists a rationalizable foresight path from the latter to the former. A set of action distributions is said to be a *stable set under rationalizable foresight*, or an *RF-stable set*, if it is closed under accessibility and any two elements of the set are mutually accessible. The situation we have in mind is the one in which the behavior pattern is subject to fluctuation through rationalizable, but not necessarily correct, beliefs formed by new entrants. An RF-stable set is stable in the sense that no rationalizable foresight drives the behavior pattern away from it. If an RF-stable set is a singleton, its element is said to be a *stable state under rationalizable foresight*, or an *RF-stable state*.[4]

We study the relationship between the stability concepts under rationalizable foresight and the corresponding concepts under perfect foresight.

[3]We do not have to take literally that a new entrant makes a life-time decision. One may interpret this as a certain commitment for a random short period of time.

[4]In Gilboa and Matsui (1991), the set-valued and the point-valued stability concepts under the best response dynamics are called 'cyclically stable sets' and 'socially stable strategies', respectively.

The concept of perfect foresight embodies the concept of equilibrium in our dynamic framework. A perfect foresight path is defined to be a path of action distribution to which every entrant takes a best response. A stable set and a stable state under perfect foresight (a *PF-stable set* and a *PF-stable state*) are defined analogously to those under rationalizable foresight. As in a one-shot game, a perfect foresight path is a rationalizable foresight path, but not vice versa. Therefore, for a given state, it is easier to escape from the state under rationalizable foresight than under perfect foresight. Using this logic, we show that every RF-stable state is a PF-stable state. The converse, however, is not true in general. We construct an example in which the stability under rationalizable foresight provides a sharper prediction than that under perfect foresight: in this example, the set of RF-stable states is a proper subset of the set of PF-stable states, and no other stable set exists.

In our analysis, inertia plays a key role.[5] If there is no inertia, the behavior pattern may jump around, and there is no hope for sharp prediction. To see this, we introduce the notion of rationalizability for static societal games where agents are randomly matched to play a given normal form game only once. In this environment, a *rationalizable strategy distribution* is defined in such a way that every pure strategy taken by some agents, i.e., contained in the support of this strategy distribution, is a best response to another rationalizable strategy distribution. We then show that as inertia vanishes, the unique RF-stable set of the dynamic societal game converges to the set of rationalizable strategy distributions of the static societal game.

Our stability concepts are related to the notion of p-dominance (Morris, Rob, and Shin (1995)). An action pair in a two-player game is said to be a p-dominant equilibrium if each action is the unique best response to any belief that the other player takes the action in this pair with a probability greater than p. It is shown that if the stage game has a p-dominant equilibrium with $p < 1/2$, which is a risk-dominant equilibrium in 2×2 games, then it is always contained in an RF-stable set.

A full characterization of the stability concepts is given for the class of 2×2 games. In the games with two strict Nash equilibria, the risk-dominant equilibrium is always contained in an RF-stable set, but the risk-dominated equilibrium may not, depending on the rate of time preference. In the games with a unique symmetric Nash equilibrium, which is completely mixed, the unique RF-stable set is a set of states that contains non-equilibrium states as well as the equilibrium state (independently of the rate of time preference). This fact is to be contrasted with the PF-stable set, which consists of the unique equilibrium state.

[5]We deal with a specific environment in which an adjustment cost is infinite after one chooses his action. We can modify this assumption to accommodate other environment such as the one with a finite adjustment cost.

The stability concepts of the present paper share the basic idea with those of Gilboa and Matsui (1991). In that paper, a best response path is defined to be a feasible path along which agents change their actions to the ones that are best responses to the current action distribution. An action distribution is accessible from another (under the best response dynamics) if there exists a best response path from the latter to the former. A stable set is a set of action distributions which is closed under accessibility and any two distributions in which are mutually accessible. The assumption that an action increases its frequency only when it is a best response to the current action distribution is equivalent to assuming myopia of decision makers. Indeed, the dynamics studied in the present paper becomes close to the best response dynamics if the agents are sufficiently impatient.

Our dynamic framework incorporates the same type of inertia as the one introduced in the perfect foresight dynamics, studied by Krugman (1991) and Matsuyama (1991) in the context of development economics, and Matsui and Matsuyama (1995) for societal games.[6] While discussing the possibility of escapes from locally stable states by way of agents' forward looking abilities, these papers consider the perfect foresight paths, and therefore, do not bear the idea of miscoordination of beliefs.

The present paper is also related to Burdzy, Frankel, and Pauzner (2001). They consider a continuum of rational agents who are repeatedly and randomly matched to play a 2×2 coordination game, with Poisson action revision opportunities, but whose payoffs are stochastically changed over time. They show that if there are the region of possible payoffs where one action is strictly dominant and the other region where the other action is in turn strictly dominant, then the unique outcome is chosen by way of eliminating strictly dominated strategies, which shares the idea with that of rationalizability. In their model, the existence of dominance regions is essential for the iterated elimination method to operate.

Another related work is conducted by Lagunoff and Matsui (1995), who construct a model in which finitely many agents are stochastically entitled to make decisions in the problem of public good provision. In that model, the coordination of beliefs among the agents is not assumed as in the present model. Each agent maximizes his expected discounted payoff against some feasible, but not necessarily rationalizable foresight, path. Under some condition, cooperation emerges as the unique outcome of this process.

The rest of the paper is organized as follows. Section 2 gives our basic framework. Section 3 defines the rationalizable foresight paths and stability concepts under rationalizable foresight (RF-stable sets and states) and shows the existence of RF-stable sets. Section 4 compares the RF-stability with the stability under perfect foresight (PF-stable sets and states) and demon-

[6]See Hofbauer and Sorger (1999, 2002) and Oyama (2000) for more recent developments.

strates by example that the RF-stability provides a sharper prediction than the PF-stability. Section 5 discusses some properties of the rationalizable foresight paths. Section 6 examines the relationship between the static rationalizability and the rationalizable foresight dynamics. Section 7 relates the RF-stability to the notion of p-dominance. Section 8 completely characterizes RF-stable sets for the class of 2×2 games. Section 9 concludes the paper.

2 Framework

We consider a symmetric two-player game with $n \geq 2$ actions. The set of actions and the payoff matrix, which are common to both players, are given by $A = \{a_1, \dots, a_n\}$ and $(u_{ij}) \in \mathbb{R}^{n \times n}$, respectively, where u_{ij} $(i, j = 1, \dots, n)$ is the payoff received by a player taking action a_i against an opponent playing action a_j. The set of mixed strategies is identified with the $(n-1)$-dimensional simplex, denoted by Δ, which is a subset of the n-dimensional real space endowed with a norm $|\cdot|$. We say that $x^* = (x_1^*, \dots, x_n^*) \in \Delta$ is an equilibrium state if (x^*, x^*) is a Nash equilibrium, i.e., for all $x = (x_1, \dots, x_n) \in \Delta$, $\sum_{ij} x_i^* \, u_{ij} \, x_j^* \geq \sum_{ij} x_i \, u_{ij} \, x_j^*$. We denote by $[a_i]$ the element of Δ that assigns one to the ith coordinate (and zero to others).

The above game is played repeatedly in a society with a continuum of identical anonymous agents. At each point in time, agents are matched randomly to form pairs and play the game. Each agent is replaced by his successor according to the Poisson process with parameter $\lambda > 0$. These processes are independent across the agents. Thus, during a time interval $[t, t+h)$, approximately a fraction λh of agents are replaced by the same size of entrants. Each agent is entitled to choose his action only upon entry to the society, i.e., one cannot change his action once it is chosen. An interpretation of this assumption is that there exists a large switching cost.[7]

A *path of action distribution*, or simply a *path*, is described by a function $\phi \colon [0, \infty) \to \Delta$, where $\phi(t) = (\phi_1(t), \dots, \phi_n(t))$ is the action distribution of the society at time t, with $\phi_i(t)$ denoting the fraction of the agents playing action a_i. Since during a time interval $[t, t+h)$, only entrants of fraction λh change their actions, $\phi_i(\cdot)$ is Lipschitz continuous with Lipschitz constant λ, which implies that it is differentiable at almost all $t \in [0, \infty)$.

Definition 1 A path of action distribution $\phi \colon [0, \infty) \to \Delta$ is *feasible* if it is Lipschitz continuous with Lipschitz constant λ and satisfies the condition that for almost all t, there exists $\alpha(t) \in \Delta$ such that

$$\dot{\phi}(t) = \lambda(\alpha(t) - \phi(t)). \tag{2.1}$$

[7] Another interpretation is that each agent lives forever and revises his action only occasionally at random points in time which follow the Poisson process with the parameter λ, and his belief may change as well when his revision opportunity arises.

The above condition is equivalent to the condition that for all $i = 1, \ldots, n$,

$$\dot{\phi}_i(t) \geq -\lambda\phi_i(t) \quad \text{a.e.} \tag{2.2}$$

Note, for example, that $\dot{\phi}_i(t) = -\lambda\phi_i(t)$ implies that (almost) all the entrants at time t take actions other than a_i. Note also that $\dot{\phi}_i(t) \geq -\lambda\phi_i(t)$ together with $\phi(t) \in \Delta$ implies $\dot{\phi}_i(t) \leq \lambda(1-\phi_i(t))$. The set of feasible paths is denoted by Φ^0.

An entrant anticipates a future path of action distribution and chooses an action that maximizes the expected discounted payoff. For a given anticipated path ϕ, the expected discounted payoff for an entrant at time t from taking action a_i is calculated as

$$\begin{aligned} V_i(\phi)(t) &= (\lambda+\theta)\int_0^\infty\int_t^{t+s} e^{-\theta(z-t)}\sum_{k=1}^n \phi_k(z)u_{ik}\,dz\,\lambda e^{-\lambda s}\,ds \\ &= (\lambda+\theta)\int_t^\infty e^{-(\lambda+\theta)(s-t)}\sum_{k=1}^n \phi_k(s)u_{ik}\,ds, \end{aligned}$$

where $\theta > -\lambda$ is the common rate of time preference while $\lambda+\theta > 0$ is viewed as the *effective* discount rate. Note that this expression is well defined since $\phi_k(\cdot)$ is bounded for each k. We write $V_i(\cdot) = V_i(\cdot)(0)$.

Given a feasible path ϕ, let $BR(\phi)(t)$ be the set of best responses in pure strategies to ϕ at time t, i.e.,

$$BR(\phi)(t) = \{a_i \in A \,|\, V_i(\phi)(t) \geq V_j(\phi)(t) \text{ for all } j\}.$$

We write $BR(\cdot) = BR(\cdot)(0)$. Note that two games (u_{ij}) and (v_{ij}) are equivalent in terms of their best response properties if there exist $\alpha > 0$ and $(w_i) \in \mathbb{R}^n$ such that $u_{ij} = \alpha v_{ij} + w_i$ holds for all i and j. The analyses below are invariant under positive affine transformations of this form.

Finally, we denote the *degree of friction* by $\delta = \theta/\lambda > -1$.

3 Rationalizable Foresight and Stability Concepts

After a certain history, changes in behavior pattern depend on what beliefs agents form and how they behave under these beliefs. We assume that the agents form their beliefs in a *rationalizable* manner. In particular, they do not necessarily coordinate their beliefs with each other.

To express this idea, we define *rationalizable foresight paths*. First, let Φ^0 be the set of all feasible paths, i.e., the set of Lipschitz continuous paths satisfying (2.2). Then for a given positive integer k, let Φ^k be the set of the paths in Φ^{k-1} along which every agent takes a best response to *some* path in Φ^{k-1}. Formally, define Φ^k as

$$\Phi^k = \big\{\phi \in \Phi^{k-1} \,|\, \forall\, i : \big[\dot{\phi}_i(t) > -\lambda\phi_i(t) \\ \Rightarrow \exists\, \psi \in \Phi^{k-1} : \psi(t) = \phi(t) \text{ and } a_i \in BR(\psi)(t)\big] \ \text{ a.e.}\big\}.$$

In this definition, $\dot{\phi}_i(t) > -\lambda\phi_i(t)$ implies that at least some positive fraction of the entire population take a_i at time t upon entry.

From this definition, it is easy to verify that $\Phi^k \subset \Phi^{k-1}$ holds and that $\Phi^k = \Phi^{k-1}$ implies $\Phi^{k+1} = \Phi^k$. Let $\Phi^* = \bigcap_{k=0}^{\infty} \Phi^k$.

Definition 2 A path in Φ^* is a *rationalizable foresight path.*

A path in Φ^* is rationalizable in the sense that each agent can construct an infinite hierarchy of beliefs which are consistent with the "rationality hypothesis". Note that every path that rests at a one-shot equilibrium state is always in Φ^k's, and therefore, it is in Φ^*. The existence of one-shot equilibrium states thus implies the nonemptiness of Φ^*. The following claim simply states this observation.

Claim 3.1 *If $x^* \in \Delta$ is an equilibrium state, then the path ϕ such that $\phi(t) = x^*$ for all t is a rationalizable foresight path.*

The action distribution is subject to fluctuation through, say, belief changes. Still, it is conceivable that the action distribution stays in a certain set and never goes out of it no matter how beliefs may change. To incorporate this point, we introduce stability concepts under rationalizable foresight.

We first define the notion of accessibility in a recursive manner. A state $y \in \Delta$ is defined to be *accessible under rationalizable foresight* from another state $x \in \Delta$ if one of the following conditions is satisfied:

(i) there exists a rationalizable foresight path ϕ such that $\phi(0) = x$ and $\phi(t) = y$ for some $t \geq 0$;

(ii) there exists a sequence of states $\{y^k\}$ converging to y such that y^k is accessible under rationalizable foresight from x for all k; and

(iii) y is accessible under rationalizable foresight from some z which is in turn accessible under rationalizable foresight from x.

Using this concept of accessibility, we define the following stability concepts.

Definition 3 A nonempty subset F^* of Δ is a *stable set under rationalizable foresight*, or an *RF-stable set*, if for any x in F^*, y is accessible from x under rationalizable foresight if and only if y is in F^*.

An action distribution $x^* \in \Delta$ is a *stable state under rationalizable foresight*, or an *RF-stable state*, if $\{x^*\}$ is a stable set under rationalizable foresight.

An RF-stable set F^* is stable in the sense that once the action distribution falls into F^*, another action distribution may be realized if and only if it is contained in F^*.

It is a direct application of Matsui (1992) to show the existence of stable sets.

Theorem 3.2 *Every game has at least one RF-stable set.*

Proof. For each $x \in \Delta$, we define $R(x)$ to be

$$R(x) = \{y \in \Delta \,|\, y \text{ is accessible from } x \text{ under rationalizable foresight}\}.$$

Observe that $R(x)$ is closed and $x' \in R(x)$ implies $R(x') \subset R(x)$ by the definition of accessibility. The nonemptiness of $R(x)$ will be shown in Section 5.

We consider the partially ordered set $(\{R(x)\}_{x\in\Delta}, \subset)$. Take any totally ordered subset of $\{R(x)\}_{x\in\Delta}$ and denote it by $\{R(x)\}_{x\in\Delta'}$. Since for each $x \in \Delta'$, $R(x)$ is a closed and nonempty subset of a compact set, $\bigcap_{x\in\Delta'} R(x)$ is nonempty. Choose any $y \in \bigcap_{x\in\Delta'} R(x)$. Since $R(y) \subset R(x)$ holds for all $x \in \Delta'$, $R(y)$ is a lower bound of $\{R(x)\}_{x\in\Delta'}$ in $\{R(x)\}_{x\in\Delta}$. Therefore, by Zorn's lemma, there exists a minimal element $R^* = R(x^*)$ in $\{R(x)\}_{x\in\Delta}$.

We claim that R^* is an RF-stable set. Indeed, for any $x \in R^*$, $R(x) \subset R^*$ holds, and since R^* is a minimal set, $R(x) = R^*$ holds. It follows that no state outside R^* is accessible from any state in R^*, and every state in R^* is accessible from any state in R^* under rationalizable foresight. ▌

On the other hand, as Example 3.1 shows, RF-stable states do not always exist.

By utilizing an argument analogous to the proof of Theorem 3.2, we also have the following.

Theorem 3.3 *Let $\mathcal{F}^*$ be the family of all the RF-stable sets. For every $x^0 \in \Delta$, there exists $y \in \bigcup_{F^*\in\mathcal{F}^*} F^*$ such that y is accessible from x^0 under rationalizable foresight.*

Proof. Take any $x^0 \in \Delta$. It is sufficient to show that there is an RF-stable set that is contained in $R(x^0)$, where $R(x^0)$ is defined in the proof of Theorem 3.2.

Define $\mathcal{R} = \{R(x) \,|\, x \in \Delta,\ R(x) \subset R(x^0)\}$. By an argument analogous to the proof of Theorem 3.2, $\mathcal{R}$ has a minimal element R^*, which is an RF-stable set and satisfies that $R^* \subset R(x^0)$. ▌

Example 3.1 A 2×2 Game

We present an example in which the unique RF-stable set contains nonequilibrium states. Consider the following 2×2 game:

$$(u_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

We divide our analysis into two cases, $\delta > 0$ and $-1 < \delta \leq 0$. If $-1 < \delta \leq 0$, then the entire space is the unique RF-stable set. If, on the other hand, $\delta > 0$, then we need the following preparation.

For a given $\delta > 0$, let $\alpha^* = \alpha^*(\delta)$ be the unique solution to

$$\frac{1+\delta}{2+\delta}(1-\alpha^*) + \frac{1}{2+\delta}\frac{(\alpha^*)^{2+\delta}}{(1-\alpha^*)^{1+\delta}} = \frac{1}{2}, \quad 0 < \alpha^* < \frac{1}{2}. \tag{3.2}$$

The left hand side of (3.2) is the expected discounted payoff to a_1 at time 0 along the path that moves from $(\alpha^*, 1-\alpha^*)$ towards $[a_1]$ until it reaches $(1-\alpha^*, \alpha^*)$, and then stays there, i.e., the path ϕ with $\phi(0) = (\alpha^*, 1-\alpha^*)$ such that

$$\phi(t) = \begin{cases} [a_1] - ([a_1] - \phi(0))\, e^{-\lambda t} & \text{if } t < T^*, \\ (1-\alpha^*, \alpha^*) & \text{if } t \geq T^*, \end{cases} \tag{3.3}$$

where T^* is given by $e^{-\lambda T^*} = \alpha^*/(1-\alpha^*)$. Since $V_1(\phi)(t) + V_2(\phi)(t) = 1$ holds for the game (3.1), equation (3.2) represents that $V_1(\phi) = V_2(\phi)$ holds. Using α^* defined above, we identify the unique RF-stable set.

Proposition 3.4 *Let the stage game be given by* (3.1)*. For a given* $\delta > -1$*, the unique RF-stable set is*

$$F^*(\delta) = \begin{cases} \{(\alpha_1, \alpha_2) \in \Delta \mid \alpha^*(\delta) \leq \alpha_1 \leq 1 - \alpha^*(\delta)\} & \textit{if } \delta > 0, \\ \Delta & \textit{if } -1 < \delta \leq 0. \end{cases}$$

Proof. See Appendix. ▮

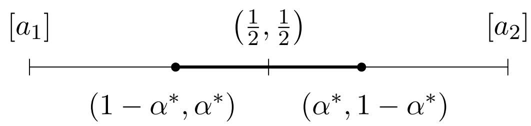


Figure 1: RF-stable set for $\delta > 0$

This RF-stable set contains non-equilibrium states for any $\delta > -1$. In this example, one needs a rationalizable foresight path along which agents constantly misforecast in order to reach a non-equilibrium state. If $\delta > 0$, such a path can be constructed in the following manner. From a state in F^*, suppose that newborns expect that the action distribution will move towards $[a_1]$ until it reaches $(1-\alpha^*, \alpha^*)$, and will stay there. Under such an expectation, a_2 is a best response, and therefore, the newborn will take a_2. If the new generations keep believing that the path moves towards $[a_1]$, then they keep taking a_2 until it reaches $(\alpha^*, 1-\alpha^*)$. Beyond this state, nobody is willing to take a_2 even if he expects the action distribution to move towards

$[a_1]$. Notice that along this path, agents constantly misforecast the future. By a symmetric argument, a_1 is a best response to the path that moves towards $[a_2]$ until it reaches $(\alpha^*, 1-\alpha^*)$, and stays there. In sum, at each state in F^*, the left-moving path and the right-moving path rationalize each other.

If $\delta \leq 0$ holds, then at any state, a_1 (resp. a_2) is a best response to the path that moves to $[a_2]$ (resp. $[a_1]$). Thereby every state can be accessible from any other state under rationalizable foresight.

Now we state the following comparative statics result.

Proposition 3.5 *$\alpha^*(\delta)$ is increasing in δ; $\alpha^*(\delta) \searrow 0$ as $\delta \searrow 0$, and $\alpha^*(\delta) \nearrow 1/2$ as $\delta \nearrow \infty$.*

Proof. See Appendix. ∎

This proposition states that as the degree of friction goes to zero from above, the stable set expands to the whole space Δ, and that as the degree of friction goes to infinity, the stable set shrinks to the singleton $\{(1/2, 1/2)\}$. Intuition behind this result is as follows. The further the action distribution is away from $(1/2, 1/2)$, the larger is the instantaneous gain from taking the action that is not taken by the majority. If the degree of friction is small, the future state is relatively more important than the current state, and therefore, it is relatively easy for the action distribution to move around. On the other hand, if the degree of friction is sufficiently large, the gain from taking the action of the minority dominates any future loss, which implies that the path moves towards $(1/2, 1/2)$ without fail.

4 Rationalizable Foresight versus Perfect Foresight

This section discusses some properties of the rationalizable foresight dynamics in comparison with the perfect foresight dynamics. A perfect foresight path is defined to be a feasible path to which every entrant takes a best response.

Definition 4 A feasible path ϕ is a *perfect foresight path* if for all $i = 1, \ldots, n$ and almost all $t \geq 0$, $\dot{\phi}_i(t) > -\lambda\phi_i(t)$ implies $a_i \in BR(\phi)(t)$.

Oyama (2000, Theorem 1) proved the existence of perfect foresight paths.

We state an immediate, but important observation for reference.

Claim 4.1 *A perfect foresight path is a rationalizable foresight path.*

We define stability concepts under perfect foresight in a similar manner: we say that a state $y \in \Delta$ is *accessible under perfect foresight* from another state $x \in \Delta$ if one of the following conditions is satisfied:

(i) there exists a perfect foresight path ϕ such that $\phi(0) = x$ and $\phi(t) = y$ for some $t \geq 0$;

(ii) there exists a sequence of states $\{y^k\}$ converging to y such that y^k is accessible under perfect foresight from x for all k; and

(iii) y is accessible under perfect foresight from some z which is in turn accessible under perfect foresight from x.

Note that the notion of accessibility makes more sense in the rationalizable foresight dynamics than in the perfect foresight dynamics. In particular, if two paths are concatenated, the new path itself becomes a rationalizable foresight path as long as the original paths are rationalizable, but this is not necessarily the case for perfect foresight paths. In the case of the rationalizable foresight dynamics, concatenation simply implies that the beliefs and the behavior pattern change at the point of concatenation because forecast error is allowed, while in the case of the perfect foresight dynamics, old agents no longer optimize against the altered future path.

Definition 5 A nonempty subset F^{**} of Δ is a *stable set under perfect foresight*, or a *PF-stable set*, if for any x in F^{**}, y is accessible from x under perfect foresight if and only if y is in F^{**}.

An action distribution $x^{**} \in \Delta$ is a *stable state under perfect foresight*, or a *PF-stable state*, if $\{x^{**}\}$ is a PF-stable set.

The same logic as that in the existence proof of RF-stable sets is applied to show the existence of PF-stable sets. It is thus stated without a proof.

Theorem 4.2 *Every game has at least one PF-stable set.*

Since the set of rationalizable foresight paths contains the set of perfect foresight paths from Claim 4.1, an RF-stable set is closed under accessibility in the perfect foresight dynamics. Accordingly, repeating the existence proof of RF-stable sets, but now using $\{R(x)\}_{x \in F^*}$ in place of $\{R(x)\}_{x \in \Delta}$ where F^* is an RF-stable set in question, we have the following.

Theorem 4.3 *Every RF-stable set contains at least one PF-stable set.*

It follows from this theorem that if an RF-stable set is a singleton, then it must be a PF-stable set.

Corollary 4.4 *An RF-stable state is a PF-stable state.*

Since an RF-stable state is an action distribution from which no rationalizable foresight path, *a fortiori* no perfect foresight path, departs, it is also a PF-stable state.

The converse of Corollary 4.4 is not true in general. Examples are (symmetric) potential games for $\delta > 0$. A PF-stable state is known to uniquely exist for a δ close to zero.

Fact 4.5 (Hofbauer and Sorger (1999)) *Assume that $u_{ij} = u_{ji}$, a fortiori, that the stage game is a potential game. Suppose that x^* is the unique* *maximizer of the potential function* $p(x) = (1/2)\sum_{ij} x_i u_{ij} x_j$ *over* Δ*. Then* x^* *is a PF-stable state if* $\delta > 0$*. If the degree of friction is sufficiently close to zero, then no other states are contained in any PF-stable set.*

However, there exist potential games such that the unique maximizer of the potential function is not RF-stable independently of the degree of friction. Indeed, in the potential game example of Example 3.1, there exists no RF-stable state for any degree of friction.

The converse of Theorem 4.3 is not true, either. We have the following counter-example.

Example 4.1 A 3×3 Game

Since the set of rationalizable foresight paths is larger than the set of perfect foresight paths, it is conceivable that there are some states from which the action distribution escapes under rationalizable foresight but not under perfect foresight. Using this logic, we construct an example in which the rationalizable foresight dynamics serves a sharper prediction than the perfect foresight dynamics.

Consider the following 3×3 game:

$$(u_{ij}) = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.1}$$

We show that for some degree of friction δ, the set of RF-stable states is a proper subset of the set of PF-stable states, and no other stable set exists.

Proposition 4.6 *Let the stage game be given by* (4.1)*. Then there exists a nonempty open set of the degrees of friction for which*

(a) $\{[a_1]\}$ *and* $\{(0, 1/2, 1/2)\}$ *are PF-stable sets, and no other PF-stable set exists; and*

(b) $\{[a_1]\}$ *is the unique RF-stable set.*

Proof. See Appendix. ▮

As in Proposition 3.4, it can be verified that the equilibrium state $(0, 1/2, 1/2)$ is not an RF-stable state. But there may exist an RF-stable set that contains this state. A natural candidate is

$$\{(\alpha_1, \alpha_2, \alpha_3) \in \Delta \mid \alpha_1 = 0,\ \alpha^* \leq \alpha_2 \leq 1 - \alpha^*\},$$

where $\alpha^* \in (0, 1/2)$ is given by (3.2). We would like to show that such a set is not RF-stable for some appropriate degree of friction. Looking at the payoff matrix given by (4.1), one may realize that if the action distribution moves from $(0, 1/2, 1/2)$ towards $[a_3]$, it is more likely than otherwise that a_1

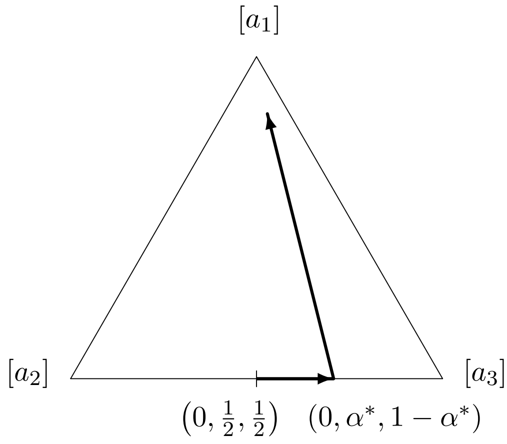


Figure 2: An escape path from $(0, 1/2, 1/2)$

becomes a best response on condition that some other agents are expected to take a_1 as well. A rationalizable foresight path performs this task (see Fig. 2), i.e., it brings the action distribution near $(0, \alpha^*, 1-\alpha^*)$. At this state, if the agents are reasonably patient, they may start taking a_1 if they form a belief that others will do the same. A direct departure from $(0, 1/2, 1/2)$ is harder than that from $(0, \alpha^*, 1-\alpha^*)$ since the agents taking a_1 incur more loss in the beginning along the path from $(0, 1/2, 1/2)$ to $[a_1]$ than along the path from $(0, \alpha^*, 1-\alpha^*)$ to $[a_1]$. Thus, we can find an open interval of the degrees of friction for which $(0, 1/2, 1/2)$ remains PF-stable, but belongs to no RF-stable set.

5 Properties of Rationalizable Foresight Paths

This section provides two characterizations of the set of rationalizable foresight paths. First, we have the following intuitive, but nontrivial statement.

Proposition 5.1 *A feasible path ϕ is contained in Φ^* if and only if for all i and almost all t such that $\dot{\phi}_i(t) > -\lambda\phi_i(t)$, there exists a path ψ in Φ^* such that $\psi(t) = \phi(t)$ and $a_i \in BR(\psi)(t)$.*

This proposition states that one can construct an infinite hierarchy of beliefs under which agents behave in a "rationalizable" manner if and only if the path is in Φ^*. This statement is nontrivial since $\phi \in \Phi^* = \bigcap_{k=0}^{\infty} \Phi^k$ merely implies that for all k, for all i, and for almost all t, there exists $\psi^{k-1} \in \Phi^{k-1}$ that satisfies a certain condition; that is to say, ψ^k's are different in general, and therefore, they may not be in Φ^*. In order to construct a desirable path $\psi \in \Phi^*$ by taking a subsequence of ψ^k's, we need to show that Φ^* is compact and the payoff function is continuous, which we demonstrate in Appendix.

Proof. See Appendix. ▮

Another property of rationalizable foresight paths is that we can view a point in Δ as a state variable, so that irrespective of the past history, only the present action distribution determines a possible future course of evolution. This is obvious once we observe that the environment is stationary and that newborns' beliefs are not bound by the past history.

The above observation suggests another way of constructing Φ^*. First, define correspondences $H^0 \colon \Delta \to \Delta$ and $\Psi^0 \colon \Delta \to \Phi^0$ as

$$H^0(z) = \Delta,$$

and

$$\Psi^0(z) = \{\phi \in \Phi^0 \mid \phi(0) = z\}.$$

For $k = 1, 2, 3, \cdots$, define H^k and Ψ^k recursively as

$$H^k(z) = \{\alpha \in \Delta \mid \alpha_i > 0 \Rightarrow \exists\, \psi \in \Psi^{k-1}(z) : a_i \in BR(\psi)\},$$

and

$$\Psi^k(z) = \{\phi \in \Psi^{k-1}(z) \mid \phi(0) = z \text{ and } \dot{\phi}(t) = \lambda(h(t) - \phi(t)), \quad h(t) \in H^k(\phi(t)) \ \text{ a.e.}\}.$$

Repeating this procedure inductively and taking the limit to obtain $H^*(z) = \bigcap_{k=0}^{\infty} H^k(z)$, we immediately have the following.

Proposition 5.2 *A feasible path ϕ is a rationalizable foresight path with initial state $x^0 \in \Delta$ if and only if $\phi(0) = x^0$, and*

$$\dot{\phi}(t) = \lambda(h(t) - \phi(t)), \quad h(t) \in H^*(\phi(t)) \quad a.e. \tag{5.1}$$

For each $z \in \Delta$, $H^*(z)$ is nonempty and compact, since $H^k(z)$ is nonempty and compact, and $H^{k+1}(z) \subset H^k(z)$ holds. Moreover, H^* is upper semi-continuous and convex-valued as H^k's are. Therefore, by the existence theorem for differential inclusion (see, e.g., Theorem 2.1.4 in Aubin and Cellina (1984, p. 101)), for each $x^0 \in \Delta$, there exists at least one rationalizable foresight path ϕ with $\phi(0) = x^0$.

6 Rationalizability in Static Societal Games

This section examines the relationship between the static rationalizability and the rationalizable foresight dynamics in a large population. We show that as δ goes to -1, i.e., as inertia vanishes, the RF-stable set becomes unique and coincides with the set of rationalizable strategy distributions.

Given a symmetric game (u_{ij}) played in a large population, construct inductively the set of rationalizable strategy distributions as follows. Let $\overline{H}^0 = \Delta$. Given $\overline{H}^{k-1}$ $(k = 1, 2, 3, \dots)$, define $\overline{H}^k$ to be

$$\overline{H}^k = \left\{ x \in \overline{H}^{k-1} \,|\, \forall i : \left[x_i > 0 \Rightarrow \exists\, y \in \overline{H}^{k-1} : a_i \in \overline{BR}(y) \right] \right\},$$

where $\overline{BR}(y)$ is the set of one-shot best responses to y in pure strategies, i.e.,

$$\overline{BR}(y) = \left\{ a_i \in A \,\Big|\, \sum_{k=1}^{n} y_k u_{ik} \geq \sum_{k=1}^{n} y_k u_{jk} \text{ for all } j \right\}.$$

Let $\overline{H}^* = \bigcap_{k=0}^{\infty} \overline{H}^k$. A strategy distribution in $\overline{H}^*$ is called a *rationalizable strategy distribution in the static societal game.* Note that a strategy distribution is rationalizable if and only if every pure strategy in the support survives iterated strict dominance. Note also that $\overline{H}^*$ is the convex hull of the pure rationalizable strategy distributions.

We now have the following result.[8]

Proposition 6.1 *For all generic (symmetric) games, there exists $\bar{\delta} > -1$ such that for all $\delta \in (-1, \bar{\delta})$, an RF-stable set uniquely exists and coincides with $\overline{H}^*$.*

Proof. In order to show that $\overline{H}^*$ is the unique RF-stable set, it is sufficient to verify that for δ sufficiently close to -1, any two strategy distributions in $\overline{H}^*$ are mutually accessible under rationalizable foresight. It is easy to see that no distribution outside $\overline{H}^*$ is contained in any RF-stable set.

Take any $[a_i] \in \overline{H}^*$. Then, we can take a strategy distribution $z^i \in \overline{H}^*$ such that $\overline{BR}(z^i) = \{a_i\}$, due to the genericity of the payoffs. Take any $w \in \Delta$, and consider the linear path ψ^i from w to z^i given by $\psi^i(t) = (1 - e^{-\lambda t}) z^i + e^{-\lambda t} w$, $t \geq 0$. As δ goes to -1, $V_j(\psi^i)$ converges to $\sum_k z^i_k u_{jk}$. Therefore, by way of the choice of z^i, $v_i(w|\delta) = V_i(\psi^i) - \max_{j \neq i} V_j(\psi^i)$ converges to a positive number as δ goes to -1. Since the set of such functions $\{v_i(\cdot|\delta)\}_{i,\delta}$ is equicontinuous and each function is defined on a compact set, there exists $\bar{\delta} > -1$ such that for all $\delta \in (-1, \bar{\delta})$, $v_i(w|\delta) > 0$ holds for all $[a_i] \in \overline{H}^*$ and all $w \in \Delta$. Take such a δ.

[8]We say that a certain property holds for all generic (symmetric) games if for any open set S of games in $\mathbb{R}^{n \times n}$, there exists a game $(u_{ij}) \in S$ that satisfies this property.

Take any two $x, y \in \overline{H}^*$. Consider a path from x to y: $\phi(t) = (1 - e^{-\lambda t})y + e^{-\lambda t}x$. We show that any such path is a rationalizable foresight path. For each t and each i with $y_i > 0$, let $\psi^{i,t}$ be a path given by $\psi^{i,t}(\tau) = (1 - e^{-\lambda\tau})z^i + e^{-\lambda\tau}\phi(t)$, where z^i satisfies $\{a_i\} = \overline{BR}(z^i)$. By way of the choice of δ, $a_i \in BR(\psi^{i,t})(t)$. Thus, ϕ is in Φ^1. Since ϕ is a path from x to y where x and y are arbitrarily chosen, every such path is in Φ^1. Repeating this procedure, we establish that every path connecting two distributions in $\overline{H}^*$ is in Φ^k for all k, and hence, in Φ^*. Thus, every distribution in $\overline{H}^*$ is accessible from any distribution in $\overline{H}^*$, which implies that the unique RF-stable set is identical with $\overline{H}^*$. ▌

Consider an agent anticipating a path ψ that moves from x to y. If δ is close to -1, then he puts almost all weight on the distant future, i.e., $V_i(\psi)$ is approximated by $\sum_k y_k u_{ik}$. The above proof essentially utilizes this observation. Note also that as δ goes to infinity, $V_i(\psi)$ converges to $\sum_k x_k u_{ik}$, and that if δ is close to zero, $V_i(\psi)$ is approximated by the average of the one-shot payoffs along the trajectory; in particular, $V_i(\psi)$ is close to $\sum_k((x_k + y_k)/2)u_{ik}$ if the path moves straight towards y from x.

Recall that along a rationalizable foresight path, each agent believes that a single path of action distribution will realize with probability one, and chooses a pure strategy. Still, mixed action distributions can be observed in the society since there are a continuum of agents who entertain different beliefs. In this way, mixed strategies and mixed beliefs in the standard rationalizability (Bernheim (1984) and Pearce (1984)) are replaced by the population distributions of actions and beliefs.

Note that our definition of rationalizable strategy distributions is different from the standard definition of rationalizable strategies. In the definition of the standard rationalizability, $\overline{H}^k$'s are replaced by $\hat{H}^0 = \Delta$ and

$$\hat{H}^k = \left\{x \in \hat{H}^{k-1} \,|\, \exists\, y \in \operatorname{co} \hat{H}^{k-1} : \left[x_i > 0 \Rightarrow a_i \in \overline{BR}(y)\right]\right\}$$

for $k \geq 1$, where the symbol "co" stands for convex hull.

Because of this difference, the set of rationalizable strategy distributions may not be the same as the set of standard rationalizable strategies. Let us consider the game:

$$(u_{ij}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 2 & 2 \end{pmatrix}. \tag{6.1}$$

The mixed strategy, or strategy distribution, $(1/2, 1/2, 0)$ is a rationalizable strategy distribution in the societal game, but not a standard rationalizable strategy. In the societal game, it is rationalizable since the half of the population believe that action a_1 will be chosen, while the other half believe that action a_2 will be chosen. On the other hand, it is not rationalizable in the standard definition since there is no mixed strategy to which $(1/2, 1/2, 0)$ is a best response.

7 p-Dominance and RF-Stability

This section relates the RF-stability to p-dominance. The notion of p-dominance, introduced by Morris, Rob, and Shin (1995), is a generalization of risk-dominance for games with more than two actions.[9]

Definition 6 Action profile (a_i, a_i) is a *p-dominant equilibrium*[10] of symmetric $n \times n$ game (u_{ij}) if for all $j \neq i$, and all $\pi \in \Delta$ with $\pi_i > p$,

$$\sum_{k=1}^{n} \pi_k u_{ik} > \sum_{k=1}^{n} \pi_k u_{jk}.$$

The following proposition provides some sufficient conditions under which a p-dominant equilibrium corresponds to an RF-stable state.

Proposition 7.1 *Suppose that (a_i, a_i) is a p-dominant equilibrium of the stage game.*

(a) *If $p < (1+\delta)/(2+\delta)$, then $\{[a_i]\}$ is an RF-stable set.*

(b) *If $p \leq 1/(2+\delta)$, then $[a_i]$ is contained in the unique RF-stable set.*

(c) *If $p < \min\{(1+\delta)/(2+\delta), 1/(2+\delta)\}$, then $\{[a_i]\}$ is the unique RF-stable set.*

The condition in (a) assures that there is no rationalizable foresight path away from $[a_i]$, while the condition in (b) implies that $[a_i]$ is accessible under rationalizable foresight from any state in Δ. The condition in (c) combines these two conditions.

Proof. (a) The proof is a direct application of Lemma 2 in Oyama (2000) to the rationalizable foresight dynamics.

We first show that $H^1([a_i]) = \{[a_i]\}$. Take any feasible path ψ with $\psi(0) = [a_i]$. It is sufficient to show that if $p < (1+\delta)/(2+\delta)$, then $BR(\psi) = \{a_i\}$.

The expected discounted payoff to action a_j at time 0 along the path ψ is written as

$$V_j(\psi) = \sum_{k=1}^{n} \pi_k u_{jk},$$

[9] In 2×2, a p-dominant equilibrium with $p < 1/2$ coincides with a risk-dominant equilibrium.

[10] This is called a *strict* p-dominant equilibrium in Kajii and Morris (1997, Definition 5.4).

where $\pi \in \Delta$ is given by

$$\pi_k = (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} \psi_k(s)\, ds.$$

Since $\psi(0) = [a_i]$ holds, we have $\psi_i(s) \geq e^{-\lambda s}$, and therefore,

$$\begin{aligned} \pi_i &\geq (\lambda + \theta) \int_0^\infty e^{-(\lambda+\theta)s} e^{-\lambda s}\, ds \\ &= \frac{1+\delta}{2+\delta} > p. \end{aligned}$$

It follows that $V_i(\psi) > V_j(\psi)$ for all $j \neq i$ from the assumption that (a_i, a_i) is a p-dominant equilibrium.

We then have $H^k([a_i]) = \{[a_i]\}$ for $k = 2, 3, \ldots$, so that $H^*([a_i]) = \{[a_i]\}$. Therefore, the unique rationalizable foresight path from $[a_i]$ is the path ϕ such that $\phi(t) = [a_i]$ for all t.

(b) This follows from Lemma 1 in Oyama (2000), which exhibits that if $p \leq 1/(2+\delta)$, then $[a_i]$ is accessible from any state in Δ under perfect foresight, *a fortiori* under rationalizable foresight.

(c) This follows from (a) and (b). ▮

Proposition 7.1 implies that a p-dominant equilibrium with $p < 1/2$ is always contained in some RF-stable set.

8 Complete Characterization for 2×2 Games

This section completely characterizes RF-stable sets for the class of games with two actions:

$$(u_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{8.1}$$

If one action weakly dominates the other, the state in which every agent takes the dominant action constitutes the unique RF-stable set as well as the unique PF-stable set. Also, if $a = c$ and $b = d$ hold, then the entire space becomes a stable set under either dynamics.

There are two nontrivial cases to consider: (i) $a > c$ and $d > b$, i.e., coordination games; and (ii) $a < c$ and $d < b$, i.e., games with a unique symmetric Nash equilibrium.

8.1 Coordination Games

This subsection studies coordination games. In this case, (8.1) can be normalized to

$$(u_{ij}) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad a > 0,\, d > 0. \tag{8.2}$$

Let $\mu = d/(a+d)$. Note that (a_1, a_1) is μ-dominant. We assume without loss of generality that $\mu \leq 1/2$. The following is a direct application of Proposition 7.1, where (a) makes use of the connectedness of RF-stable sets as well.

Proposition 8.1 *Let the stage game be given by* (8.2). *Suppose that* $\mu = d/(a+d) \leq 1/2$. *Then we have the following:*

(a) *If* $\delta > (1-2\mu)/\mu$, *then* $\{[a_1]\}$ *and* $\{[a_2]\}$ *are RF-stable sets, and no other RF-stable set exists.*

(b) *If* $-(1-2\mu)/(1-\mu) < \delta \leq (1-2\mu)/\mu$, *then* $\{[a_1]\}$ *is the unique RF-stable set.*

(c) *If* $-1 < \delta \leq -(1-2\mu)/(1-\mu)$, *then* Δ *is the RF-stable set.*

If the degree of friction, δ, is large, both strict Nash equilibrium states are RF-stable. If the friction is not too large, but still positive (or negative but close to zero), then the rationalizable foresight dynamics selects $[a_1]$, the risk-dominant equilibrium, over $[a_2]$ provided that $\mu < 1/2$. If the agents care more about the future than the present, i.e., θ is close to $-\lambda$, then the action distribution moves around and the entire space becomes the unique RF-stable set.

8.2 Games with a Unique Symmetric Equilibrium

This subsection considers the case where $a < c$ and $d < b$. In this case, (8.1) can be normalized to

$$(u_{ij}) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad b > 0,\, c > 0. \tag{8.3}$$

We assume without loss of generality that $b < c$ holds.[11] Denote by $\hat{x} = (\hat{x}_1, \hat{x}_2) \in \Delta$ the unique equilibrium state, i.e.,

$$(\hat{x}_1, \hat{x}_2) = \left(\frac{b}{b+c}, \frac{c}{b+c}\right).$$

Note that $\hat{x}_1 < 1/2 < \hat{x}_2$.

The RF-stable set is of the form:

$$\{(x_1, x_2) \in \Delta \,|\, \hat{\alpha} \leq x_1 \leq 1 - \hat{\beta}\}.$$

Our task is to determine the endpoints $(\hat{\alpha}, 1-\hat{\alpha})$ and $(1-\hat{\beta}, \hat{\beta})$. The state $(\hat{\alpha}, 1-\hat{\alpha})$ (resp. $(1-\hat{\beta}, \hat{\beta})$) is the closest to $[a_2]$ (resp. $[a_1]$) such that a_2 (resp. a_1) is a best response to the path that starts there, moves towards to $[a_1]$ (resp. $[a_2]$) at the maximum speed, and stays at $(1-\hat{\beta}, \hat{\beta})$ (resp. $(\hat{\alpha}, 1-\hat{\alpha})$) once reached.

[11] The case of $b = c$ has already been analyzed in Example 3.1.

For this purpose, let F and G be two functions from $[0, \hat{x}_1] \times [0, \hat{x}_2]$ into $\mathbb{R}$ defined as:

$$F(\alpha, \beta) = \frac{1+\delta}{2+\delta}(1-\alpha)^{2+\delta} + \frac{1}{2+\delta}\beta^{2+\delta} - \hat{x}_2(1-\alpha)^{1+\delta},$$
$$G(\alpha, \beta) = \frac{1+\delta}{2+\delta}(1-\beta)^{2+\delta} + \frac{1}{2+\delta}\alpha^{2+\delta} - \hat{x}_1(1-\beta)^{1+\delta}.$$

The sign of $F(\alpha, \beta)$ is identical with that of the payoff difference $V_1(\phi) - V_2(\phi)$ along the path ϕ that starts with $(\alpha, 1-\alpha)$, moves towards $[a_1]$ at the maximum speed, and stays at $(1-\beta, \beta)$ once reached. Similarly, $G(\alpha, \beta)$ has the same sign as the payoff difference $V_2(\psi) - V_1(\psi)$ where ψ is the path that starts with $(1-\beta, \beta)$, moves towards $[a_2]$ at the maximum speed, and stays at $(\alpha, 1-\alpha)$ once reached.

Let $(\alpha^\star, \beta^\star) \in (0, \hat{x}_1) \times (0, \hat{x}_2)$ be the unique solution to:

$$F(\alpha^\star, \beta^\star) = 0,$$
$$G(\alpha^\star, \beta^\star) = 0, \tag{8.4}$$

if it exists (see Fig. 3(a)).[12] If it does not, then solve the following system (see Fig. 3(b)):

$$F(0, \beta^{\star\star}) \leq 0,$$
$$G(0, \beta^{\star\star}) = 0. \tag{8.5}$$

In the latter case, we have a unique solution $\beta^{\star\star} = 1 - \hat{x}_1(2+\delta)/(1+\delta) \in (0, \hat{x}_2)$ to (8.5) if and only if $\delta > -(1-2\hat{x}_1)/(1-\hat{x}_1)$. Write

$$\delta^{\star\star} = -\frac{1-2\hat{x}_1}{1-\hat{x}_1}.$$

If $\delta \leq \delta^{\star\star}$, then $F(0,0) < 0$ and $G(0,0) < 0$, so that the entire space Δ becomes the RF-stable set.

[12]It can be verified that $F(\hat{x}_1, \hat{x}_2) = G(\hat{x}_1, \hat{x}_2) = 0$ holds. We also have

$$\frac{\partial F}{\partial \alpha} = -(1+\delta)(1-\alpha)^\delta(\hat{x}_1 - \alpha),$$

which is negative for $\alpha \in (0, \hat{x}_1)$, and

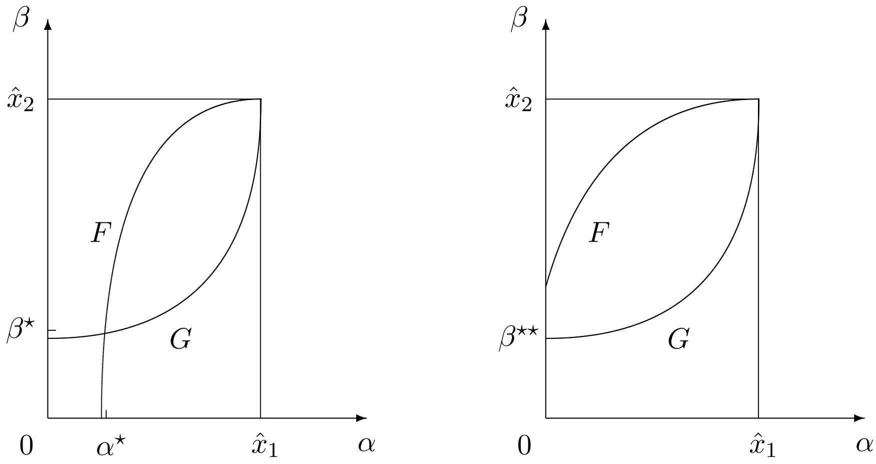
$$\frac{\partial F}{\partial \beta} = \beta^{1+\delta} > 0.$$

Let $\hat{\alpha}(\beta)$ satisfy $F(\hat{\alpha}(\beta), \beta) = 0$. Then, one can verify that $\hat{\alpha}$ is well defined for $\beta \in (0, \hat{x}_2)$, that

$$\frac{d\hat{\alpha}}{d\beta} = -\frac{(\partial F/\partial\beta)(\hat{\alpha}(\beta), \beta)}{(\partial F/\partial\alpha)(\hat{\alpha}(\beta), \beta)}$$

is positive, and that $\hat{\alpha}(\beta)$ is convex in β with $\hat{\alpha}'(0) = 0$ and $\lim_{\beta \to \hat{x}_2} \hat{\alpha}'(\beta) = \infty$.

Similarly, let $\hat{\beta}(\alpha)$ satisfy $G(\alpha, \hat{\beta}(\alpha)) = 0$. In a similar manner, one can verify that $\hat{\beta}$ is well-defined, and that it is increasing and convex in α with $\hat{\beta}'(0) = 0$ and $\lim_{\alpha \to \hat{x}_1} \hat{\beta}'(\alpha) = \infty$.



(a) $\delta > \delta^\star$

(b) $\delta^{\star\star} < \delta < \delta^\star$

Figure 3: Graphs of $F(\alpha, \beta) = 0$ and $G(\alpha, \beta) = 0$

Let $\delta^\star$ be the unique solution to

$$F\left(0, 1 - \hat{x}_1 \frac{2 + \delta^\star}{1 + \delta^\star}\right) = 0. \tag{8.6}$$

Indeed, it can be verified that $F(0, 1 - \hat{x}_1(2 + \delta)/(1 + \delta))$ is increasing in δ, and has a negative value, $-2(\hat{x}_2 - 1/2)(1 - \hat{x}_2)$, at $\delta = 0$, and a positive value, $1 - \hat{x}_2$, in the limit as $\delta \to \infty$. Observe that (8.4) has the solution in $(0, \hat{x}_1) \times (0, \hat{x}_2)$ if and only if $\delta > \delta^\star$.

Proposition 8.2 *Let the stage game be given by* (8.3). *Suppose that* $b < c$. *Then we have the following:*

(a) *If* $\delta > \delta^\star$, *then the unique RF-stable set is*

$$\{(x_1, x_2) \in \Delta \mid \alpha^\star \leq x_1 \leq 1 - \beta^\star\}.$$

(b) *If* $\delta^{\star\star} < \delta \leq \delta^\star$, *then the unique RF-stable set is*

$$\{(x_1, x_2) \in \Delta \mid 0 \leq x_1 \leq 1 - \beta^{\star\star}\}.$$

(c) *If* $-1 < \delta \leq \delta^{\star\star}$, *then the unique RF-stable set is* Δ.

We omit the proof of this proposition since it is essentially the same as, albeit a little more complicated than, that of Proposition 3.4. As in Example 3.1, the RF-stable set always contains non-equilibrium states, and one needs a rationalizable foresight path that is not a perfect foresight path in order to escape from the equilibrium state. It is worth noting that $\{\hat{x}\}$ is the unique PF-stable set for any degree of friction. This follows from the facts that independently of the friction, $\hat{x}$ is accessible under perfect foresight from any state in Δ, and that there exists no perfect foresight path that escapes from $\hat{x}$, which can be proved in the same way as the proof of Lemma A.2 in Appendix.

9 Conclusion

We have proposed the rationalizable foresight dynamics and defined the stability concepts under the dynamics. We have then discussed its properties, including the existence of stable sets. The rationalizable foresight dynamics is intended to overcome, albeit in limited situations, some of the shortcomings that three theories, the equilibrium theory, the theory of rationalizability, and the evolutionary game theory, have in different ways. By introducing the concept of rationalizable foresight, we have abandoned the requirement that agents' beliefs should be coordinated as assumed in the equilibrium theory. We have reclaimed the notion of rationality that the evolutionary game theory had discarded. Finally, we have mitigated the poor performance of the theory of rationalizability as prediction device by incorporating inertia into the system as in the evolutionary game theory.

We have illustrated by way of an example that the RF-stability gives a sharper prediction than the PF-stability. A key observation for this result is that, in general, it is easier to escape from an action distribution under rationalizable foresight than under perfect foresight. Accordingly, there may exist a state from which a rationalizable foresight path escapes but no perfect foresight path does.

In our analysis, inertia plays a key role. If there is no inertia, then the behavior pattern may jump around, and there is no hope for sharp prediction. Indeed, for a sufficiently small effective discount rate, the unique RF-stable set coincides with the set of rationalizable strategy distributions of the corresponding static societal game. Note here that mixed strategies and mixed beliefs in the standard rationalizability are replaced by the distributions of actions and beliefs.

We have limited our analysis to a special class of dynamic environments since our aim is to present a conceptual framework that allows us to examine the situations in which rationality is common knowledge among infinitesimal agents, but beliefs may not be coordinated with each other, as opposed to providing a universal framework. How the present analysis can be extended to general situations remains to be seen in the future.

Appendix

A.1 Proofs of Propositions 3.4 and 3.5

Proof of Proposition 3.4. We divide the proof into two cases, (i) $\delta > 0$ and (ii) $-1 < \delta \leq 0$.
(i) $\delta > 0$: We would like to show that $H^*(\cdot)$ is given by

$$H^*(z) = \begin{cases} \{[a_1]\} & \text{if } z_1 < \alpha^*, \\ \Delta & \text{if } \alpha^* \leq z_1 \leq 1 - \alpha^*, \\ \{[a_2]\} & \text{if } z_1 > 1 - \alpha^*, \end{cases}$$

where α^* satisfies (3.2). For this purpose, it suffices to show that

$$H^k(z) = \begin{cases} \{[a_1]\} & \text{if } z_1 < \alpha^k, \\ \Delta & \text{if } \alpha^k \leq z_1 \leq 1 - \alpha^k, \\ \{[a_2]\} & \text{if } z_1 > 1 - \alpha^k \end{cases} \tag{A.1}$$

holds for all $k = 0, 1, 2, \ldots$, where $\{\alpha^k\}_{k=0}^{\infty}$ is given by $\alpha^0 = 0$ and

$$\frac{1+\delta}{2+\delta}(1 - \alpha^k) + \frac{1}{2+\delta}\frac{(\alpha^{k-1})^{2+\delta}}{(1-\alpha^k)^{1+\delta}} = \frac{1}{2}, \quad 0 < \alpha^k < \frac{1}{2}, \tag{A.2}$$

and that $\lim_{k\to\infty} \alpha^k = \alpha^*$.

Let $H^0(\cdot) \equiv \Delta$, and let $H^k(\cdot)$ be given by (A.1). We construct H^{k+1} from H^k for $k = 0, 1, 2, \ldots$. By symmetry, we consider only those z's with $z_1 \in [0, 1/2]$. Take any such $z = (z_1, z_2)$.

Consider first the path $\psi \in \Psi^0(z)$ given by

$$\psi(t) = \begin{cases} [a_1] - ([a_1] - z)\, e^{-\lambda t} & \text{if } t < T_1, \\ \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if } t \geq T_1, \end{cases}$$

where T_1 satisfies $z_2\, e^{-\lambda T_1} = 1/2$. Since $V_1(\psi) \geq V_2(\psi)$ holds, $[a_1]$ is always in $H^{k+1}(z)$.

We then check if there exists a path to which a_2 is a best response. The best scenario for a_2 is expressed by the path ψ such that

$$\psi(t) = \begin{cases} [a_1] - ([a_1] - z)\, e^{-\lambda t} & \text{if } t < T_2, \\ (1 - \alpha^k, \alpha^k) & \text{if } t \geq T_2, \end{cases} \tag{A.3}$$

where $T_2 \in (0, \infty]$ is given by

$$(1 - z_1)\, e^{-\lambda T_2} = \alpha^k.$$

The expected discounted payoffs along this path are calculated as:

$$\begin{aligned} V_1(\psi) &= (\lambda+\theta)\int_0^{T_2} e^{-(\lambda+\theta)s}\, z_2\, e^{-\lambda s}\, ds + (\lambda+\theta)\int_{T_2}^{\infty} e^{-(\lambda+\theta)s}\, \alpha^k\, ds \\ &= \frac{1+\delta}{2+\delta}\,(1-z_1) + \frac{1}{2+\delta}\,\frac{(\alpha^k)^{2+\delta}}{(1-z_1)^{1+\delta}}; \\ V_2(\psi) &= (\lambda+\theta)\int_0^{T_2} e^{-(\lambda+\theta)s}\,\{1-(1-z_1)\}\, e^{-\lambda s}\, ds \\ &\qquad\qquad + (\lambda+\theta)\int_{T_2}^{\infty} e^{-(\lambda+\theta)s}\,(1-\alpha^k)\, ds \\ &= 1 - \frac{1+\delta}{2+\delta}\,(1-z_1) - \frac{1}{2+\delta}\,\frac{(\alpha^k)^{2+\delta}}{(1-z_1)^{1+\delta}}. \end{aligned}$$

It follows that $V_1(\psi) \leq V_2(\psi)$ if and only if $z_1 \geq \alpha^{k+1}$, where $\alpha^{k+1} \in (0, 1/2)$ is given by

$$\frac{1+\delta}{2+\delta}\,(1-\alpha^{k+1}) + \frac{1}{2+\delta}\,\frac{(\alpha^k)^{2+\delta}}{(1-\alpha^{k+1})^{1+\delta}} = \frac{1}{2}.$$

Note that the left hand side is decreasing in $\alpha^{k+1} \in (0, 1/2)$. Thus, we have proved that (A.1) holds for all $k = 0, 1, 2, \ldots$.

We now show by induction that $\alpha^0 < \alpha^1 < \cdots (< 1/2)$, and hence, $\{\alpha^k\}_{k=0}^{\infty}$ has the limit, which is equal to the unique solution α^* to (3.2). First, $\alpha^0 = 0 < \alpha^1 = \delta/(2+2\delta)$. Suppose next that $\alpha^{k-1} < \alpha^k$. For $(\alpha, \beta) \in [0, 1/2] \times [0, 1/2]$, define

$$f(\alpha, \beta) = \frac{1+\delta}{2+\delta}(1-\alpha) + \frac{1}{2+\delta}\frac{\beta^{2+\delta}}{(1-\alpha)^{1+\delta}} - \frac{1}{2}.$$

Recall that $f(\alpha^{k+1}, \alpha^k) = 0$. Since $f(\alpha, \beta)$ is increasing in β,

$$f(\alpha^k, \alpha^k) > f(\alpha^k, \alpha^{k-1}) = 0,$$

so that $f(\alpha^k, \alpha^k) > f(\alpha^{k+1}, \alpha^k)\,(= 0)$. Since $f(\alpha, \beta)$ is decreasing in α, we have $\alpha^k < \alpha^{k+1}$.

(ii) $-1 < \delta \leq 0$: It suffices to verify that $[a_1]$ is accessible from $[a_2]$ and vice versa. Similarly as above, at any $z \in \Delta$, a_1 (resp. a_2) is a best response to the path ψ given by $\psi(t) = [a_2] - ([a_2] - z)\, e^{-\lambda t}$ (resp. $\psi(t) = [a_1] - ([a_1] - z)\, e^{-\lambda t}$). ▮

Proof of Proposition 3.5. We show that $\alpha^*(\delta)$ is increasing in δ. Define

$$g(\alpha, \delta) = (1-\alpha)\left\{\frac{1+\delta}{2+\delta} + \frac{1}{2+\delta}\left(\frac{\alpha}{1-\alpha}\right)^{2+\delta} - \frac{1}{2(1-\alpha)}\right\}.$$

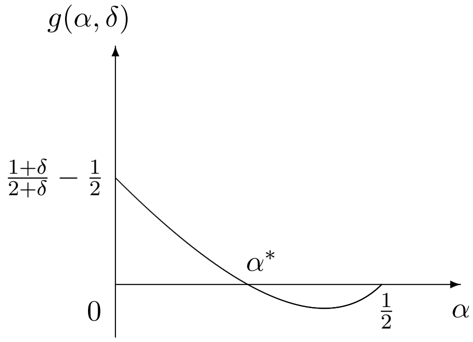


Figure 4: Graph of $g(\alpha, \delta)$

Recall that, by (3.2), $\alpha^*(\delta)$ satisfies $g(\alpha^*(\delta), \delta) = 0$. We have

$$\frac{\partial g}{\partial \alpha}(\alpha, \delta) = -\frac{1+\delta}{2+\delta} + \left(\frac{\alpha}{1-\alpha}\right)^{1+\delta} + \frac{1+\delta}{2+\delta}\left(\frac{\alpha}{1-\alpha}\right)^{2+\delta},$$

which is increasing in α. Since $g(1/2, \delta) = 0$, $(\partial g/\partial \alpha)(0, \delta) < 0$, and $(\partial g/\partial \alpha)(1/2, \delta) > 0$, we have

$$\frac{\partial g}{\partial \alpha}(\alpha^*(\delta), \delta) < 0,$$

as depicted in Fig 4.

On the other hand,

$$\frac{\partial g}{\partial \delta}(\alpha, \delta) = \frac{(1-\alpha)}{(2+\delta)^2}\left[1 - \left(\frac{\alpha}{1-\alpha}\right)^{2+\delta}\left\{1 - (2+\delta)\log\left(\frac{\alpha}{1-\alpha}\right)\right\}\right] > 0$$

for all $\delta > 0$ and all $\alpha \in (0, 1/2)$, since

$$\left(\frac{\alpha}{1-\alpha}\right)^{-(2+\delta)} > 1 - (2+\delta)\log\left(\frac{\alpha}{1-\alpha}\right). \tag{A.4}$$

It follows that

$$\frac{d\,\alpha^*(\delta)}{d\,\delta} = -\frac{(\partial g/\partial \delta)(\alpha^*(\delta), \delta)}{(\partial g/\partial \alpha)(\alpha^*(\delta), \delta)} > 0. \tag{A.5}$$

Next, due to $g(0, 0) = 0$, the continuity of g, and (A.5), we have

$$\lim_{\delta \to 0} \alpha^*(\delta) = 0.$$

Finally, we have

$$\lim_{\delta \to \infty} \alpha^*(\delta) = \frac{1}{2},$$

since from the proof of Proposition 3.4, $\alpha^1 < \alpha^*(\delta) < 1/2$ and $\alpha^1 = \delta/(2 + 2\delta) \to 1/2$ as $\delta \to \infty$. ▌

A.2 Proof of Proposition 4.6

In order to prove Proposition 4.6, we need a few lemmata, which are given below.

Lemma A.1 *If $\delta > 1$, then $[a_1]$ is both PF-stable and RF-stable.*

Proof. This lemma follows from Proposition 7.1. ▌

Lemma A.2 *Any perfect foresight path ϕ with $\phi_2(0) = \phi_3(0)$ satisfies $\phi_2(t) = \phi_3(t)$ for all $t > 0$.*

Proof. Take any perfect foresight path ϕ with $\phi_2(0) = \phi_3(0)$. We suppose that $\phi_2(t^0) < \phi_3(t^0)$ for some $t^0 > 0$. Define $\underline{t}$ to be

$$\underline{t} = \inf\{t < t^0 \,|\, \forall\, s \in (t, t^0) : \phi_2(s) < \phi_3(s)\}.$$

Note that $\underline{t} < t^0$ and $\phi_2(\underline{t}) = \phi_3(\underline{t})$ due to the continuity of the perfect foresight path.

Claim A.1. There exists $t \in (\underline{t}, t^0)$ such that $V_2(t) \leq V_3(t)$.
If $V_2(t) > V_3(t)$ for all $t \in (\underline{t}, t^0)$, then

$$\begin{aligned}\phi_2(t^0) &\geq \phi_2(\underline{t})\, e^{-\lambda(t-\underline{t})},\\ \phi_3(t^0) &= \phi_3(\underline{t})\, e^{-\lambda(t-\underline{t})},\end{aligned}$$

implying that $\phi_2(t^0) \geq \phi_3(t^0)$. This contradicts the definition of t^0, completing the proof of Claim A.1.

We denote by T^1 such a t in Claim A.1, i.e., $T^1 \in (\underline{t}, t^0)$, and

$$V_2(T^1) - V_3(T^1) \leq 0. \tag{A.6}$$

Claim A.2. There exists $t > t^0$ such that $\phi_2(t) \geq \phi_3(t)$.
Suppose the contrary. Then $\phi_2(t) < \phi_3(t)$ for all $t > T^1$. It follows that

$$V_2(T^1) - V_3(T^1) = (\lambda + \theta) \int_{T^1}^{\infty} e^{-(\lambda+\theta)(s-T^1)}\, (\phi_3(s) - \phi_2(s))\, ds > 0.$$

This contradicts (A.6), completing the proof of Claim A.2.

Define $\bar{t}\,(> t^0)$ to be

$$\bar{t} = \sup\{t > t^0 \,|\, \forall\, s \in (t^0, t) : \phi_2(s) < \phi_3(s)\},$$

which is finite due to Claim A.2. Note again that $\phi_2(\bar{t}) = \phi_3(\bar{t})$.

Claim A.3. There exists $t \in (t^0, \bar{t})$ such that $V_2(t) \geq V_3(t)$.

If $V_2(t) < V_3(t)$ for all $t \in (t^0, \bar{t})$, then

$$\phi_2(\bar{t}) = \phi_2(t^0)\, e^{-\lambda(\bar{t}-t^0)},$$
$$\phi_3(\bar{t}) \geq \phi_3(t^0)\, e^{-\lambda(\bar{t}-t^0)},$$

implying that $\phi_2(t^0) \geq \phi_3(t^0)$. This contradicts the definition of t^0, completing the proof of Claim A.3.

We denote by T^2 such a t in Claim A.3, i.e., $T^2 \in (t^0, \bar{t})$, and

$$V_2(T^2) - V_3(T^2) \geq 0. \tag{A.7}$$

Since $\phi_2(t) < \phi_3(t)$ for all $t \in (T^1, T^2)$,

$$\begin{aligned} V_2(T^1) - V_3(T^1) &= (\lambda+\theta) \int_{T^1}^{\infty} e^{-(\lambda+\theta)(s-T^1)} (\phi_3(s) - \phi_2(s))\, ds \\ &= (\lambda+\theta) \int_{T^1}^{T^2} e^{-(\lambda+\theta)(s-T^1)} (\phi_3(s) - \phi_2(s))\, ds \\ &\qquad + e^{-(\lambda+\theta)(T^2-T^1)}(V_2(T^2) - V_3(T^2)) \\ &> e^{-(\lambda+\theta)(T^2-T^1)}(V_2(T^2) - V_3(T^2)) \geq 0, \end{aligned}$$

where the last inequality follows from (A.7). This contradicts (A.6). ▌

Lemma A.3 *If $\delta > 1$, then $(0, 1/2, 1/2)$ is a PF-stable state.*

Proof. Take any perfect foresight path ϕ with $\phi(0) = (0, 1/2, 1/2)$. Due to Lemma A.2, it must satisfy $\phi_2(t) = \phi_3(t)$ for all t. Hence,

$$\begin{aligned} V_1(\phi) &= (\lambda+\theta) \int_0^{\infty} e^{-(\lambda+\theta)s}\, \phi_1(s)\, ds \\ &\leq (\lambda+\theta) \int_0^{\infty} e^{-(\lambda+\theta)s}\, (1 - e^{-\lambda s})\, ds = \frac{1}{2+\delta}, \end{aligned}$$

and

$$\begin{aligned} V_2(\phi) = V_3(\phi) &= (\lambda+\theta) \int_0^{\infty} e^{-(\lambda+\theta)s}\, \phi_2(s)\, ds \\ &\geq (\lambda+\theta) \int_0^{\infty} e^{-(\lambda+\theta)s}\, \frac{1}{2}\, e^{-\lambda s}\, ds = \frac{1}{2} \cdot \frac{1+\delta}{2+\delta}, \end{aligned}$$

so that $V_1(\phi) < V_2(\phi) = V_3(\phi)$ for $\delta > 1$. It follows that $\phi_1(t) = 0$ and, therefore, $\phi_2(t) = \phi_3(t) = 1/2$. ▌

Lemma A.4 *There exists $\bar{\delta} > 1$ such that if $1 < \delta \leq \bar{\delta}$, then no RF-stable set contains $(0, 1/2, 1/2)$.*

Proof. Using the same logic as the one in Example 3.1, we can verify that $(0, \alpha^*(\delta), 1-\alpha^*(\delta))$ is accessible under rationalizable foresight from $(0, 1/2, 1/2)$. It is therefore sufficient to show that the linear path ϕ from $(0, \alpha^*(\delta), 1-\alpha^*(\delta))$ to $[a_1]$ is a rationalizable foresight path for a $\delta > 1$ sufficiently close to 1. Here, $\alpha^* = \alpha^*(\delta)$ is given by $g(\alpha^*, \delta) = 0$, where

$$g(\alpha, \delta) = \frac{1+\delta}{2+\delta}(1-\alpha) + \frac{1}{2+\delta}\frac{(\alpha)^{2+\delta}}{(1-\alpha)^{1+\delta}} - \frac{1}{2}, \quad 0 < \alpha < \frac{1}{2}. \tag{A.8}$$

Recall from the proof of Proposition 3.5 that $g(\alpha, \delta) < 0$ if and only if $\alpha^* < \alpha < 1/2$. Along the path ϕ,

$$\begin{aligned} V_1(\phi) &= 1 - 2\alpha^* \frac{1+\delta}{2+\delta}, \\ V_2(\phi) &= (1-\alpha^*)\frac{1+\delta}{2+\delta}, \\ V_3(\phi) &= \alpha^* \frac{1+\delta}{2+\delta}. \end{aligned}$$

It is sufficient to demonstrate that there exists $\bar{\delta}$ such that if $1 < \delta \leq \bar{\delta}$, then $V_1(\phi) \geq V_2(\phi)$, or equivalently,

$$\alpha^* \leq \frac{1}{1+\delta}. \tag{A.9}$$

We find the range of $\delta\,(>1)$ such that $g(1/(1+\delta), \delta) \leq 0$. Since

$$g\Big(\frac{1}{1+\delta}, \delta\Big) = -\frac{1}{(1+\delta)(2+\delta)}\Big\{\frac{(1+\delta)(2-\delta)}{2} - \delta^{-(1+\delta)}\Big\},$$

$g(1/(1+\delta), \delta) \leq 0$ if and only if

$$(1+\delta)(2-\delta)\delta^{1+\delta} - 2 \geq 0.$$

Write

$$h(\delta) = (1+\delta)(2-\delta)\delta^{1+\delta} - 2.$$

Then, we have

$$h'(\delta) = \delta^{\delta}\{\delta(1+\delta)(2-\delta)\log\delta - (-2 - 4\delta + 2\delta^2 + \delta^3)\}. \tag{A.10}$$

Since $h(1) = 0$ and $h'(1) > 0$, there exists $\bar{\delta} > 1$ such that for all $\delta \in (1, \bar{\delta}]$, $h(\delta) \geq 0$, i.e., $g(1/(1+\delta), \delta) \leq 0$. Thus, for such a δ, the linear path from $(0, \alpha^*(\delta), 1-\alpha^*(\delta))$ to $[a_1]$ is a rationalizable foresight path. From Lemma A.1, no RF-stable set contains $(0, 1/2, 1/2)$. ▮

Proof of Proposition 4.6. Combining these lemmata, we complete the proof of the proposition. ▮

A.3 Proof of Proposition 5.1

We introduce a Banach space X, the set of bounded functions $f\colon [0,\infty) \to \mathbb{R}^n$ with the norm

$$\|f\|_r = \sup_{t\geq 0} e^{-rt}|f(t)|$$

for $r > 0$.

Lemma A.5 *$\Phi^0 \subset X$ is compact.*

Proof. Observe first that due to the Ascoli-Arzelà theorem,

$$K = \{\phi\colon [0,\infty) \to \Delta \subset \mathbb{R}^n \mid \phi \text{ is Lipschitz with constant } \lambda\}$$

is a compact subset of X. Thus, it is sufficient to show that Φ^0, which is a subset of K, is closed.

Take a sequence $\{\phi^m\}$ such that $\phi^m \in \Phi^0$ for all m, and assume $\phi^m \to \phi$. Suppose that there exist i and t such that

$$\dot{\phi}_i(t) < -\lambda\phi_i(t).$$

Then, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, $(\phi_i(t+\varepsilon) - \phi_i(t))/\varepsilon < -\lambda\phi_i(t)$. It follows that for a sufficiently large m, $(\phi_i^m(t+\varepsilon) - \phi_i^m(t))/\varepsilon < -\lambda\phi_i^m(t)$, or

$$\phi_i^m(t+\varepsilon) < \phi_i^m(t)(1 - \lambda\varepsilon). \tag{A.11}$$

On the other hand, since $\dot{\phi}_i^m(s) \geq -\lambda\phi_i^m(s)$ holds for any s and any m, we have

$$\begin{aligned}\phi_i^m(t+\varepsilon) &\geq \phi_i^m(t)\, e^{-\lambda\varepsilon} \\ &> \phi_i^m(t)(1-\lambda\varepsilon),\end{aligned}$$

which contradicts (A.11). ∎

Lemma A.6 *For all k, Φ^k is closed.*

Proof. First, by Lemma A.5, Φ^0 is closed. Suppose next that Φ^{k-1} is closed. Let $\{\phi^m\}$ be such that $\phi^m \in \Phi^k$ for all m, and assume $\phi^m \to \phi$. Take any i and t such that $\dot{\phi}_i(t) > -\lambda\phi_i(t)$. Observe that for any $\varepsilon > 0$, there exists M such that for all $m \geq M$,

$$\dot{\phi}_i^m(t^m) > -\lambda\phi_i^m(t^m)$$

holds for some $t^m \in (t-\varepsilon, t+\varepsilon)$. Take a sequence $\{\varepsilon^\ell\}$ such that $\varepsilon^\ell > 0$ and $\varepsilon^\ell \to 0$. Then we can take a subsequence $\{\phi^{m_\ell}\}$ of $\{\phi^m\}$ such that $\dot{\phi}_i^{m_\ell}(t^\ell) > -\lambda\phi_i^{m_\ell}(t^\ell)$ holds for some $t^\ell \in (t-\varepsilon^\ell, t+\varepsilon^\ell)$. For each ϕ^{m_ℓ}, since it is contained in Φ^k, there exists $\psi^\ell \in \Phi^{k-1}$ such that $\psi^\ell(t^\ell) = \phi^{m_\ell}(t^\ell)$ and $a_i \in BR(\psi^\ell)(t^\ell)$. Since Φ^0 is compact and, by the hypothesis, Φ^{k-1} is closed, a subsequence (again denoted by) ψ^ℓ converges to some $\psi \in \Phi^{k-1}$, which satisfies $\psi(t) = \phi(t)$. Moreover, since the payoff $V(\cdot)(\cdot)$ is continuous, and hence, $BR(\cdot)(\cdot)$ is upper semi-continuous, we have $a_i \in BR(\psi)(t)$, so that $\phi \in \Phi^k$. ▌

Proof of Proposition 5.1. Take any $\phi \in \Phi^*$, and any i and t such that $\dot{\phi}_i(t) > -\lambda\phi_i(t)$. Since $\phi \in \Phi^k$ for all k, we can take a sequence $\{\psi^k\}$ with $\psi^k \in \Phi^k\,(\subset \Phi^0)$ such that $\psi^k(t) = \phi(t)$ and $a_i \in BR(\psi^k)(t)$. Since Φ^0 is compact due to Lemma A.5, a subsequence (again denoted by) ψ^k converges to some $\psi \in \Phi^0$ with $\psi(t) = \phi(t)$. For each k, $\{\psi^{k'}\}_{k' \geq k}$ is contained in Φ^k, so that the limit ψ is in Φ^k since Φ^k is closed due to Lemma A.6. Therefore, $\psi \in \Phi^*\,(= \bigcap_{k=0}^\infty \Phi^k)$. Moreover, due to the upper semi-continuity of BR, we have $a_i \in BR(\psi)(t)$.

Conversely, take any $\phi \notin \Phi^*$. Then, $\phi \notin \Phi^k$ for some k. For such a k, there exist i and t such that no path $\psi \in \Phi^k\,(\supset \Phi^*)$ with $\psi(t) = \phi(t)$ satisfies $a_i \in BR(\psi)(t)$. ▌

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