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Evolution and Learning
in Games with Randomly Disturbed Payoffs

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Evolution and Learning in Games with Randomly Disturbed Payoffs*

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Abstract

We consider four models of evolution and learning in games which rely on perturbations of payoffs, including stochastic fictitious play. In all cases, we establish global stability results for zero-sum games, games with an interior ESS, potential games, and supermodular games.

1. Introduction

A common theme of many recent models of evolution and learning in games is the abandonment of the assumption that players always evaluate payoffs using the same fixed payoff matrix. Some stochastic models of learning (Fudenberg and Kreps (1993), Kaniovski and Young (1995), Benaïm and Hirsch (1999a)) and evolution (Blume (1993, 1997), Young (1998)) consider players whose payoffs are perturbed by random shocks which are i.i.d. over time. Other deterministic models of learning (Ellison and Fudenberg (2000)) and evolution (Ely and Sandholm (2000)) look at populations of players with heterogenous preferences. All of these papers can be viewed as successors of the seminal work of Harsanyi (1973a), who introduced random payoff perturbations as the basis for his model of the purification of mixed equilibrium.

In this paper, we consider the performance of these models in four important classes of games: zero sum games, games with an interior ESS, potential games, and supermodular games. For all combinations of models and games, we are able to establish global convergence results.

Of all the models we analyze, the most thoroughly studied is stochastic fictitious play. Brown (1951) introduced standard fictitious play as a method of computing Nash equilibria. In fictitious play, each player chooses best responses to his beliefs about his opponents, which are given by the time average of past play. Convergence of beliefs to Nash equilibrium has been established for two-player zero sum games (Robinson (1951)), 2 x 2 games (Miyasawa (1961)), potential games (Monderer and Shapley (1996a)), games with an interior ESS (Hofbauer (1995b)), and certain supermodular games (Milgrom and Roberts (1991), Krishna (1992)).[1] However, even when players' beliefs converge to Nash equilibrium, their actual behavior may not. In particular, since best responses are generically pure, behavior cannot converge to the mixed equilibrium of a game. For this reason, the appropriateness of fictitious play as a model of learning mixed equilibrium has been called into question.

To contend with this, Fudenberg and Kreps (1993) introduced stochastic fictitious play. In this model, each player's payoffs are perturbed in each period by shocks *a la* Harsanyi (1973a). As a consequence, each player's anticipated behavior in each period is a genuine mixed strategy. Fudenberg and Kreps (1993), Kaniovski and

[1] Examples in which beliefs fail to converge have been offered by Shapley (1964), Foster and Young (1998), and Krishna and Sjöström (1998).

Young (1995), and Benaïm and Hirsch (1999a) are therefore able to extend Miyasawa's (1961) result for 2 x 2 games to stochastic fictitious play, proving not only convergence of beliefs to equilibrium, but also convergence of behavior. Benaïm and Hirsch (1999a) also establish convergence in certain *p*-player, two strategy games. However, because of the complications created by the random payoff perturbations, results for other classes of games have proved difficult to obtain. In particular, very little is known about convergence in games with more than two strategies per player.

In this paper, we establish the convergence of stochastic fictitious play in all of the classes of games noted above in which standard fictitious play is known to converge. We also prove convergence for these games in three other models: the stochastic evolution model of Blume (1993, 1997) and Young (1998), Ellison and Fudenberg's (2000) model of population fictitious play, and Ely and Sandholm's (2000) model of evolution with diverse preferences. While general techniques have been developed for studying these other three classes of models, few guarantees of convergence have been established for specific classes of games.[2] The present paper fills this gap.

The four classes of models listed above are defined using four different mathematical structures: the stochastic models use Markov chains with decreasing and fixed step sizes, while the deterministic models rely on ordinary and functional differential equations. Nevertheless, behavior in all four models can be characterized in terms of solutions to a single ordinary differential equation which captures the models' expected or aggregate motion. Because of complications created by the payoff perturbations, this differential equation is difficult to analyze unless the number of strategies in the underlying game is small.

The crucial differential equation is a smooth version of the best response dynamics. This perturbed dynamics is obtained by introducing a *stochastic perturbation* of the payoffs to each *pure strategy*. The first step in our analysis establishes an alternate representation of the perturbed dynamics which utilizes a non-linear, *deterministic* perturbation of the payoffs to each *mixed strategy*.[3]

While the perturbed best response dynamics can be quite difficult to analyze when presented in their original form, they often become easier to study when

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[2] We know of two exceptions for the stochastic evolution model: results of Blume (1993, 1997) and Young (1998) for potential games, and results of Benaïm and Hirsch (1999b) for certain *p*-player, two strategy games.

[3] Deterministic payoff perturbations were first considered by Harsanyi (1973b), while the corresponding dynamics were introduced by Fudenberg and Levine (1998).

expressed in terms of deterministic perturbations. For zero sum games, games with an interior ESS, and potential games, Hofbauer (2000) and Hofbauer and Hopkins (2000) have constructed Lyapunov functions for the deterministically perturbed dynamics. For supermodular games, we can establish that the perturbed dynamics form a *strongly monotone* dynamical system (Hirsch (1988)). These results enable us to establish convergence of the perturbed dynamics to an equilibrium from almost every initial condition. Moreover, these results can be used to characterize the *chain recurrent* states (Conley (1978)) of the perturbed dynamics in all four classes of games.[4] This inclusive notion of recurrence is critical to understanding the long run behavior of stochastic fictitious play.

Recall that the perturbed best response dynamics arise as a description of expected or aggregate motion in more complex evolution and learning models. Fortunately, techniques from the theories of stochastic approximation, convergence of Markov processes, and functional differential equations can be used to show that in all of the models in question, an understanding of expected or aggregate motion is nearly enough to determine limit behavior. By combining these techniques with the representation theorem and the analyses of the perturbed dynamics, we are able to obtain a variety of strong convergence results.

Section 2 describes the perturbed best response dynamics for a single population model, and provides its stochastic and deterministic derivations. Section 3 uses the latter derivation to characterize these dynamics in four classes of games. Sections 4 through 7 establish stability results for four models of evolution and learning. Section 8 extends these results to multipopulation settings. Section 9 concludes. Proofs omitted from the text can be found in the Appendix.

2. The Perturbed Best Response Dynamics

We first consider players who are paired to play a two-player, n strategy symmetric game. Let $A \in \mathbf{R}^{n\times n}$ denote the payoff matrix for such a game, so that $A_{ij} = e_i \cdot A e_j$ is the payoff received by a player who plays strategy i against an opponent who plays strategy j, while $y \cdot Ax$ is the expected payoff of mixed strategy y against

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[4] The chain recurrent states of a deterministic flow are those states which can arise in the long run if the flow is subjected to small shocks which occur at isolated moments in time. See Section 3 for a formal definition.

mixed strategy x. Let $\Delta = \{x \in \mathbf{R}^n_+ : \sum_j x_j = 1\}$ denote the simplex, which represents both the set of mixed strategies and the set of strategy distributions.

Traditional game theoretic analyses focus on best responses. Here, the *best response correspondence* $B: \Delta \Rightarrow \Delta$ is defined by

$$B(x) = \arg\max_{y \in \Delta} \; y \cdot Ax.$$

A (*symmetric*) *Nash equilibrium* is a fixed point of B: $x^* \in B(x^*)$. The best response correspondence is also used to define the *best response dynamics* (Gilboa and Matsui (1991)):

$$\text{(BR)} \qquad \dot{x} \in B(x) - x.$$

Under these dynamics, the strategy distribution x always moves in the direction of a current best response.[5]

As we shall see, behavior in a number of recent models of evolution and learning in games can be characterized in terms of smoothed versions of the best response dynamics. We express these *perturbed best response dynamics* as

$$\text{(P)} \qquad \dot{x} = \tilde{B}(x) - x,$$

where the *perturbed best response function* $\tilde{B}: \Delta \to \Delta$ is a smooth approximation of the best response correspondence. Our analysis relies on the fact that the function $\tilde{B}$ can be derived using two distinct approaches.

2.1 The Stochastic Derivation of $\tilde{B}$

The first derivation of the perturbed best response function is obtained by stochastically perturbing the payoffs to each pure strategy, an approach pioneered by Harsanyi (1973a). Suppose that when playing strategy i against an opponent playing strategy j, a player receives a payoff of $A_{ij} + b_i$. Here, A_{ij} is the appropriate entry from the player's payoff matrix, while b_i represents a random payoff term. The random variables b_i are i.i.d. with some fixed distribution function F whose density function

[5] Since B is set valued, equation (BR) may admit multiple solution trajectories from a single initial condition – see Hofbauer (1995b).

f is strictly positive and bounded. We define the perturbed best response function $\tilde{B}$: $\Delta \to \Delta$ by letting $\tilde{B}_i(x)$ equal the probability that strategy i is optimal:

$$\tilde{B}_i(x) \equiv P(\text{argmax}_j\ (Ax)_j + b_j = i). \tag{1}$$

2.2 The Deterministic Derivation of $\tilde{B}$

One can also define a perturbed best response function by deterministically perturbing the payoffs to each mixed strategy. Harsanyi (1973b) introduced this sort of perturbation in a study of the number of Nash equilibria in generic games, augmenting the payoffs to each mixed strategy $y \in \text{int}(\Delta)$ by $\sum_i \ln y_i$. Van Damme (1991) uses such deterministic perturbations to represent control costs, and investigates the connections between these costs and refinements of Nash equilibrium. More recently, Fudenberg and Levine (1998) consider deterministic perturbations of payoffs in the context of learning in games.

We suppose that a player who chooses mixed strategy y against mixed strategy x receives a payoff of $y \cdot Ax - V(y)$. The expression $y \cdot Ax$ is the usual random matching payoff, while the function $V(y)$ is a deterministic perturbation which depends nonlinearly on the mixed strategy the player chooses. We call the perturbation V: $\text{int}(\Delta) \to \mathbf{R}$ *admissible* if $D^2 V(y)$ is positive definite on $\mathbf{R}^n_0 = \{z \in \mathbf{R}^n\!: z \cdot \mathbf{1} = 0\}$ for all y, and $\|\nabla V(y)\|$ approaches infinity as y approaches the boundary of Δ. Any admissible perturbation V also defines a perturbed best response function, namely

$$\tilde{B}(x) = \underset{y \in \text{int}(\Delta)}{\arg\max} \left(y \cdot Ax - V(y) \right). \tag{2}$$

2.3 The Characterization Theorem

Theorem 2.1 is fundamental to much of the analysis that follows. It shows that any perturbed best response function defined using stochastic perturbations of payoffs can be represented in terms of an appropriate deterministic perturbation.

Theorem 2.1: *Fix the bias distribution F. Then the function $\tilde{B}$ defined in equation* (1) *satisfies equation* (2) *for some admissible deterministic perturbation V.*

To prove Theorem 2.1, it is useful to express the dynamics (BR) and (P) in a somewhat different way. Notice that in defining the best response $B(x)$ and the perturbed best response $\tilde{B}(x)$, the strategy distribution x is only used to compute the payoff vector Ax, which in turn is used to determine choice probabilities. We can therefore decompose the dynamics (BR) and (P) as follows:

(BR') $\dot{x} \in M(Ax) - x$;
(P') $\dot{x} = C(Ax) - x$.

Here, $M(a) = \text{argmax}_{y \in \Delta}\, y \cdot a$ denotes the maximizer correspondence; the *choice probability function* C: $\mathbf{R}^n \to \text{int}(\Delta)$ is a perturbed version of M. Theorem 2.1 follows immediately from this decomposition and Theorem 2.2.

Theorem 2.2: *Fix the bias distribution F, and define the choice probability function C by*

$$C_i(a) \equiv P(\text{argmax}_j\, a_j + b_j = i). \tag{1'}$$

Then for some admissible disturbance V,

$$C(a) = \underset{y \in \text{int}(\Delta)}{\arg\max} \left(y \cdot a - V(y) \right) \tag{2'}$$

The proof of this result proceeds as follows. First, we establish that the derivative matrix DC is symmetric and has negative off-diagonal terms. These properties of DC imply that the vector field C admits a convex potential function, which we call W.[6] We show that the required disturbance function V can be obtained as the Legendre transform of W. This choice of V ensures that the functions $(\nabla V)^{-1}$ and $\nabla W \equiv C$ are identical (in a sense to be made precise below), so that C satisfies the first order conditions for the maximization problem (2').

Proof: Fix a strategy i and a payoff vector a. Since the density function for $a_i + b_i$ is $f(t - a_i)$, and since the distribution function for $\max_{k \neq i}(a_k + b_k)$ is $\prod_{k \neq i} F(t - a_k)$, a convolution yields

$$\begin{aligned} C_i(a) &= P(a_i + b_i \geq \max_{k \neq i}(a_k + b_k)) \\ &= P(\max_{k \neq i}(a_k + b_k) - (a_i + b_i) \leq 0) \end{aligned} \tag{3}$$

[6] These facts are well known in the literature on discrete choice theory: see McFadden (1981) or Anderson, de Palma, and Thisse (1992, Chapters 2 and 3). However, the remainder of our argument appears to be new.

$$= \int_{-\infty}^{\infty} f(t-a_i)\left(\prod_{k\neq i} F(t-a_k)\right)dt.$$

Hence, when $i \neq j$,

$$\frac{\partial C_i}{\partial a_j}(a) = -\int_{-\infty}^{\infty} f(t-a_i)f(t-a_j)\left(\prod_{k\neq i,j} F(t-a_k)\right)dt < 0. \tag{4}$$

(The fact that f is bounded allows us to differentiate under the integral sign, and also ensures that the resulting integral is finite and depends continuously upon a.)

Equations (3) and (4) have a number of important consequences. First, equation (4) implies that the derivative matrix $DC(a) \in \mathbf{R}^{n\times n}$ is symmetric:

$$\frac{\partial C_i}{\partial a_j}(a) = \frac{\partial C_j}{\partial a_i}(a) \quad \text{for all } i \text{ and } j. \tag{5}$$

Second, $DC(a)$ is positive definite on $\mathbf{R}^n_0$. To see this, note that by equation (4), the off-diagonal terms of $DC(a)$ are strictly negative. Moreover, since $\sum_j C_j(a) = 1$ by definition, it follows that $\sum_j \frac{\partial C_j}{\partial a_i}(a) = 0$ for each i, and so that

$$\frac{\partial C_i}{\partial a_i}(a) = -\sum_{j\neq i} \frac{\partial C_j}{\partial a_i}(a). \tag{6}$$

Hence, equations (5) and (6) imply that

$$DC(a)\,\mathbf{1} = 0. \tag{7}$$

Moreover, if z is not proportional to $\mathbf{1}$, then if we let $d_{ij} = \frac{\partial C_i}{\partial a_j}(a)$, equations (6), (5), and (4) imply that

$$\begin{aligned} z \cdot DC(a)\, z &= \sum_i \sum_j d_{ij} z_i z_j \\ &= \sum_j \sum_{i\neq j} d_{ij} z_i z_j - \sum_j (\sum_{i\neq j} d_{ij}) z_j^2 \\ &= \sum_j \sum_{i\neq j} d_{ij}(z_i z_j - z_j^2) \\ &= \sum_j \sum_{i<j} d_{ij}(2z_i z_j - z_i^2 - z_j^2) \\ &= \sum_j \sum_{i<j} -d_{ij}(z_i - z_j)^2 > 0. \end{aligned} \tag{8}$$

These observations imply that C is one-to-one on $\mathbf{R}_0^n$ and satisfies $C(a + c\mathbf{1}) = C(a)$ for all $c \in \mathbf{R}$: shifting payoffs by a constant vector does not affect choice probabilities.

Finally, we make an observation about the range of the function C: if components a_j, $j \in J$ stay bounded while the remaining components approach infinity, then $C_j(a) \to 0$ for all $j \in J$: that is, $C(a)$ converges to a subface of the simplex. It follows that there are points in the range of C arbitrarily close to each corner of the simplex.

Since the derivative matrix $DC(a)$ is symmetric, the vector field C admits a potential function W: $\mathbf{R}^n \to \mathbf{R}$ (that is, a function which satisfies $\nabla W \equiv C$). Indeed,

$$W(a) = -\int_{-\infty}^{\infty}\left(\prod_k F(t - a_k) - F(t)^n\right)dt.$$

Equation (8) implies that W is strictly convex on $\mathbf{R}_0^n$.

Now consider the restrictions of W and $C \equiv \nabla W$ to $\mathbf{R}_0^n$, and let V: $\text{int}(\Delta) \to \mathbf{R}$ denote the Legendre transform of W:

$$V(y) = y \cdot C^{-1}(y) - W(C^{-1}(y)). \tag{9}$$

Since W: $\mathbf{R}_0^n \to \mathbf{R}$ is strictly convex and C: $\mathbf{R}_0^n \to \text{int}(\Delta)$ takes values at points arbitrarily close to each corner of the simplex, Theorem 26.5 of Rockafellar (1970) implies that the following statements are true. First, the domain of V is convex and equals the range of C, which therefore must be all of $\text{int}(\Delta)$. Second, V and W solve the dual optimization problems

$$V(y) = \max_{a \in \mathbf{R}_0^n}\big(y \cdot a - W(a)\big) \text{ and} \tag{10}$$

$$W(a) = \max_{y \in \text{int}(\Delta)}\big(y \cdot a - V(y)\big). \tag{11}$$

Third, ∇V: $\text{int}(\Delta) \to \mathbf{R}_0^n$ is invertible, with $(\nabla V)^{-1} \equiv \nabla W \equiv C$ on $\mathbf{R}_0^n$.[7]

We conclude by establishing the required properties of V. First, since $(\nabla V)^{-1} \equiv C$, the observation three paragraphs above shows that $\|\nabla V(y)\|$ approaches infinity as y approaches the boundary of Δ. Furthermore, since $C(\nabla V(y)) = y$, differentiating

[7] Since the domain of V is $\text{int}(\Delta)$, the partial derivatives of V are not well defined. Consequently, $\nabla V(y)$ is defined to be the unique vector in $\mathbf{R}_0^n$ such that $V(y + hz) = V(y) + (\nabla V(y) \cdot z)h + o(h)$ for all unit length vectors z in $\mathbf{R}_0^n$.

yields $DC(\nabla V(y))\ D^2V(y) = I$, where all expressions are interpreted as linear operators on $\mathbf{R}^n_0$. Since $DC(\nabla V(y))$ is symmetric and positive definite on $\mathbf{R}^n_0$ and inverts $D^2V(y)$ on $\mathbf{R}^n_0$, it follows that $D^2V(y)$ is also positive definite on $\mathbf{R}^n_0$.

Finally, solving for the maximizer

$$y^* = \operatorname*{arg\,max}_{y \in \mathrm{int}(\Delta)} \left(y \cdot a - V(y)\right),$$

we find that $a = \nabla V(y^*)$, and hence that $y^* = C(a)$. This completes the proof of the theorem. ■

Theorem 2.2 is proved for cases where the stochastic derivation of C uses i.i.d. perturbations to the payoffs to each of the n pure strategies. The restriction to i.i.d. perturbations is actually unnecessary: this theorem and all of our subsequent results continue to hold so long as the vector $(b_1, \ldots, b_n)$ of payoff disturbances admits a strictly positive density on $\mathbf{R}^n$.

To summarize: The perturbed dynamics (P) can be derived in two distinct ways. In the models we study in Sections 4 through 7, perturbed dynamics arise which are based on stochastic perturbations b_i described by some distribution function F.

$$\text{(P-}F\text{)} \qquad \dot{x}_i = P(\operatorname{argmax}_j (Ax)_j + b_j = i) - x.$$

Alternatively, one can define perturbed dynamics using some deterministic perturbation V.

$$\text{(P-}V\text{)} \qquad \dot{x} = \operatorname*{arg\,max}_{y \in \mathrm{int}(\Delta)} \left(y \cdot Ax - V(y)\right) - x.$$

Theorem 2.1 shows that any dynamic of the form (P-F) can be represented as a dynamic of the form (P-V). In order to understand the former dynamics, it is enough to characterize the latter.

2.4 Discussion

2.4.1 The Converse of Theorem 2.2

It is natural to ask whether the converse of Theorem 2.2 also holds: that is, whether the choice function derived from any admissible deterministic perturbation of payoffs can be derived from an appropriate stochastic perturbation.

Proposition 2.3, which considers the logarithmic deterministic perturbations studied by Harsanyi (1973b), shows that such a reconstruction is not always possible.

Proposition 2.3: *When* $n \geq 4$, *there is no stochastic perturbation of payoffs which yields the same choice probabilities as the admissible deterministic perturbation* $V(y) = -\sum_j \ln y_j$.

More generally, we have the following characterizations of the two types of choice functions. The Legendre transform argument in the proof of Theorem 2.2 shows that the choice function C: $\mathbf{R}^n \rightarrow \text{int}(\Delta)$ can be derived from an admissible deterministic payoff perturbation V if and only if $DC(a)$ is symmetric, positive definite on $\mathbf{R}^n_0$, and satisfies $DC(a)\,\mathbf{1} = 0$. On the other hand, the Williams-Daly-Zachary Theorem (see McFadden (1981)) implies that the choice functions C which can be derived from some stochastic payoff perturbation $(b_1, \ldots, b_n)$ with a strictly positive density on $\mathbf{R}^n$ are characterized by these requirements, plus the additional requirement that the partial derivatives of C satisfy

$$(-1)^k \frac{\partial^k C_{i_0}}{\partial a_{i_1} \ldots \partial a_{i_k}} > 0$$

for each $k = 1, \ldots, n-1$ and each set of $k+1$ distinct indices $\{i_0, i_1, \ldots, i_k\}$.

2.4.2 The Logit Dynamics

A class of perturbed dynamics for which the connections above can be described explicitly is the *logit dynamics*,

$$\text{(L)} \qquad \dot{x} = L(Ax) - x,$$

which are defined in terms of the *logit choice function*

$$L_i(a) = \frac{\exp(\varepsilon^{-1} a_i)}{\sum_j \exp(\varepsilon^{-1} a_j)}.$$

We call the parameter $\varepsilon \in (0, \infty)$ the *noise level*. When ε approaches zero, L approaches the maximizer correspondence M; when ε approaches infinity, L approaches the uniform probability vector $(\frac{1}{n}, \ldots, \frac{1}{n})$.

The logit choice function is widely used in microeconomics (Anderson, de Palma, and Thisse (1992)), macroeconomics (Durlauf (1997)), and econometrics (Amemiya (1981, 1986)), and has been studied in game theoretic contexts by Blume (1993, 1997), McKelvey and Palfrey (1995), Chen, Friedman, and Thisse (1997), Fudenberg and Levine (1998), Young (1998), and Anderson, Goeree, and Holt (1999). We now present its stochastic and deterministic derivations.

Proposition 2.4: (*i*) *If the distribution function F is the extreme value distribution* $F(b) = \exp(-\exp(-\varepsilon^{-1}b - \gamma))$ (*where γ is Euler's constant*), *then* $P(\text{argmax}_j\ a_j + b_j = i) = L_i(a)$.

(*ii*) *If the deterministic disturbance V is the entropy function* $V(y) = \varepsilon\sum_j y_j \ln y_j$, *then* $\arg\max_{y\in \text{int}(\Delta)} \left(y \cdot a - V(y)\right) = L(a)$.

Proof: For a proof of part (*i*), see Anderson, de Palma, and Thisse (1992, Theorem 2.2). Part (*ii*) is established by evaluating the first order conditions of the maximization problem – see Rockafellar (1970, p. 148), Anderson, de Palma, and Thisse (1992, Theorem 3.7), or Fudenberg and Levine (1998, p. 119). ■

To relate these results to Legendre transforms, observe that the potential function associated with the logit choice function L is $W(a) = \varepsilon\ln(\sum_j(\exp(\varepsilon^{-1}a_j))$. If we restrict L to $\mathbf{R}^n_0$, then it is invertible, with inverse $(L^{-1})_i(y) = \varepsilon(\ln y_i - \frac{1}{n}(\sum_j \ln y_j))$. Then computing the Legendre transform of W using equation (9) yields $V(y) = \varepsilon\sum_j y_j \ln y_j$. It can be checked that V and W solve the dual maximization problems (10) and (11), and that[8] $\nabla V \equiv L^{-1}$.

In order to model boundedly rational choice in a simple fashion, Chen, Friedman, and Thisse (1997) consider choice functions of the form

$$C_i(a) = \frac{w(a_i)}{\sum_j w(a_j)}, \tag{12}$$

[8] If one computes the gradient of V by taking its partial derivatives, one obtains $L^{-1}(y)$ plus a constant vector. However, since the domain of V is actually $\text{int}(\Delta)$, we view $\nabla V(y)$ as a vector in $\mathbf{R}^n_0$, which can be obtained from the vector of partial derivatives by subtracting this same constant vector. Thus, $\nabla V(y)$ is equal to $L^{-1}(y)$.

where the weighting function w: $\mathbf{R} \to (0, \infty)$ is some increasing and differentiable function of payoffs. We conclude this section by noting that the only choice function of this form which can be derived from either stochastic or deterministic perturbations of payoffs is the logit choice function L.

Proposition 2.5: *Suppose that the choice function C satisfies condition* (12) *and either condition* (1') *or condition* (2'). *Then* $C \equiv L$ *for some noise level* $\varepsilon > 0$.

3. Analysis of the Perturbed Dynamics for Four Classes of Games

Theorem 2.1 provides a representation of the perturbed dynamics (P-F) in terms of a deterministic payoff perturbation. We now use this representation to analyze the perturbed dynamics arising in certain important classes of games. In subsequent sections, we use this analysis as a foundation for studying four models of evolution and learning.

3.1 Preliminaries

3.1.1 Rest Points and Nash Equilibria

The main results of this section establish the stability of rest points of the dynamics (P-F). These rest points will ultimately constitute our predictions of play in the evolution and learning models. For these predictions to accord with standard game theoretic analyses, the stable rest points should approximate Nash equilibria of the underlying game. The following result ensures that this is true whenever the perturbations generating the dynamics (P-F) are sufficiently small.[9]

Theorem 3.1: *Fix a game A. For each* $k \in \mathbf{Z}_+$, *let* F^k *be a distribution function, and let* x^k *be a rest point of* (P-F^k) *under this noise distribution. Suppose the sequence* $\{F^k\}$ *converges weakly to a mass point at zero. If the sequence* $\{x^k\}$ *converges to* x^*, *then* x^* *is a Nash equilibrium of A.*

[9] To apply Theorem 3.1 to the logit dynamics (L), we observe that the extreme value distribution $F(t) = \exp(-\exp(-\varepsilon^{-1}t - \gamma))$ has mean zero and variance $\varepsilon^2\pi^2/6$ (Anderson *et. al*. (1992, p. 40)). Hence, rest points of (L) converge to Nash equilibria as the noise level ε approaches zero.

3.1.2 ω-Limit Sets and Chain Recurrence

In order to state our stability results, we must introduce two notions of recurrent behavior for deterministic flows: ω-limit sets and chain recurrence. The proofs of our results also require other notions of recurrence; a full account is provided in the Appendix.

Let ϕ: $\mathbf{R}_+ \times \Delta \to \Delta$ denote the (*semi-*) *flow* for the dynamics (P): $\phi(t, x)$ is the position at time t of the solution to (P) which begins at x. The set of rest points of (P) can be defined as $RP = \{x \in \Delta\text{: } \phi(t, x) = x \text{ for all } t \geq 0\} = \{x \in \Delta\text{: } \tilde{B}(x) = x\}$.

The *ω-limit set* of the state x, $\omega(x) = \{z \in X\text{: } \lim_{k\to\infty}\phi(t_k, x) = z \text{ for some } t_k \to \infty\}$, is the set of limit points of the solution trajectory starting at x. We let $\Omega = \bigcup_{x\in\Delta}\omega(x)$ denote the union of the ω-limit sets. Clearly, $RP \subset \Omega$.

Knowledge of Ω is generally not sufficient to characterize limit behavior under stochastic fictitious play. To accomplish this, we require a more general notion of recurrent behavior for deterministic flows, one which includes the states which can arise in the long run if the flow is subject to small shocks occurring at isolated points in time. This form of robustness is captured by chain recurrence, a notion of recurrence due to Conley (1978).

We call a sequence $\{x = x_0, x_1, \ldots , x_n = y\}$ an *ε-chain* from x to y if for each $i \in \{1, \ldots, n\}$, there is a $t_i \geq 1$ such that $\left|\phi(t_i, x_{i-1}) - x_i\right| < \varepsilon$. The ε-chain specifies $n + 1$ segments of solution trajectories to (P). The first begins at x, and the last is simply the point y; the jumps between the ends and beginnings of consecutive segments are never longer than ε. We call the state x *chain recurrent* if there is an ε-chain from x to itself for all $\varepsilon > 0$, and we let CR denote the set of chain recurrent points. The set CR contains all rest points, periodic orbits, quasiperiodic motions, and chaotic orbits of the flow. It can be shown that $\Omega \subset CR$, and that in general this inclusion is strict.[10] See the Appendix for further discussion of these notions of recurrence.

3.2 Zero Sum Games and Games with an Interior ESS

We now provide stability results for the perturbed best response dynamics (P) for a number of classes of games. As usual, any result which holds for the game A also holds for the game $A + \mathbf{1}v'$, $v \in \mathbf{R}^n$, since adding the vector v' to each row of A does

[10] For a simple example, consider a flow on a circle which moves clockwise everywhere except at a finite number of rest points. Then while the rest points are the only ω-limit points, all points on the circle are chain recurrent.

not affect players' incentives. More notably, our convergence results which hold for A also hold for $A + w\mathbf{1}'$, which is obtained by adding the vector w to each column of A.[11] Of course, the latter transformation alters the sets of Nash equilibria and rest points of (P), while the former one does not.

We now turn to the classes of games to be studied. In the current symmetric setting, A is a *zero sum game* if the matrix A is skew-symmetric ($A^T = -A$). In such games, the sum of the payoffs received by the players in any match is zero.

The notion of an *evolutionarily stable strategy* (Maynard Smith and Price (1973)) is the original solution concept of evolutionary game theory. A mixed strategy $x^* \in \Delta$ is an ESS if $x^* \cdot Ax > x \cdot Ax$ for all mixed strategies x in a neighborhood of x^*. An ESS is a mixed strategy with the property that after any invasion by a mutant mixed strategy, the ESS performs better than the mutant in the post-entry population.

Hofbauer (2000) studies global properties of the perturbed best response dynamics for these classes of games.[12] He defines the dynamics (P-V) for some deterministic perturbation of payoffs V, and then uses V to construct a *strict Lyapunov function* Λ: $\Delta \to \mathbf{R}$ for the dynamics: that is, a function whose value increases strictly along every non-constant solution trajectory. The existence of a strict Lyapunov function implies global convergence to the set of rest points of (P-V).[13] Moreover, if the Lyapunov function is strictly concave, as is true in the cases under consideration, then the maximizer of this function is globally stable and is the unique chain recurrent point.

Theorem 3.2 (Hofbauer (2000)): *If A is zero sum or admits an interior ESS, then the strictly concave function*

$$\Lambda(x) = \left(x \cdot Ax - V(x)\right) - \left[\max_y \left(y \cdot Ax - V(y)\right)\right] = x \cdot Ax - (V(x) + W(Ax))$$

is a strict Lyapunov function for the dynamics (P-V). *The unique maximizer of* Λ, $x^*(\Lambda)$, *is the unique and globally stable rest point of* (P-V). *Moreover,* $CR = \{x^*(\Lambda)\}$.

[11] To see why, let b be the (random) disturbance vector with joint distribution F. Then the choice function generated by the game $A + w\mathbf{1}'$ and the disturbance vector b is identical to the choice function generated by the game A and the disturbance vector $b + w$. In other words, the effect of adding w to each column of A can be mimicked by a corresponding shift in the payoff disturbances.

[12] Local stability of rest points of (P-V) in these games was established by Hopkins (1999).

[13] For a proof of this implication, see, e.g., Hofbauer and Sigmund (1988, Theorem 7.6).

Together, Theorems 3.2 and 2.1 establish the existence of a global attractor for the stochastically perturbed dynamics (P-F).

3.3 Potential Games

We say that the game A is a *potential game* if the matrix A is symmetric, so that the payoffs received by matched players are identical.[14] Potential games include all pure coordination games, and also arise in applications of evolutionary techniques to implementation problems – see Sandholm (2000c).

Potential games are known to have strong convergence properties under evolutionary dynamics. Hofbauer (1995a) and Sandholm (2000b) show that the average payoff function $a(x) = \frac{1}{2}x \cdot Ax$ is a Lyapunov function for these games under a broad class of unperturbed evolutionary dynamics, and that its local maximizers are precisely the dynamics' locally stable rest points. We now show that under a mild regularity condition, analogous convergence and equilibrium selection results hold for the perturbed dynamics (P-V), and hence for the dynamics (P-F).

To begin, we present another result due to Hofbauer (2000).

Theorem 3.3 (Hofbauer (2000)): *If A is a potential game, then the function*

$$\Pi(x) = \tfrac{1}{2}x \cdot Ax - V(x).$$

is a strict Lyapunov function for the dynamics (P-V). *It follows that every solution trajectory of* (P-V) *converges to a connected set of rest points of* (P-V), *and so that $\Omega = RP$.*

Theorem 3.3 provides a strict Lyapunov function for the dynamics (P-V), and therefore guarantees that Ω, the set of limit points for the solution trajectories of (P-V), is equal to the set of rest points of (P-V). But to analyze stochastic fictitious play, we must also characterize the chain recurrent set CR. Remarkably, the existence of a strict Lyapunov function is not enough to ensure that $CR = RP$.[15] However, this

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[14] These games have also been dubbed partnership games (Hofbauer and Sigmund (1988)), doubly symmetric games (Weibull (1995)), and games with identical interests (Monderer and Shapley (1996a)). Some authors (e.g., Monderer and Shapley (1996b)) also include games of the form $A + \mathbf{1}v'$ as potential games, where A is again a symmetric matrix and $\mathbf{1}v'$ represents an incentive-irrelevant payoff shift. Since the shift $\mathbf{1}v'$ does not affect the dynamics (P) (see Section 3.2), all of our results continue to hold under this broader definition of potential games.

[15] For counterexamples, see Akin (1993, p. 25-26 and 55-56) and Benaïm (1999, p. 27).

equivalence can be established under slightly stronger assumptions. To explain these, we first note that since the matrix A is symmetric, the first order condition for a maximizer of Π on Δ is

$$Ax^* - \nabla V(x^*) = c\mathbf{1}.$$

Since the choice function satisfies $C(a + c\mathbf{1}) = C(a)$ and inverts ∇V, points satisfying the first order condition also satisfy $C(Ax^*) = C(Ax^* - c\mathbf{1}) = C(\nabla V(x^*)) = x^*$. In other words, the critical points of the Lyapunov function Π are precisely the rest points of the dynamics (P-V).

With this observation in hand, it is possible to show that $CR = RP$ under either of two mild assumptions. One possibility is to require that the dynamics (P-V) be sufficiently smooth. To accomplish this, it is enough to require smoothness of the disturbance function.

Proposition 3.4: *If A is a potential game and V is C^{n-1}, then $CR = RP$.*

If we begin with the stochastically perturbed dynamics (P-F), the deterministic perturbation V which corresponds to F will be C^{n-1} if F is sufficiently smooth.

We can reach stronger conclusions if we instead impose a generic regularity condition. To make this point more precisely, we rewrite the first order condition above as

$$I_0(Ax^* - \nabla V(x^*)) = \mathbf{0},$$

where I_0 is the matrix which projects $\mathbf{R}^n$ onto $\mathbf{R}^n_0$.[16] A second order sufficient condition for x^* to maximize Π is that the second derivative matrix $I_0\,(A - D^2V(x^*))$ is negative definite on $\mathbf{R}^n_0$. We will impose our regularity condition on this matrix.

(R) $I_0\,(A - D^2V(x^*))$ has full rank on $\mathbf{R}^n_0$ for all $x^* \in RP$.

Our local stability analysis of the dynamics (P-V) focuses on the derivative of these dynamics at the rest point x^*, $D\dot{x} = DC(Ax^*)\,A - I$. We call the rest point x^* *linearly stable* if all eigenvalues of $D\dot{x}$ corresponding to directions in $\mathbf{R}^n_0$ have strictly negative real part. Linear stability implies both Lyapunov and asymptotic

[16] That is, I_0 is a linear map which is the identity on $\mathbf{R}^n_0$ and which annihilates the vector $\mathbf{1}$.

stability. Similarly, we call x^* *linearly unstable* if some eigenvalue of $D\dot{x}$ corresponding to a direction in $\mathbf{R}_0^n$ has strictly positive real part.

Theorem 3.5 summarizes the connections between critical points of the Lyapunov function Π and the rest points of (P-V). In particular, part (*iii*) of the theorem shows that under the regularity condition (R), all rest points of the perturbed dynamics are linearly stable or linearly unstable, according to whether the point is a local maximizer of Π. This implies that the number of rest points is finite, and guarantees that these points are the only chain recurrent states.

Theorem 3.5: *If A is a potential game, then*:

(*i*) *The critical points of* Π *are precisely the rest points of* (P-V).

(*ii*) *The strict local maximizers of* Π *are precisely the asymptotically stable rest points of* (P-V).

(*iii*) *If condition* (R) *holds, then the local maximizers of* Π *are precisely the linearly stable rest points, while other rest points are linearly unstable. Moreover, RP is finite, and CR = RP.*

3.4 Supermodular Games

Supermodular games describe interactions in which different players' actions are strategic complements. In these games, an order relation is placed on the strategy sets; strategic complementarity means that the advantage of switching to a higher strategy increases when opponents choose higher strategies. Supermodular games arise in many economic applications; for examples, see Milgrom and Roberts (1990), Vives (1990), and Fudenberg and Tirole (1992).

We say that A is a (*strictly*) *supermodular game* if whenever $j > i$, the difference $A_{jk} - A_{ik}$ is strictly increasing in k. In words, the advantage of strategy j over strategy $i < j$ is greater the higher is the opponent's strategy k.[17]

A fundamental property of supermodular games is that they possess increasing best response correspondences. This property can be used to show that pure strategy Nash equilibria of these games exist and possess a very simple structure, and is also important for studying learning processes.[18] A related monotonicity property is

[17] Of course, it is enough for this property to hold after the names of the strategies have been permuted in an appropriate way.

[18] For the properties of pure strategy equilibria, see Topkis (1979) or the aforementioned references. Milgrom and Roberts (1991) show that fictitious play converges in supermodular games with a unique Nash equilibrium, while Krishna (1992) proves convergence in supermodular games satisfying a

fundamental for studying the perturbed best response dynamics. To state this property, we define the invertible linear operator S: $\mathbf{R}^n_0 \to \mathbf{R}^{n-1}$ by

$$(Sx)_i = \sum_{j=i+1}^{n} x_j .$$

That is, the ith component of Sx equals the proportion of players choosing strategies larger than i. If we view points in the simplex as probability distributions on the strategy set $\{1, 2, \ldots , n\}$, then $Sy \geq Sx$ if and only if y stochastically dominates x.

Proposition 3.6 establishes that in supermodular games, the perturbed best response function is monotone with respect to the stochastic dominance order.

Proposition 3.6: *Suppose that A is supermodular. If* $Sy \geq Sx$, *then* $S\tilde{B}(y) \geq S\tilde{B}(x)$.

The remainder of the analysis in this paper only directly utilizes the stochastically perturbed dynamics (P-*F*). For this reason, we will henceforth refer to these dynamics simply as (P).

In the present context, it is easier to study the dynamics (P) after applying the change of coordinates S. The next result shows that this yields the dynamics

$$\text{(S)} \qquad \dot{v} = S\tilde{B}(S^{-1}v) - v$$

on the set $S(\Delta) = \{v \in \mathbf{R}^{n-1}\colon\ 1 \geq v_1 \geq \ldots \geq v_{n-1} \geq 0\}$.

Proposition 3.7: *The dynamics* (P) *and the dynamics* (S) *are linearly conjugate*: $\{x_t\}_{t\geq 0}$ *solves* (P) *if and only if* $\{Sx_t\}_{t\geq 0}$ *solves* (S).

The differential equation $\dot{v} = g(v)$ on $S(\Delta) \subset \mathbf{R}^{n-1}$ is called *cooperative* if $\frac{\partial g_i}{\partial v_j}(v) \geq 0$ for all v and all distinct i and j: that is, an increase in any component of the state increases the rate of change of all other components. The equation is *irreducible* if for each subset I of $\{1, \ldots , n-1\}$, there is an $i \in I$ and a $j \in I^C$ such that $\frac{\partial g_i}{\partial v_j}(v) \neq 0$ for all v. The transformed dynamics (S) are of interest because they possess both of these properties.

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diminishing returns condition. Kandori and Rob (1995) use the monotonicity of best responses in characterizing the stochastically stable states of supermodular games in the Kandori, Mailath, and Rob (1993) model.

Theorem 3.8: *If A is supermodular, the dynamics* (S) *are cooperative and irreducible.*

Observe that if ι_j is a standard basis vector in $\mathbf{R}^{n-1}$, then $S^{-1}\iota_j = e_{j+1} - e_j$, where the latter vectors are standard basis vectors in $\mathbf{R}^n$. In light of this observation and Proposition 3.7, the fact that (S) is cooperative has the following interpretation for the perturbed best response dynamics (P): if some players increase their strategy from j to $j + 1$, the growth rate of strategy $i + 1$ increases relative to that of strategy i for all $i \neq j$.

Theorem 3.8 is important because dynamics which are cooperative and irreducible have desirable monotonicity and convergence properties. In the next result, we list a number of useful implications of Theorem 3.8 for the perturbed best response dynamics (P).

Theorem 3.9: *If A is supermodular, then*

(*i*) *The dynamics* (P) *are strongly monotone with respect to the stochastic dominance order*: *if* $\{x_t\}_{t\geq 0}$ *and* $\{y_t\}_{t\geq 0}$ *are two solutions to* (P) *with* $Sy_0 \geq Sx_0$ *and* $y_0 \neq x_0$, *then* $Sy_t > Sx_t$ *for all* $t > 0$.

(*ii*) *There is an open dense set of initial conditions from which solutions to* (P) *converge to unique limit points in RP.*

(*iii*) *The remaining initial conditions are contained in a finite or countable union* $\bigcup_i M_i$ *of invariant manifolds of codimension* 1, *and hence have measure zero.*

(*iv*) *Chain recurrent points are either rest points or are contained in these invariant manifolds*: $CR \subset RP \cup \bigcup_i M_i$.

Proof: In light of Proposition 3.7 and Theorem 3.8, part (*i*) follows from Theorem 4.1.1 of Smith (1995), part (*ii*) from Theorem 2.4.7 of Smith (1995), part (*iii*) (after a reversal of time) from Theorem 1.1 of Hirsch (1988), and part (*iv*) from Theorems 1.6 and 1.7 of Hirsch (1999) (also see Theorem 3.3 and Corollary 3.4 of Benaïm and Hirsch (1999b)). ■

4. Stochastic Fictitious Play

Stochastic fictitious play is studied by Fudenberg and Kreps (1993), Kaniovski and Young (1995), and Benaïm and Hirsch (1999a). We now describe a symmetric variant of their models.

In each discrete time period, a pair of players drawn from a larger group plays the symmetric game A. In standard fictitious play, each player chooses a best response to his belief about how his opponent will behave; this belief is determined by the time average of past play. In stochastic fictitious play, players make these choices after their payoffs are subjected to random shocks.

The state variable in stochastic fictitious play is the time average of play, represented by the sequence of random variables $\{Z_t\}_{t=1}^{\infty}$. This sequence is defined by

$$Z_t = \tfrac{1}{2t}\sum_{s=1}^{t}\left(Y_s^1 + Y_s^2\right), \tag{13}$$

where Y_t^1 and Y_t^2 are the choices made by the two matched players at time t. The initial choices Y_1^1 and Y_1^2 are arbitrary pure strategies, while subsequent choices are best responses to beliefs Z_t. Best responses are determined after payoffs have been subject to disturbances, $(b_t^\alpha)_i$, which are independent over time t and across players $\alpha \in \{1, 2\}$ and strategies $i \in \{1, \ldots, n\}$; the disturbances all follow the common distribution function F. Hence, players' choice probabilities are described by the perturbed best response function $\tilde{B}$:

$$P\left(Y_{t+1}^\alpha = e_i \,\middle|\, Z_t = z\right) = P(\operatorname{argmax}_k\, (Az)_k + (b_{t+1}^\alpha)_k = i) = \tilde{B}_i(z). \tag{14}$$

By rearranging equation (13), we can obtain a recursive definition of the process Z_t:

$$Z_{t+1} = \tfrac{1}{2t+2}(2t\, Z_t + Y_{t+1}^1 + Y_{t+1}^2).$$

Then using equation (14), we can compute the expected increments of Z_t:

$$E\left(Z_{t+1} - Z_t \,\middle|\, Z_t = z\right) = \tfrac{1}{t+1}\left[E\left(\tfrac{1}{2}(Y_{t+1}^1 + Y_{t+1}^2) \,\middle|\, Z_t = z\right) - z\right] = \tfrac{1}{t+1}(\tilde{B}(z) - z).$$

Thus, the expected change in the time average is governed by the perturbed best response dynamics; since current behavior has a diminishing impact on the time average, its rate of change falls over time.

Using techniques from stochastic approximation theory, Fudenberg and Kreps (1993), Kaniovski and Young (1995), and Benaïm and Hirsch (1999a) show how the behavior of the time average Z_t in any stochastic fictitious play can be characterized in terms of the differential equation defined by its expected motion. However, as the perturbations make this equation difficult to analyze, Fudenberg and Kreps (1993) and Kaniovski and Young (1995) only establish convergence in 2 x 2 games, while Benaïm and Hirsch (1999a) also prove convergence in certain *p*-player, two strategy games.

By combining our analysis of the perturbed best response dynamics with results of Pemantle (1990), Benaïm and Hirsch (1999a), and Benaïm (2000), we can establish convergence of beliefs and choice probabilities in four important classes of games. While our results are stated for beliefs Z_t, since $\tilde{B}$ is continuous, corresponding results hold for the choice probabilities $\tilde{B}(Z_t)$ as well.[19]

To state our results, we let $LS \subset RP$ denote the set of Lyapunov stable rest points of (P), and let $LU \subset RP$ denote the set of linearly unstable rest points of (P).

Theorem 4.1: *Consider the process of stochastic fictitious play described above, and fix any initial condition.*

(*i*) *If A is a zero sum game or has an interior ESS, then* $P(\lim_{t\to\infty} Z_t = x^*(\Lambda)) = 1$.

(*ii*) *Suppose A is a potential game. If f is sufficiently smooth, then* $P(\omega(Z_t)$ is a connected subset of $RP) = 1$. *If condition* (R) *holds and* f' *exists and is bounded, then* $P(\lim_{t\to\infty} Z_t \in LS) = 1$.

(*iii*) *If A is a supermodular game, then* $P(\omega\{Z_t\} \subset RP$ or $\omega\{Z_t\} \subset M_i$ for some $i) = 1$. *If in addition, n equals* 2 *or* 3 *and* f' *exists and is bounded, then* $P(\lim_{t\to\infty} Z_t \in RP - LU) = 1$.

Proof: Theorem 3.3 of Benaïm and Hirsch (1999a) shows that with probability one, the process Z_t converges to an attractor-free set of the dynamics (P). Proposition C in the Appendix shows that this set must be a connected subset of *CR*. Part (*i*) follows from this result and our Theorems 2.1 and 3.2. The proofs of the remaining parts, which rely on our analyses from Sections 3.3 and 3.4 and on results of

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[19] In particular, if beliefs converge to some rest point x^* of (P), then choice probabilities also converge to $\tilde{B}(x^*) = x^*$.

Pemantle (1990), Benaïm and Hirsch (1999a), and Benaïm (2000), can also be found in the Appendix. ■

Part (*i*) of the theorem guarantees convergence of stochastic fictitious play to the unique rest point of (P) in games which are zero sum or which admit an interior ESS. Part (*ii*) shows that in potential games, convergence to the set of rest points is ensured if the disturbance distribution is smooth; under regularity condition (R), convergence is always to a unique limit point which is Lyapunov stable under (P), and hence a local maximizer of the Lyapunov function Π.

Part (*iii*) of the theorem only guarantees convergence to rest points of (P) in supermodular games with 2 or 3 strategies. When there are more strategies, we cannot rule out convergence to one of the unstable invariant manifolds M_i. Benaïm (2000) conjectures (and proves under additional assumptions) that such manifolds cannot be limits of stochastic approximation processes. If this conjecture is correct, convergence to rest points of (P) can be established in all supermodular games.

5. Evolution with Stochastic Decision Rules

Blume (1993, 1997) and Young (1998) study the evolution of play in large populations whose members follow stochastic decision rules. They obtain tight long run predictions of behavior when players play a potential game and where players' decision probabilities are determined using logit choice functions. Unfortunately, as we shall explain below, their method of analysis does not seem to extend beyond these classes of games and choice functions.[20]

More recently, Binmore and Samuelson (1999), Sandholm (2000a), and Benaïm and Weibull (2000) have developed general techniques for analyzing evolutionary models with stochastic decision rules. These papers do not offer predictions of play for specific classes of games. However, by combining the techniques developed in these papers and others with our analysis of the perturbed best response dynamics, we can generate medium and long run predictions of play in four classes of games, predictions which do not depend on the specification of payoff disturbances.

A population of N players is repeatedly matched to play a symmetric game A. Players occasionally receive opportunities to change their behavior, with each player's revision opportunities arriving via independent, rate 1 Poisson processes.

[20] On the other hand, Blume (1993, 1997) and Young (1998) also obtain results for local interaction models, which we do not consider here.

When a player receives a revision opportunity, he evaluates the current expected payoff to each of his pure strategies, but these evaluations are subject to shocks which are independent across strategies and over time and which follow a common distribution function F. The player selects the strategy which he evaluates as best.

Aggregate behavior in this model is described by a continuous time Markov process $\{X_t^N\}_{t\geq 0}$ on the state space $\Delta^N = \{x \in \Delta\text{: } Nx_i \in \mathbf{Z} \text{ for all } i\}$. The initial condition X_0 is arbitrary. Let τ^r be the random time at which the rth revision opportunity is received, and let b^r be a random vector representing the payoff disturbance which occurs during this opportunity. For a switch from strategy i to strategy j to occur, the player granted the revision opportunity must be playing strategy i, and the realization of his payoff disturbance must render strategy j a best response. Transitions of X_t^N are therefore described by

$$\begin{aligned} P\left(X_{\tau^{r+1}}^N = x + \tfrac{1}{N}(e_j - e_i) \,\middle|\, X_{\tau^r}^N = x\right) &= x_i\, P(\text{argmax}_k\, (Ax)_k + b_k^r = j) \\ &= x_i\, \tilde{B}_j(x) \end{aligned}$$

for all $i \neq j$. With the remaining probability of $\sum_i x_i \tilde{B}_i(x)$, no change in state occurs.

The expected increment in X_t^N during a single revision opportunity is given by

$$\begin{aligned} E\left(X_{\tau^{r+1}}^N - X_{\tau^r}^N \,\middle|\, X_{\tau^r}^N = x\right) &= \sum_i \sum_j \tfrac{1}{N}(e_j - e_i) x_i \tilde{B}_j(x) \\ &= \tfrac{1}{N}\left(\sum_j e_j \tilde{B}_j(x) \sum_i x_i - \sum_i e_i\, x_i \sum_j \tilde{B}_j(x) \right) \\ &= \tfrac{1}{N}(\tilde{B}(x) - x). \end{aligned}$$

Since the expected number of revision opportunities per time unit is N, the expected motion of X_t^N is captured by the perturbed best response dynamics (P).

Binmore and Samuelson (1999), Sandholm (2000a), and Benaïm and Weibull (2000) show that over finite time spans, the stochastic evolution of a large population closely mirrors a solution to the differential equation which describes its expected motion. We can therefore show that in the classes of games we consider, a large population must approach and then remain near the rest points of the dynamics (P) for some long (but finite) amount of time.

To state this result, we consider a sequence of Markov processes X_t^N whose initial conditions $X_0^N \in \Delta^N$ converge to some state $x_0 \in \Delta$ as N approaches infinity. We say that these processes *converge in the medium run* to the set $A \subset \Delta$ from the initial condition x_0 if for each $\varepsilon > 0$, there is a time $T_0 = T_0(x_0)$ such that for all $T \in [T_0, \infty)$,

$$\lim_{N\to\infty} P\left(\sup_{t\in[T_0,T]} \inf_{x^*\in A} \left| X_t^N - x^* \right| < \varepsilon \right) = 1.$$

In words: if a large population begins play near x_0, then with probability close to 1, the population approaches the set A and remains nearby for a long, finite time span.

Theorem 5.1: *Consider the model of stochastic evolution described above.*

(*i*) *If A is zero sum or admits an interior ESS, then* X_t^N *converges in the medium run to* $RP = \{x^*(\Lambda)\}$ *from every initial condition* $x_0 \in \Delta$.

(*ii*) *If A is a potential game, then* X_t^N *converges in the medium run to RP from every initial condition* $x_0 \in \Delta$.

(*iii*) *If A is a supermodular game, then* X_t^N *converges in the medium run to RP from almost every initial condition* $x_0 \in \Delta$.

Proof: Theorem 4.1 of Sandholm (2000a) shows that over any finite horizon, the stochastic process X_t^N stays within $\frac{\varepsilon}{2}$ of the solution trajectory of (P) with the same initial condition with probability close to 1 when N is large. Theorems 2.1, 3.2, and 3.3 show that in the games considered in parts (*i*) and (*ii*), all solution trajectories of (P) converge to *RP*; Theorem 3.9 shows that in supermodular games, this is true of trajectories starting from almost every initial condition. Combining these results proves the theorem. ■

In case (*i*), where A is zero sum or admits an interior ESS, one can establish a stronger result: the time T_0 of convergence to a neighborhood of $x^*(\Lambda)$ can be chosen independently of the initial condition x_0. A statement and proof of this claim is presented in the Appendix (Proposition 7).

The finite horizon description of behavior provided by Theorem 5.1 suffices for many economic applications. However, in settings where behavior over very long time spans is of interest, it may be more natural to consider an infinite horizon description of behavior. Unfortunately, Theorem 5.1 cannot be directly extended to an infinite horizon result ($T = \infty$): since the process X_t^N is irreducible, all states in Δ^N

are visited infinitely often with probability one, so large deviations from all rest points are certain to occur.[21]

To characterize infinite horizon behavior, one needs to describe the stationary distribution μ^N of the process X_t^N. Since X_t^N is irreducible and aperiodic, the stationary distribution is unique. It describes the long run behavior of X_t^N in two distinct ways. Regardless of initial behavior, μ^N approximates the probability distribution of X_t^N after a long enough time has passed; μ^N also almost surely describes the limiting time average of play.

We now provide our characterizations of μ^N.

Theorem 5.2: *Consider the model of stochastic evolution described above.*

(*i*) *Suppose A is zero sum or admits an interior ESS. If Q is an open set containing* $x^*(\Lambda)$, *then* $\lim_{N\to\infty} \mu^N(Q) = 1$.

(*ii*) *Suppose A is a potential game, and let O be an open set containing RP. Then* $\lim_{N\to\infty} \mu^N(O) = 1$. *If condition* (R) *holds, and Q is an open set containing LS, then* $\lim_{N\to\infty} \mu^N(Q) = 1$.

(*iii*) *Suppose that A is supermodular. If Q is an open set containing LS, then* $\lim_{N\to\infty} \mu^N(Q) = 1$.

Part (*i*) of the theorem shows that when A is zero sum or admits an interior ESS, then in the long run a large population spends nearly all time in a neighborhood of the rest point $x^*(\Lambda)$. In fact, we show in the Appendix that in these cases, long run behavior can be described by a normal distribution centered at $x^*(\Lambda)$. Part (*ii*) shows that if A is a potential game, then in the long run, a large population will nearly always stay near rest points of (P); if the regularity condition (R) holds, the population only stays near Lyapunov stable rest points. Part (*iii*) shows that if A is supermodular, a large population must again stay near Lyapunov stable rest points of (P). The proof of the theorem, which combines our earlier analysis with results of Kurtz (1976), Benaïm (1998), Benaïm and Hirsch (1999b), and Benaïm and Weibull (2000), is contained in the Appendix.

Blume (1993, 1997), Young (1998), and Ianni (1999) study evolution in potential games under the logit choice rule. They show that in this setting, the Markov process describing the evolution of play is reversible, regardless of whether interactions are global or local in nature. This allows them to establish that the

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[21] This observation underlies analyses of games based on stochastic stability, an approach pioneered by Foster and Young (1990), Kandori, Mailath, and Rob (1993), and Young (1993).

stationary distribution is proportional to $\exp(\frac{1}{\varepsilon}p(\cdot))$, where $p(\cdot)$ is a suitably chosen function on the relevant state space, and ε is the noise level. If the interaction is global, it can be shown that $p(x)$ is nearly proportional to the potential function $a(x) = \frac{1}{2}x \cdot Ax$ when the population size is large. Thus, when the noise level ε is small, the stationary distribution places most of its mass near the global maximizer of potential whenever a unique maximizer exists. This result provides a unique prediction of long run behavior.

When it is applicable, reversibility is a powerful tool for studying infinite horizon behavior. Unfortunately, this tool seems only to work when studying potential games, and then only under logit choice functions: these are the only choice functions which both generate a reversible Markov process and can be derived from an explicit model of payoff perturbations. In contrast, our analysis can be employed to study other classes of games, and is not sensitive to the choice of disturbance distribution.

The equilibrium selection results of Blume (1993, 1997), Young (1998), and Ianni (1999) are obtained under the logit choice rule, which is generated by extreme value distributed disturbances. It is natural to ask whether these results can be generalized to arbitrary disturbance distributions. Blume (1999) shows that this is possible in 2 x 2 games: the risk dominant equilibrium, which is also the global maximizer of potential, is selected under any i.i.d. payoff disturbances. Whether an analogous selection result can be established for n x n games is an open question.

6. Fictitious Play in a Diverse Population

In the stochastic models we have considered so far, the distribution F has been used to describe random payoff shocks. We now turn to two deterministic models of learning (Ellison and Fudenberg (2000)) and evolution (Ely and Sandholm (2000)) in which this distribution is used to describe the preference composition of a fixed population. Our notation is borrowed from Ely and Sandholm (2000).

In these models, players are randomly matched to play an $n \times n$ game, but different players evaluate the outcome of a match using different payoff matrices $\pi \in \Pi = \mathbf{R}^{n\times n}$. To obtain the distribution of payoff matrices in the population, we suppose that all players' preferences are based on the same payoff matrix A, with variation around this matrix determined by biases towards each strategy. Specifically, we suppose that these biases are described by i.i.d. random variables with distribution F. Letting b denote a vector of such random variables, we define the distribution of payoff matrices in the population by $P(\pi \in S) = P(A + b\mathbf{1}' \in S)$.

There are a continuum of players with each preference π in the support of P. The behavior of the subpopulation with preference π is a strategy distribution $\sigma(\pi)$ in the simplex Δ. Hence, a complete description of the population's behavior is given by a *Bayesian strategy* $\sigma: \Pi \to \Delta$. Observe that $E\sigma = \int_{\Pi} \sigma \, dP \in \Delta$ represents the aggregate behavior of the population as a whole.

Since players are randomly matched with players drawn from the entire population, each player's best responses are defined with respect to aggregate behavior $x = E\sigma \in \Delta$. Thus, the best response function, $\mathbf{B}: \Delta \to \Sigma$, is defined by

$$\mathbf{B}(x)(\pi) = \underset{y \in \Delta}{\arg\max} \; y \cdot \pi x.$$

Best responses are unique up to a set of preferences with measure zero.

In Ellison and Fudenberg's (2000) two population fictitious play model, players always choose a best response to the time average of past play. We now describe a single population version of their model. Let $c_t \in \Delta$ denote aggregate behavior at time t, and let $z_t = \frac{1}{t}\int_0^t c_t \, dt \in \Delta$ denote the time average of play. By the definition of the fictitious play process, $c_t = E(\mathbf{B}(z_t))$.[22] Hence, by differentiating the definition of z_t with respect to t, we obtain

$$\begin{aligned} \dot{z}_t &= \tfrac{1}{t} c_t - \tfrac{1}{t^2}\int_0^t c_t \, dt \\ &= \tfrac{1}{t}(E(\mathbf{B}(z_t)) - z_t). \end{aligned} \tag{15}$$

Thus, the time average always moves in the direction of the aggregate best response $E(\mathbf{B}(z_t))$, doing so more slowly as time passes. A computation reveals that

$$\begin{aligned} E(\mathbf{B}(z))_i &= \int_{\Pi} \mathbf{B}_i(z) \, dP \\ &= P(\pi\colon \; \mathbf{B}_i(z)(\pi) = e_i) \\ &= P\big(\pi : \; \arg\max_k \, (\pi z)_k = i\big) \\ &= P\big(\arg\max_k \, (Az)_k + b_k = i\big) \\ &= \tilde{B}_i(z). \end{aligned} \tag{16}$$

[22] To avoid technical difficulties at time zero, one can assume that c_t and z_t are constant during some initial interval $[0, t_0]$.

Thus, the equation of motion for beliefs in the fictitious play process is again a reparameterized version of the perturbed best response dynamics (P).

Since equation (15) is an ordinary differential equation, local stability of its rest points can be determined by computing the eigenvalues of its derivative matrix. Using this approach, Ellison and Fudenberg (2000) characterize local stability of mixed equilibria of asymmetric 2 x 2 games under all bias distributions, and prove local stability results for certain classes of 3 x 3 games and bias distributions. We now establish *global* convergence results for four families of n x n games and all bias distributions. As before, we state results for beliefs z_t; since $E(\mathbf{B}(\cdot))$ is continuous, corresponding results hold for aggregate behavior $E(\mathbf{B}(z_t))$.

Theorem 6.1: *Consider population fictitious play.*

(*i*) *If A is zero sum or admits an interior ESS, then beliefs z_t converge to $x^*(\Lambda)$ from all initial conditions.*

(*ii*) *If A is a potential game, then $\omega\{z_t\} \subset RP$ regardless of the initial condition z_0. If condition* (R) *holds, convergence is always to a single point, and only rest points in LS are locally stable.*

(*iii*) *If A is supermodular, then beliefs converge to a point in RP from almost every initial condition in Δ.*

Proof: Follows directly from Theorems 2.1, 3.2, 3.3, 3.5, and 3.9. ■

7. Evolution in a Diverse Population

Ely and Sandholm (2000) study evolution in populations whose preferences are diverse, focusing on best response dynamics. Under the standard best response dynamics (BR), aggregate behavior always moves in the direction of the current best response. To extend this idea to the diverse population setting, one supposes that this occurs in each subpopulation π. This leads to the *Bayesian best response dynamics*

(B) $\dot{\sigma} = \mathbf{B}(E(\sigma)) - \sigma,$

which is defined on the space $\Sigma = \{\sigma\colon \Pi \to \Delta\}$ of Bayesian strategies.

Since Σ is a function space, in order to interpret the Bayesian dynamics (B) one must specify the norm on Σ with respect to which the dynamics are defined.[23] Ely and Sandholm (2000) show that if (B) is interpreted using the L^1 norm,

$$\|\sigma\| \equiv \sum_{i=1}^{n} E|\sigma_i|,$$

then equation (B) defines a Lipschitz continuous law of motion, and so possesses unique solution trajectories.

Dynamics on the space Σ of Bayesian strategies are difficult to analyze directly. For this reason, Ely and Sandholm (2000) establish close connections between the Bayesian dynamics (B) and the *aggregate best response dynamics*, defined on the simplex by

$$\text{(AB)} \quad \dot{x} = E(\mathbf{B}(x)) - x.$$

These dynamics describe the evolution of aggregate behavior $E\sigma$ under the dynamics (B). Ely and Sandholm (2000) show that aggregate behavior x^* is a rest point of (AB) if and only if the Bayesian strategy $\sigma^* = \mathbf{B}(x^*)$ is a rest point of (B). Similarly, x^* is stable under (AB) if and only if $\mathbf{B}(x^*)$ is stable under (B), where "stable" can refer to Lyapunov, asymptotic, or global stability.

Equation (16) shows that equation (AB) is identical to the perturbed best response dynamics (P). We can therefore establish the following results concerning the convergence of solutions to the Bayesian dynamics (B).

Theorem 7.1: *Consider evolution in a diverse population under the dynamics* (B).

(*i*) *If A is zero sum or admits an interior ESS, then the Bayesian strategy* $\mathbf{B}(x^*(\Lambda))$ *is globally stable.*

(*ii*) *If A is a potential game, then* $\omega\{\sigma_t\} \subset \mathbf{B}(RP)$ *regardless of the initial condition* σ_0. *If condition* (R) *holds, convergence is always to a single point, and only points in* $\mathbf{B}(LS)$ *are locally stable.*

(*iii*) *If A is supermodular, there is an open dense set of initial conditions* $\Sigma_0 \subset \Sigma$ *from which convergence to a single point in* $\mathbf{B}(RP)$ *is guaranteed.*

[23] The trajectory $\{\sigma_t\}$ is a *solution of* $\dot{\sigma} = f(\sigma)$ *with respect to norm* $\|\cdot\|$ if $\lim_{h\to 0} \|h^{-1}(\sigma_{t+h} - \sigma_t) - f(\sigma_t)\| = 0$ for each time t.

Proof: Theorem 6.6 of Ely and Sandholm (2000) shows that aggregate behavior x^* is globally stable under (AB) = (P) if and only if the Bayesian strategy $\mathbf{B}(x^*)$ is globally stable under (B). Part (*i*) of the theorem follows from this result and Theorems 2.1 and 3.2. Parts (*ii*) and (*iii*) of the theorem are proved in the Appendix. ■

8. Multiplayer and Multipopulation Models

We now describe how the results from the previous sections can be extended to multipopulation models. For convenience, we consider only two-player games; however, our results for potential games and supermodular games can be extended to the *p*-player case.

An (asymmetric) two-player game is described by a pair of $m \times n$ payoff matrices (A^1, A^2); A^k_{ij} is the payoff received by player k when player 1 plays strategy $i \in \{1, \dots, m\}$ and player 2 plays strategy $j \in \{1, \dots, n\}$. Thus, if player 1 chooses mixed strategy $x \in \Delta^1 = \{x \in \mathbf{R}^m_+ \colon \sum_i x_i = 1\}$ and player 2 mixed strategy $y \in \Delta^2 = \{y \in \mathbf{R}^n_+ \colon \sum_j y_j = 1\}$, their expected payoffs are $x \cdot A^1 y$ and $x \cdot A^2 y$, respectively.

Since best response functions for each player are defined with respect to the other player's behavior, the perturbed best response dynamics are defined as follows:

$$\text{(P2)} \qquad \begin{aligned} \dot{x} &= \tilde{B}^1(y) - x \equiv C^1(A^1 y) - x; \\ \dot{y} &= \tilde{B}^2(x) - y \equiv C^2((A^2)' x) - y. \end{aligned}$$

The choice probability functions C^1 and C^2 are defined via equation (1') in terms of stochastic perturbations with distributions F^1 and F^2. By Theorem 2.2, C^1 and C^2 can also be represented using equation (2') in terms of some admissible deterministic perturbations, V^1 and V^2.

As in the single population case, the perturbed dynamics associated with zero sum games and potential games can be characterized by taking advantage of their deterministic representation.[24] A two player game (A^1, A^2) is *zero sum* if $A^1 = -A^2$, while it is a *potential game* if $A^1 = A^2$. The following results of Hofbauer and Hopkins (2000) show that the perturbed dynamics for these two classes of games can be described using Lyapunov functions.

[24] Our results for games with an interior ESS only extend to the multipopulation case in a trivial sense: any ESS of a multipopulation game must be a strict (and hence pure) Nash equilibrium (Selten (1980)).

Theorem 8.1 (Hofbauer and Hopkins (2000)):

(*i*) *If* $(A^1, A^2) = (A, -A)$ *is zero sum, then the strictly concave function*

$$\Lambda(x, y) = -\left(V^1(x) + W^1(Ax) + V^2(y) + W^2(-A'x)\right)$$

is a strict Lyapunov function for the dynamics (P2).

(*ii*) *If* $(A^1, A^2) = (A, A)$ *is a potential game, then the function*

$$\Pi(x, y) = x \cdot Ay - V^1(x) - V^2(y).$$

is a strict Lyapunov function for the dynamics (P2).

Theorem 8.1 (*i*) implies that the dynamics (P2) corresponding to any zero sum game $(A, -A)$ admit a unique chain recurrent point. It can therefore be shown that all of our results for zero sum games from Sections 4 through 7 can be established in the two population case. To extend our results on potential games, we must once again impose additional assumptions to characterize the chain recurrent set. In particular, all of our earlier results can be established under an appropriate regularity condition on the matrix

$$\begin{bmatrix} -D^2V^1(x) & A \\ A' & -D^2V^2(y) \end{bmatrix},$$

which is the Hessian matrix of the Lyapunov function $\Pi(x, y)$.

All of our results for supermodular games can be established in the multipopulation case as well. We call the game (A^1, A^2) (*strictly*) *supermodular* if both players' payoffs satisfy an increasing differences property: when $j > i$, $A^1_{jk} - A^1_{ik}$ and $A^2_{kj} - A^2_{ki}$ are strictly increasing in k. Given our results from Section 3.4, it is easily verified that after an appropriate change in coordinates, the dynamics (P2) derived from any supermodular game are cooperative and irreducible, and hence strongly monotone. We can therefore extend Theorem 3.9 and our results for supermodular games from Sections 4 through 7 to multipopulation settings.

9. Conclusion

We studied four models of evolution and learning in games which rely on perturbations of the payoffs to each pure strategy. For each of these models, we established global convergence results in four important classes of games.

While most of this paper has focused on the similarities among the four models, we should also point out the differences. In both of the stochastic models, payoff disturbances are i.i.d. over time, and so represent influences on behavior which are realized anew each time the game is played. In contrast, in the deterministic models different players experience different payoffs, but the payoffs of each individual are fixed over time. Such payoff diversity seems natural in many economic applications involving large populations of players.[25]

We can also make distinctions between the predictions of the stochastic and deterministic models. The most noteworthy of these is the relative strengths of their long run predictions. Because stochastic fictitious play and stochastic evolution explicitly allow for randomness in behavior, unstable limit points of the perturbed best response dynamics can be eliminated from consideration as long run predictions of play. In the deterministic models these unstable limit points cannot be ruled our *a priori*; however, they can only arise after very unusual initial conditions.

Finally, we observe an important difference between the possibilities for long run prediction in the two stochastic models. Regardless of the initial conditions, every attractor of the dynamics (P) is a possible limit behavior under stochastic fictitious play (Benaïm and Hirsch (1999, Theorem 5.4)). In contrast, it is well known that in models of stochastic evolution, the limiting stationary distribution can place all of its mass on a single attractor, even when multiple attractors exist.[26] Theorem 5.2 showed that in the games under consideration, the limiting distribution only places mass on the stable rest points of (P), but our result does not make a selection among them. As we noted earlier, Blume (1999) has obtained a selection result for 2 x 2 games which does not depend on the disturbance distribution. Establishing equilibrium selection results for more general strategic environments is an important topic for future research.

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[25] See, for example, Sandholm (2000c).

[26] There can also be more subtle differences between the limit behaviors of the two models – see Benaïm (1998, p. 70).

Appendix

A.1 Notions of Recurrence for Deterministic Flows

Our analyses of long run behavior in stochastic fictitious play and in the stochastic evolution model require a variety of notions of recurrence for deterministic flows. In this section, we introduce the concepts we need and explain the relationships between them.

Consider a Lipschitz continuous differential equation $\dot{x} = f(x)$ which is defined on the simplex Δ and whose solutions do not leave Δ. This equation defines a (*semi-*) *flow* ϕ: $\mathbf{R}_+ \times \Delta \to \Delta$, where $\phi(t, x)$ is the position at time t of the solution to $\dot{x} = f(x)$ starting at x. For each set $A \subset \Delta$, we define $\phi(t, A) = \{\phi(t, x):\ x \in A\}$. The set A is *forward invariant* if $\phi(t, A) \subset A$ for all $t \geq 0$, and it is *invariant* if $\phi(t, A) = A$ for all $t \geq 0$.

The classical notions of recurrence and asymptotic behavior of a flow are based on the concept of the ω-limit set. The *ω-limit set* of the state $x \in \Delta$ is the set of limit points of the trajectory with initial condition x: $\omega(x) = \{z \in \Delta:\ \lim_{k\to\infty}\phi(t_k, x) = z$ for some $t_k \to \infty\}$. We let $\Omega = \bigcup_{x\in\Delta}\omega(x)$ denote the union of all ω-limit sets. Also, we call state x *recurrent* if $x \in \omega(x)$, and we let R denote the set of recurrent states. The set $cl(R)$ is known as the *Birkhoff center of attraction*.

For the analysis of the two stochastic models two other notions of recurrence are more useful. One is a refinement of the aforementioned concepts, the other a coarsening.

A *center of attraction* is a compact invariant set A such that for all open sets U containing A and all $x \in \Delta$, we have that $\frac{1}{t}\int_0^t 1_U(\phi(t,x))dt$ approaches 1 as t approaches infinity.[27] That is, all orbits spend almost all of their time arbitrarily close to A. It can be shown that the intersection of all centers of attraction is nonempty and itself a center of attraction – the *minimal center of attraction MCA*.

There is a different description of MCA in terms of measures which are invariant under the flow ϕ. Let μ be a (Borel) probability measure on Δ. The support of this measure is defined by $supp(\mu) = \{x \in \Delta:\ \mu(O) > 0$ for every open set O containing $x\}$. Alternatively, $supp(\mu)$ is the smallest compact subset of Δ with μ-measure 1. We say that the measure μ is *invariant* under ϕ if $\mu(\phi(t, A)) = \mu(A)$ for all A and all $t \geq 0$. Let $M(\phi)$ denote the set of all measures which are invariant under ϕ.

[27] Here, 1_U: $\Delta \to \{0, 1\}$ denotes the indicator function for the set $U \subset \Delta$.

Then the minimal center of attraction of ϕ can be characterized as $MCA = cl(\bigcup_{\mu \in M(\phi)} supp(\mu))$. For further details concerning the minimal center of attraction, see Nemytskii and Stepanov (1960, Sections 5.6 and 6.9) or Akin (1993).

Our final and most general notion of recurrence is chain recurrence, a concept due to Conley (1978) (also see Akin (1993), Robinson (1995), or Benaïm (1999, Section 5.1)). This notion of recurrence describes the states which can occur in the long run if the dynamics are subject to small shocks which occur at isolated moments in time. An *ε-chain* from x to y is a sequence $\{x = x_0, x_1, \ldots, x_n = y\}$ such that $\left|\phi(t_i, x_{i-1}) - x_i\right| < \varepsilon$ for some $t_i \geq 1$, where $i \in \{1, \ldots, n\}$. The state x is *chain recurrent* if there is an ε-chain from x to itself for all $\varepsilon > 0$. We let CR denote the set of chain recurrent states.

The relationships among the four notions of recurrence described above are well known. We summarize them the following proposition.

Proposition A: $MCA \subset cl(R) \subset cl(\Omega) \subset CR$.

Proof: The first inclusion follows from the Poincaré Recurrence Theorem; see Nemytskii and Stepanov (1960, p. 368). The second inclusion follows from the fact that $R \subset \Omega$.

To prove the third inclusion, we first show that $\Omega \subset CR$. Let $y \in \omega(x)$. Then there exists an unbounded sequence of times $\{t_k\}_{k \in \mathbf{N}}$ such that $\left|\phi(t_k, x) - y\right| \to 0$. Fix $\varepsilon > 0$. By the continuity of ϕ, we can choose an l large enough that $\left|\phi(t_l + 1, x) - \phi(1, y)\right| < \varepsilon$. Then since $\left|\phi(t_k, x) - y\right| < \varepsilon$ for some $t_k > t_l + 1$, $\{y, \phi(t_l + 1, x), y\}$ is an ε-chain, and so $y \in CR$.

Finally, we show that CR is closed. Suppose that $\{x_k\}_{k \in \mathbf{N}} \subset CR$ and that $x_k \to x$. Fix $\varepsilon > 0$. Choose l large enough that $\left|\phi(t, x_l) - \phi(t, x)\right| < \frac{\varepsilon}{2}$ for all $t \in [0, 2]$. Let $\{x_l = y_0, y_1, \ldots, y_{n-1}, y_n = x_l\}$ be an $\frac{\varepsilon}{2}$-chain; without loss of generality, we can suppose that $t_1 \in [1, 2]$. Then by applying the triangle inequality twice, we find that $\{x, y_1, \ldots, y_{n-1}, x\}$ is an ε-chain. Hence, $x \in CR$, and so CR is closed, allowing us to conclude that $cl(\Omega) \subset CR$. ■

There is an alternate description of CR in terms of the concept of an attractor. An *attractor* A for a semiflow ϕ is a nonempty, compact, invariant set which is asymptotically stable. (The definition stated in Benaïm (1999, p. 22) is equivalent, and attractors so defined are sometimes called *uniform attractors*.) We let $\mathbf{A}(\phi)$ denote the set of attractors under ϕ. Each attractor A possesses a basin of attraction

$B(A) = \{x \in \Delta\colon \omega(x) \subset A\}$, which is an open set containing A. Its complement $R(A) = \Delta - B(A)$, which is called the *dual repellor of* A, is a compact, invariant set.

We now provide the alternate description of the chain recurrent set. Begin with the simplex Δ; then, for each attractor A, remove the set of "transient" states $B(A) - A$. What remains is the chain recurrent set CR.

Proposition B: $CR = \bigcap_{A \in \mathbf{A}(\phi)} A \cup R(A) = \Delta - \left(\bigcup_{A \in \mathbf{A}(\phi)} B(A) - A \right)$.

Proof: See Robinson (1995, Theorem 9.1.3).

Benaïm and Hirsch (1999a) introduce the notion of an attractor-free set. The set A is *attractor-free* if it is nonempty, compact, and forward invariant, and if no proper subset of A is an attractor with respect to the flow restricted to A. The last proposition shows that every point in an attractor-free set is chain recurrent.

Proposition C: *The set A is attractor-free if and only if A is connected and every point in A is chain recurrent with respect to the flow restricted to A. In particular, if A is attractor-free, then $A \subset CR$.*

Proof: See Benaïm (1998, Proposition 5.3).

A.2 Proofs Omitted from the Text

The Proof of Proposition 2.3

Substituting $V(y) = -\sum_j \ln y_j$ into equation (2'), we find that this selection of V yields the choice probability function $C_i(a) = (c(a) - a_i)^{-1}$, where $c(a)$ is the unique number satisfying $c(a) > \max_j a_j$ and $\sum_j (c(a) - a_i)^{-1} = 1$. Now suppose that $n \geq 4$, and let i, j, and k be distinct strategies. A computation reveals that

$$\frac{\partial^2 C_i}{\partial a_j \, \partial a_k} = \frac{2 C_i^2 C_j^2 C_k^2}{\left(\sum_l C_l^2\right)^3} \left(\left(C_i + C_j + C_k \right) \sum_l C_l^2 - \sum_l C_l^3 \right).$$

This expression is negative whenever C_i, C_j, and C_k are all close enough to zero.

Now suppose that the choice function $\hat{C}$ is derived from a stochastic perturbation of payoffs. Then differentiating expression (4) with respect to a_k reveals that $\frac{\partial^2 \hat{C}_i}{\partial a_j \partial a_k}$ must always be strictly positive. (In fact, this can be established even when the stochastic payoff perturbations are not i.i.d.) We therefore conclude that the choice function $C_i(a) = (c(a) - a_i)^{-1}$ cannot be derived from a stochastic perturbation of payoffs. ■

The Proof of Proposition 2.5

Suppose that the choice function C satisfies equation (12) and equation (2') for some admissible V. (Theorem 2.2 implies that if C satisfies (1'), it satisfies (2') as well.) Then if we define W: $\mathbf{R}_0^n \to \mathbf{R}$ via equation (11), Theorem 26.5 of Rockafellar (1970) implies that $\nabla W \equiv C$ on $\mathbf{R}_0^n$. Moreover, it is clear from equation (2') that $C(a + c\mathbf{1}) = C(a)$ for all $c \in \mathbf{R}$. Thus, if we define W on the remainder of $\mathbf{R}^n$ by $W(a + c\mathbf{1}) = W(a) + c$ for any $a \in \mathbf{R}_0^n$ and $c \in \mathbf{R}$, it can be verified that $\nabla W \equiv C$ on all of $\mathbf{R}^n$. It follows that $DC(a)$ is symmetric for all $a \in \mathbf{R}^n$.

Now, applying equation (12), we observe that if $i \neq j$, then

$$\frac{\partial C_i}{\partial a_j}(a) = -\frac{w(a_i)\,w'(a_j)}{\left(\sum_k w(a_k)\right)^2}.$$

Thus, the symmetry of DC implies that $w(a_i)w'(a_j) = w(a_j)w'(a_i)$ for all a. Since w is strictly positive, it follows that $w'(a_i) = \varepsilon\, w(a_i)$ for some constant ε, and hence that $w(a_i) = K \exp(\varepsilon\, a_i)$ for some constants ε and K. Since w is strictly positive and increasing, it must be that ε and K are strictly positive, and hence that $C \equiv L$. ■

The Proof of Theorem 3.1

Recall that the perturbed best response function $\tilde{B}(x)$ can be written as $C(Ax)$, where C is a perturbed version of the maximizer function $M(a) = \text{argmax}_{y \in a}\, y \cdot a$. For each distribution function F^k, let C^k denote the corresponding choice probability function: $C_i^k(a) = P(\text{argmax}_j\, a_j + b_j^k = i)$, where the random variables b_j^k are i.i.d. with distribution F^k.

We first prove a lemma.

Lemma 1: *Suppose that* $F^k \Rightarrow \delta_{\{0\}}$ *and that* $a^k \to a^*$. *If* $i \notin \text{argmax}_j\, a_j^*$, *then* $C_i^k(a^k) \to 0$.

Proof: Let $l \in \text{argmax}_j\, a_j^*$, and let $\varepsilon = a_l^* - a_i^* > 0$. Then for all large enough k,

$$\begin{aligned} C_i^k(a^k) &= P(\text{argmax}_j\, a_j^k + b_j^k = i) \\ &\leq P(a_i^k + b_i^k \geq a_l^k + b_l^k) \\ &\leq P(b_i^k - b_l^k \geq \tfrac{\varepsilon}{2}). \end{aligned}$$

Therefore, since the random variables $(b_i^k - b_l^k)$ converge in distribution to the constant 0 as k approaches infinity, $C_i^k(a^k) \to 0$. ❑

Now suppose that $x^k = C^k(A\,x^k)$ and that $x^k \to x^*$. To prove the theorem, it is enough to show that $x^* \in M(Ax^*)$. Clearly, $A\,x^k \to Ax^*$. Hence, Lemma 1 tells us that if $i \notin \text{argmax}_j\, (Ax^*)_j$, then $x_i^* = \lim_{k\to\infty} x_i^k = \lim_{k\to\infty} C_i^k(A\,x^k) = 0$. Consequently, $x^* \in M(Ax^*)$. This completes the proof of the theorem. ■

The Proof of Proposition 3.4

The Lyapunov function $\Pi(x) = \frac{1}{2}x\cdot Ax - V(x)$, which is a map from the $n - 1$ dimensional space Δ to the one dimensional space $\mathbf{R}$, is C^{n-1} by assumption. Therefore, since $n - 1 > \max\{0, (n-1) - 1\}$, Sard's Theorem implies that the set of critical values of Π has measure zero. This set is equal to the set $\{\Pi(x)\colon\ x \in RP\}$ by Theorem 3.5 (*i*). Thus, Propositions 5.3 and 6.4 of Benaïm (1999) (also see Exercises 3.16 and 6.11 of Akin (1993)) imply that not only the set of ω-limit points of (P-*V*), but also every attractor-free set under (P-*V*) is contained in *RP*. Therefore, Proposition C enables us to conclude that $CR = RP$. ■

The proof of Theorem 3.5 makes use of the following lemma.

Lemma 2: *Suppose that A is a potential game and that x* is a rest point of* (P-*V*). *Then the eigenvalues of* $H = I_0\,(A - D^2V(x^*))$ *corresponding to directions in* $\mathbf{R}_0^n$ *are real, and have the same signs as the eigenvalues of* $D\dot{x} = DC(Ax^*)\,A - I$ *corresponding to directions in* $\mathbf{R}_0^n$.

Proof: Because A and $D^2V(x^*)$ are symmetric, the matrix H is symmetric on $\mathbf{R}_0^n$ (i.e., $x\cdot Hy = y\cdot Hx$ for all $x, y \in \mathbf{R}_0^n$), so the eigenvalues of H corresponding to directions in $\mathbf{R}_0^n$ are in fact real. (The projection I_0 is introduced so that H maps $\mathbf{R}_0^n$ to itself, ensuring that H can be diagonalized.)

To make the connection with the eigenvalues of $D\dot{x}$, we recall (from the proof of Theorem 2.1) that $DC(a)z = [D^2V(C(a))]^{-1}z$ for all $z \in \mathbf{R}_0^n$, and that $DC(a)\,\mathbf{1} = 0$ (because $C(a + c\mathbf{1}) \equiv C(a)$). Since $[D^2V(C(a))]^{-1}$ is only defined on $\mathbf{R}_0^n$, we can let $[D^2V(C(a))]^{-1}\,\mathbf{1} = 0$ and can write

$$D\dot{x} = [D^2V(C(Ax))]^{-1}A - I.$$

Since at the rest point, $x^* = C(Ax^*)$, and since $[D^2V(C(a))]^{-1}\,\mathbf{1} = 0$, we can continue:

$$\begin{aligned} D\dot{x} &= [D^2V(x^*)]^{-1}A - I \\ &= [D^2V(x^*)]^{-1}(A - D^2V(x^*)) \\ &= [D^2V(x^*)]^{-1}I_0\,(A - D^2V(x^*)) \\ &= QH. \end{aligned}$$

The matrix $Q \equiv [D^2V(x^*)]^{-1} = DC(Ax^*)$ annihilates $\mathbf{1}$, and maps $\mathbf{R}_0^n$ into itself. Therefore, we can find a matrix of eigenvectors, Z, whose first $n - 1$ columns are in $\mathbf{R}_0^n$ and whose nth column is $\mathbf{1}$. Since Q is symmetric and is positive definite on $\mathbf{R}_0^n$, we can write $Q = ZDZ' = PP'$, where D is a diagonal matrix whose first $n - 1$ diagonal elements are positive and whose last diagonal element is zero, and where $P = ZD^{1/2}$. Let $\mathbf{R}_{n-1}^n = \{x \in \mathbf{R}^n\colon x_n = 0\}$. Observe that P maps $\mathbf{R}_{n-1}^n$ onto $\mathbf{R}_0^n$ and maps $(0,\ldots,0,1)'$ to $\mathbf{0}$, while P' maps $\mathbf{R}_0^n$ onto $\mathbf{R}_{n-1}^n$ and maps $\mathbf{1}$ to $\mathbf{0}$.

Let $\tilde{P}^{-1}$ be a matrix which inverts P on $\mathbf{R}_0^n$. Then $QH = PP'H$, viewed as a linear operator on $\mathbf{R}_0^n$, is similar to $\tilde{P}^{-1}PP'HP = P'HP$, viewed as a linear operator on $\mathbf{R}_{n-1}^n$. Moreover, since $P\colon \mathbf{R}_{n-1}^n \to \mathbf{R}_0^n$ is full rank, the symmetric bilinear form $P'HP$ on $\mathbf{R}_{n-1}^n$ is congruent to the symmetric bilinear form H on $\mathbf{R}_0^n$. Since similarity and congruence preserve the signs of eigenvalues, we have established the result. ■

The Proof of Theorem 3.5:

Part (*i*) of the result was proved in the text, while part (*ii*) follows directly from the fact that Π is a strict Lyapunov function for (P-*V*) (Theorem 3.3). We therefore consider part (*iii*). By assumption (R), the critical point x^* is a local maximizer of Π precisely when all of the eigenvalues of $H = I_0\,(A - D^2V(x^*))$ corresponding to directions in $\mathbf{R}_0^n$ are strictly negative; when x^* is not a local maximizer, some eigenvalue must be positive. (Recall that H is symmetric on $\mathbf{R}_0^n$, so that all of these eigenvalues are real). Lemma 2 therefore implies that if x^* is a local maximizer of

Π, it is linearly stable under (P-V), while if it is not a local maximizer, it is linearly unstable.

We now show that RP is finite. The argument above shows that all rest points of (P-V) are hyperbolic: at rest points, all eigenvalues of $D\dot{x}$ are real and nonzero. Hence, all rest points of (P-V) are isolated. If $RP \subset \Delta$ is an infinite set, then as Δ is compact, RP has an accumulation point $\bar{x}$. Since each $x^* \in RP$ satisfies $\tilde{B}(x^*) = x^*$ and since $\tilde{B}$ is continuous, it follows that $\tilde{B}(\bar{x}) = \bar{x}$, and hence that $\bar{x} \in RP$. This contradicts the fact that rest points are isolated, and so we conclude that RP is finite.

Since RP is finite, so is the set $\{\Pi(x): x \in RP\}$. Therefore, by again applying Propositions 5.3 and 6.4 of Benaïm (1999), we can conclude that $CR = RP$. ■

The Proof of Proposition 3.6:

We begin with a lemma.

Lemma 3: *Suppose that* $\{u_k\}_{k=1}^n$ *is strictly increasing and that* $\{c_k\}_{k=1}^n$ *satisfies*

$$\sum_{k=1}^{j} c_k \leq 0 \text{ for all } j \leq n, \text{ with a strict inequality for some } j \text{ and equality at } j = n. \tag{17}$$

Then $\sum_{i=1}^n u_k c_k > 0$.

Proof: For all $j < n$, let $d_j = u_{j+1} - u_j > 0$. Then

$$\sum_{i=1}^{n} u_k c_k = u_n \sum_{k=1}^{n} c_k - \sum_{j=1}^{n-1} d_j \left(\sum_{k=1}^{j} c_k \right) = -\sum_{j=1}^{n-1} d_j \left(\sum_{k=1}^{j} c_k \right) > 0. \quad \square$$

The next lemma shows that the increasing differences property of supermodular games still holds when we consider ordered pairs of opponent's *mixed* strategies. Its proof makes use of the following observation:

$$Sy \geq Sx \text{ if and only if } \sum_{i=1}^{m} (y_i - x_i) \leq 0 \text{ for all } m < n. \tag{18}$$

Lemma 4: *If* $Sy \geq Sx$ *and* $y \neq x$*, then* $(Ay)_i - (Ax)_i$ *is strictly increasing in* i.

Proof: Fix $i < j$. We want to show that $(Ay)_j - (Ax)_j > (Ay)_i - (Ax)_i$, or equivalently, that

$$(Ay)_j - (Ay)_i = \sum_{k=1}^{n}(A_{jk} - A_{ik})y_k > \sum_{k=1}^{n}(A_{jk} - A_{ik})x_k = (Ax)_j - (Ax)_i.$$

Since A is strictly supermodular, $u_k = A_{jk} - A_{ik}$ is strictly increasing in k, while since $Sy \geq Sx$ and $y \neq x$, observation (18) implies that $c_k = y_k - x_k$ satisfies condition (17). Thus, Lemma 3 yields the result. ❑

Now suppose that $Sy \geq Sx$. We want to show that $S\tilde{B}(y) \geq S\tilde{B}(x)$. If $y = x$ this is obviously true, so we suppose instead that $y \neq x$. By observation (18), it is enough to show that for all $m < n$,

$$0 > \sum_{i=1}^{m}(\tilde{B}_i(y) - \tilde{B}_i(x)) = \sum_{i=1}^{m}\int_0^1 \nabla\tilde{B}_i(\lambda y + (1-\lambda)x)\cdot(y-x)\,d\lambda.$$

If we let $z = \lambda y + (1 - \lambda)x$, it is enough to show that

$$\sum_{i=1}^{m}\nabla\tilde{B}_i(z)\cdot(y-x) < 0 \text{ for all } m < n. \tag{19}$$

Since $\tilde{B}(z) = C(Az)$, we see that

$$\sum_{i=1}^{m}\nabla\tilde{B}_i(z)\cdot(y-x) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\sum_{k=1}^{n}\frac{\partial C_i}{\partial a_k}(Az)A_{kj}\right)(y_j - x_j).$$

This expression is negative if

$$\sum_{k=1}^{n}(A(y-x))_k\left(-\sum_{i=1}^{m}\frac{\partial C_i}{\partial a_k}(Az)\right) > 0. \tag{20}$$

Now $u_k = (A(y - x))_k$ is strictly increasing by Lemma 4, while equations (4) and (6) imply that $c_k = -\sum_{i=1}^{m}\frac{\partial C_i}{\partial a_k}$ satisfies condition (17). Therefore, Lemma 3 implies that inequality (20) holds for all $m < n$, and hence that $S\tilde{B}(y) \geq S\tilde{B}(x)$. This completes the proof of the proposition. ■

It is worth noting that only two properties of the choice probability function C were used to establish Proposition 3.6: to establish condition (17) we used the facts that

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\tfrac{\partial C_i}{\partial a_j} > 0 \text{ for all } k, l < n, \text{ and that } \sum_{j=1}^{n}\tfrac{\partial C_i}{\partial a_j} = 0 \text{ for all } i \le n.$$

Notably, the symmetry of *DC*, which was essential for establishing our results for other classes of games, was not needed here. In fact, all of our results for supermodular games extend immediately to dynamics based on any choice probability function satisfying the two properties noted above.

The Proof of Proposition 3.7

If $\{x_t\}$ solves (P), then

$$\tfrac{d}{dt}Sx_t = S(\tfrac{d}{dt}x_t) = S(\tilde{B}(x_t) - x_t) = S\tilde{B}(S^{-1}(Sx_t)) - Sx_t,$$

so $\{Sx_t\}$ solves (S). Conversely, if $\{Sx_t\}$ solves (S), then

$$\tfrac{d}{dt}x_t = S^{-1}(\tfrac{d}{dt}Sx_t) = S^{-1}(S\tilde{B}(S^{-1}(Sx_t)) - Sx_t) = \tilde{B}(x_t) - x_t,$$

so $\{x_t\}$ solves (P). ■

The Proof of Theorem 3.8

Define the function $\hat{B}$: $S(\Delta) \to S(\Delta)$ by $\hat{B}(v) = S\tilde{B}(S^{-1}v)$. Fix $i, j \in \{1, \ldots, n-1\}$, $i \neq j$, and fix $v \in S(\Delta)$. It is enough to show that $\tfrac{\partial \hat{B}_i}{\partial v_j}(v) > 0$.

Let $x = S^{-1}v$. Observe that if e_{j+1} and e_j are standard basis vectors in $\mathbf{R}^n$, then $S(e_{j+1} - e_j) = \iota_j$, a standard basis vector in $\mathbf{R}^{n-1}$. It follows that

$$\begin{aligned}
\frac{\partial \hat{B}_i}{\partial v_j}(v) &= \left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\hat{B}(v+\varepsilon\iota_j) - \hat{B}(v)\right)\right]_i \\
&= \left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(S\tilde{B}(S^{-1}(v+\varepsilon\iota_j)) - S\tilde{B}(S^{-1}(v))\right)\right]_i \\
&= \left[S\left(\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\tilde{B}(x+\varepsilon(e_{j+1}-e_j)) - \tilde{B}(x)\right)\right)\right]_i \\
&= \left[S(D\tilde{B}(x)(e_{j+1}-e_j))\right]_i \\
&= \sum_{k=i+1}^{n}\nabla\tilde{B}_k(x)\cdot(e_{j+1}-e_j)
\end{aligned}$$

Since $\sum_{k=1}^{n}\tilde{B}_k(y) = 1$ for all $y \in \Delta$, $\sum_{k=1}^{n}\nabla\tilde{B}_k(y)\cdot z = 0$ for all $y \in \Delta$ and all $z \in \mathbf{R}_0^n$. We can therefore conclude from equation (19) that

$$\frac{\partial \hat{B}_i}{\partial v_j}(v) = -\sum_{k=1}^{i} \nabla \tilde{B}_k(x) \cdot (e_{j+1} - e_j) > 0. \ \blacksquare$$

To prove Theorem 4.1, we need to establish that the stochastic fictitious play Z_t satisfies a global version of Pemantle's (1990) nondegeneracy condition. To state this condition, we let $U_0^n = \{\theta \in \mathbf{R}_0^n\colon \sum_i \theta_i^2 = 1\}$ denote the set of unit vectors in $\mathbf{R}_0^n$, and let $Y_t = \frac{1}{2}\left(Y_t^1 + Y_t^2\right)$.

Lemma 5: $\min_{z \in \Delta} \min_{\theta \in U_0^n} E\left(\max\left\{(Y_{t+1} - \tilde{B}(z)) \cdot \theta, 0\right\} \middle| Z_t = z\right) > 0.$

To interpret this condition, recall that if $Z_t = z$, then the increment in the beliefs process at time $t + 1$ is $Z_{t+1} - Z_t = \frac{1}{t+1}(Y_{t+1} - z)$, while the expected increment is $E\left(Z_{t+1} - Z_t \middle| Z_t = z\right) = \frac{1}{t+1}(\tilde{B}(z) - z)$. Thus, the condition in the lemma requires that there are significant random deviations of the process Z_t from its expected motion; these deviations must be possible from any current state z and in any direction θ. The proof of Lemma 5 requires this preliminary result.

Lemma 6: *If* $\theta \in U_0^n$, *then* $\max_i \theta_i \geq \frac{1}{\sqrt{n(n-1)}} > \frac{1}{n}$ *and* $\min_i \theta_i \leq -\frac{1}{\sqrt{n(n-1)}} < -\frac{1}{n}$.

Proof of Lemma 6: Since $z \in \mathbf{R}_0^n$, we know that k, the number of strictly positive components of θ, is between 1 and $n - 1$; we may suppose the first k components are strictly positive. Now suppose that $\max_i \theta_i < \frac{1}{\sqrt{n(n-1)}}$. In this case, $\sum_{i=1}^k \theta_i < \frac{k}{\sqrt{n(n-1)}}$ and $\sum_{i=1}^k \theta_i^2 < \frac{k}{n(n-1)}$. Since $\theta \in \mathbf{R}_0^n$, $\sum_{i=k+1}^n \theta_i = -\sum_{i=1}^k \theta_i$; moreover, $\sum_{i=k+1}^n \theta_i^2 < \frac{k^2}{n(n-1)}$, as this sum is maximized if exactly one term is non-zero. But then $\sum_{i=1}^n \theta_i^2 < \frac{(k+1)k}{n(n-1)} \leq 1$, contradicting that θ is of unit length. The proof of the other claim is similar. ❑

Proof of Lemma 5: Since the density of the disturbance vector b has full support on $\mathbf{R}^n$, we can place a uniform lower bound on the probability of an arbitrary strategy being the best response after the payoff disturbances are realized:

$$m \equiv \min_{x \in \Delta} \min_{i \in S} \tilde{B}_i(x) > 0.$$

Now fix $z \in \Delta$ and $\theta \in U_0^n$, and let $A^+ = \{(i, j)\colon (\frac{1}{2}(e_i + e_j) - \tilde{B}(z)) \cdot \theta > 0\}$. Then

$$
\begin{aligned}
&E\Big(\max\Big\{(Y_{t+1}-\tilde{B}(z))\cdot\theta,\,0\Big\}\Big|\,Z_t=z\Big)\\
&\quad=\sum_{i,j}P\Big(Y_{t+1}^1=e_i,\,Y_{t+1}^2=e_j\,\Big|\,Z_t=z\Big)\,\max\Big\{\Big(\tfrac{1}{2}(e_i+e_j)-\tilde{B}(z)\Big)\cdot\theta,\,0\Big\}\\
&\quad=\sum_{(i,j)\in A^+}\tilde{B}_i(z)\,\tilde{B}_j(z)\,\Big((\tfrac{1}{2}(e_i+e_j)-\tilde{B}(z))\cdot\theta\Big)\\
&\quad\geq m^2\max_i\,(e_i-\tilde{B}(z))\cdot\theta\\
&\quad=m^2\Big(\max_i\,\theta_i-(\tilde{B}(z)\cdot\theta)\Big)\\
&\quad\geq m^2\Big(\max_i\,\theta_i-\Big((1-m)\max_i\,\theta_i+m\min_i\,\theta_i\Big)\Big)\\
&\quad=m^3\Big(\max_i\,\theta_i-\min_i\,\theta_i\Big)\\
&\quad\geq\tfrac{2m^3}{n}.\ \blacksquare
\end{aligned}
$$

The Proof of Theorem 4.1 (*ii*)

Suppose that A is a potential game and that f is smooth enough that the corresponding deterministic perturbation V is C^{n-1}. Then Proposition 3.4 and Theorem 2.1 imply that $CR = RP$, while Proposition C shows that this set contains all attractor-free sets under (P). Theorem 3.3 of Benaïm and Hirsch (1999a) states that with probability one, $\omega(Z_t)$ is an attractor-free set. Proposition C implies that all attractor-free sets are connected. We can therefore conclude that $P(\omega(Z_t)$ is a connected subset of $RP) = 1$. (In addition, Proposition 6.4 of Benaïm (1999) implies that the Lyapunov function Π is almost surely constant on $\omega(Z_t)$.)

Next, suppose that A is a potential game, that condition (R) holds, and that f is C^1. Then Theorem 3.5 (*iii*) and Theorem 2.1 show that $CR = RP$ and that this set is finite. Thus, Theorem 3.3 of Benaïm and Hirsch (1999a) implies that $P(\lim_{t\to\infty}Z_t \in RP) = 1$. Furthermore, Theorem 3.5 (*iii*) shows that all points in RP are either linearly stable or linearly unstable. The fact that f' is bounded implies that equation (P) is C^2. Given this observation and Lemma 5, we can use Theorem 1 of Pemantle (1990) to show that $P(\lim_{t\to\infty}Z_t \in LU) = 0$. Consequently, $P(\lim_{t\to\infty}Z_t \in LS) = 1$. ■

The Proof of Theorem 4.1 (*iii*)

Suppose that A is supermodular. By Theorem 3.9 (*iv*), $CR = RP \cup \bigcup_i M_i$; hence, applying Proposition C and Theorem 3.3 of Benaïm and Hirsch (1999) establishes the first statement in part (*iii*). If we suppose that f' is bounded, then equation (P) is C^2. If $n = 2$, then each M_i is simply a rest point, and so our result for this case follows from the C^2 smoothness of (P), Lemma 5, and Theorem 1 of Pemantle (1990). If n = 3, we can appeal to Proposition 3.7, Theorem 3.8, and Theorem 4.3 of Benaïm (2000), which establishes that when (P) is a C^2, two dimensional, cooperative, and irreducible dynamics, Z_t converges with probability one to a rest point which is not linearly unstable. Once again, Lemma 5 provides the nondegeneracy condition which is needed to apply this result. ■

After presenting Theorem 5.1, we claimed that when A is zero sum or admits an interior ESS, we can determine an upper bound on the time before a neighborhood of the rest point $x^*(\Lambda)$ is reached which is independent of the population's initial behavior. Recalling the definition of convergence in the medium run, we say that the processes X_t^N *converge uniformly in the medium run* if the time until convergence, $T_0(x_0)$, can be chosen independently of x_0. We then have the following result.

Proposition 7: *In the model of stochastic evolution, if A is zero sum or admits an interior ESS, then* X_t^N *converges uniformly in the medium run to* $RP = \{x^*(\Lambda)\}$.

To prove this result, one combines Theorems 2.1 and 3.2, Theorem 4.1 of Sandholm (2000a), and the following classical result from dynamical systems.

Lemma 8: *Let* x^* *be a Lyapunov stable global attractor of the flow* ϕ *on the simplex* Δ. *Fix* $\gamma > 0$, *and let* $\tau(x) = \inf\{T\colon \left|\phi(t,x) - x^*\right| \leq \gamma$ for all $t \geq T\}$. *Then* $\sup_{x \in \Delta} \tau(x) < \infty$.

Proof: Since x^* is globally stable, $\tau(x) < \infty$ for all $x \in \Delta$. Now suppose that the lemma is false. Then there is a sequence of initial conditions $\{x^k\} \subset \Delta$ such that $\lim_{k\to\infty} \tau(x^k) = \infty$. Since Δ is compact, this sequence has an accumulation point $\bar{x} \in \Delta$. Because x^* is Lyapunov stable, there is an $\eta > 0$ such that whenever $\left|x - x^*\right| \leq \eta$, $\left|\phi(t,x) - x^*\right| \leq \gamma$ for all $t \geq 0$. Because x^* is a global attractor, there is a time $\bar{T} < \infty$ such that $\left|\phi(\bar{T},\bar{x}) - x^*\right| \leq \frac{\eta}{2}$. Finally, since the flow is continuous in the initial condition x, we know that for all x sufficiently close to $\bar{x}$, $\left|\phi(\bar{T},x) - \phi(\bar{T},\bar{x})\right| \leq \frac{\eta}{2}$. Therefore, for all sufficiently large k, the triangle inequality implies that $\left|\phi(\bar{T},x^k) - x^*\right| \leq \eta$, and hence that $\left|\phi(t,x^k) - x^*\right| \leq \gamma$ for all $t \geq \bar{T}$. But then $\tau(x^k) \leq \bar{T}$ for all sufficiently large k, contradicting the definition of the sequence $\{x^k\}$. ■

The next lemma, due to Hopkins (1999), establishes a local stability result needed to prove Theorem 5.2(*i*). We provide a simpler proof.

Lemma 9: *If A is zero sum or admits an interior ESS, then* $x^*(\Lambda)$, *the global attractor of* (P-V), *is linearly stable.*

Proof: The derivative of the dynamics (P-V), $\dot{x} = C(Ax) - x$, is given by $DC(Ax)A - I$. We will show that this derivative matrix is stable on $\mathbf{R}_0^n$ at all points in Δ. If A is zero sum then $z{\cdot}Az = 0$ for all $z \in \mathbf{R}$, while if A admits an interior ESS, it is negative definite on $\mathbf{R}_0^n$ (Hofbauer and Sigmund (1988, p. 122)). Either way, $z{\cdot}Az \le 0$ for all $z \in \mathbf{R}_0^n$. As we noted in Section 2.4.1, $DC(Ax)$ is symmetric, positive definite on $\mathbf{R}_0^n$, and annihilates the vector **1**. Hence, the argument on p. 128-129 of Hofbauer and Sigmund (1988) shows that all eigenvalues of $DC(Ax)A$ corresponding to directions in $\mathbf{R}_0^n$ have non-positive real parts. Moreover, $DC(Ax)A - I$ has the same eigenvectors as $DC(Ax)A$; for each eigenvalue λ of $DC(Ax)A$, the corresponding eigenvalue of $DC(Ax)A - I$ is $\lambda - 1$. Consequently, $DC(Ax)A - I$ is stable on $\mathbf{R}_0^n$. ■

The Proof of Theorem 5.2(*i*)

Our proof relies on a theorem of Kurtz (1976) (K76 hereafter). To apply this theorem, we define for each state x a random vector I^x, which represents a standardized random increment of this process X_t^N at the state x. More precisely, I^x represents the possible increments during a single revision opportunity in the *numbers* (rather than proportions) of players of choosing each strategy. The distribution of I^x, which is independent of the population size N, is given by

$$P(I^x = e_j - e_i) = x_i \tilde{B}_j(x) \quad \text{whenever } i \ne j;$$
$$P(I^x = 0) = \sum_i x_i \tilde{B}_i(x).$$

Using the notation of K76, the expected motion of X_t^N is given by

$$F(x) = \sum_l l\,P(I^x = l) = \tilde{B}(x) - x,$$

while the matrix function g defined by

$$g_{ij}(x) = \sum_l l_i\, l_j\, P(I^x = l) = \begin{cases} -(x_i\, \tilde{B}_j(x) + x_j\, \tilde{B}_i(x)) & \text{if } i \neq j; \\ \sum_{l \neq i} (x_i\, \tilde{B}_l(x) + x_l\, \tilde{B}_i(x)) & \text{if } i = j \end{cases}$$

captures the comovements of the components of this process.

K76 Theorem 2.7 shows that if x^* is a global attractor of $\dot{x} = F(x)$ (i.e., equation (P)) and certain additional conditions are satisfied, then the stationary distributions of the rescaled process $Z_t^N = \sqrt{N}(X_t^N - x^*)$ converge in distribution to $N(0, \Sigma)$, where[28]

$$\Sigma = \int_0^\infty \exp(Df(x^*)s)\; g(x^*)\; \exp((Df(x^*))'s)\, ds.$$

To prove our result, it is enough to verify the required conditions.

The condition contained in K76 Theorem 2.3 requires that $DF(x)$ be continuous in x. This follows from the fact that C is C^1, which was established in the proof of Theorem 2.2. K76 condition 2.1 and the second and third supremum conditions of K76 Theorem 2.7 follow directly from the fact that the number of possible transitions at each state are finite. The first supremum condition follows from the fact that $DF(x)$ is a continuous function on the compact set Δ. That X_t^N is irreducible on Δ^N implies that it admits a unique stationary distribution. Finally, the global stability of $x^* = x^*(\Lambda)$ follows from Theorems 2.1 and 3.2, while the linear stability of x^* follows from Theorem 2.1 and Lemma 9. This verifies all of the conditions required by K76 Theorem 2.7, completing the proof of our theorem. ■

The Proof of Theorem 5.2(ii)

To prove the first claim, we appeal to the proof of Proposition 3 of Benaïm and Weibull (2000), which shows that when N is large, no stationary distribution of a stochastic evolutionary process can place significant mass away from the minimal center of attraction of the corresponding deterministic system.[29] Proposition A shows that this set is contained in the set of ω-limit points Ω; by Theorems 2.1 and 3.3, $\Omega = RP$.

---

[28] This expression for the covariance matrix Σ corrects the expression found in Kurtz (1976) (Kurtz, personal communication).

[29] While the results of Benaïm and Weibull (2000) and of Benaïm and Hirsch (1999b) (used below) are presented for discrete time models, they still apply in our continuous time setting: stationary distributions are independent of this modeling choice.

The proof of the second claim is based on Theorem 4.3 of Benaïm (1998) (hereafter B98). Our conclusion follows immediately once we verify the hypotheses which used in this theorem.

The process X_t^N is an *urn process* as defined in B98 Example 1.1, with $p^+(x) = C(Ax)$ and $p^-(x) = x$. Hence, conditions (*i*) and (*ii*) of B98 Corollary 3.3 are satisfied, implying that B98 Hypotheses 2.1 and 2.3 hold; conditions (*ii*), (*iii*), and (*iv*) of B98 Hypothesis 3.4 are satisfied as well. Lastly, since (P) possesses a globally attracting set which is contained in int(Δ), B98 Hypothesis 3.4 (*i*) also holds.

Next, we confirm the conditions stated in B98 Theorem 4.3. Since we have assumed that (R) holds, condition (*i*) follows from our Theorem 3.5 (*iii*). Condition (*ii*) follows from B98 Remark 3.10 (*iii*) and the fact that all rest points of (P) are in int(Δ). Condition (*iii*) follows from the fact that X_t^N is defined on Δ.

Finally, we must show that under condition (R), any rest point x^* of (P) which is not Lyapunov stable under (P) is *weakly unstable* (B98 p. 68). The proof of Theorem 3.5 (*iii*) shows that under condition (R), all rest points of (P) are hyperbolic. Hence, if x^* is not Lyapunov stable, it follows from the Hartman-Grobman Theorem (Robinson (1995, Theorem 5.5.3)) that there is a solution trajectory $\{x_t\}_{t\in\mathbf{R}}$ such that $\lim_{t\to-\infty} x_t = x^*$. Moreover, Theorem 3.4 implies that $\lim_{t\to\infty} x_t = y^*$, where $y^* \neq x^*$ is another rest point of (P). Moreover, since Π is a strict Lyapunov function for (P), it must be that $\Pi(y^*) > \Pi(x^*)$, and hence that no *orbit chain* (B98, p. 68) beginning at y^* can lead to x^*. Thus, x^* is weakly unstable. This completes the proof of Theorem 5.2 (*ii*). ■

The Proof of Theorem 5.2(*iii*)

This proof relies on results of Benaïm and Hirsch (1999b) (henceforth BH). To apply these results, we let I^x be a random vector describing the increments of the process X_t^N from the state x, as described in the proof of Theorem 5.2 (*i*). Let $\Sigma^x \in \mathbf{R}^{n\times n}$ denote the covariance matrix of I^x. Since I^x takes values in $\mathbf{R}_0^n$, $\mathbf{1}\cdot\Sigma^x\mathbf{1} = Var(\sum_i X_i) = 0$. Therefore, since the matrix Σ^x is symmetric, it admits a matrix of real eigenvectors; one of these eigenvectors is $\mathbf{1}$, and the others lie in $\mathbf{R}_0^n$. Let $\lambda(x)$ be the smallest eigenvalue of Σ^x corresponding to an eigenvector in $\mathbf{R}_0^n$. To use the results of BH, we need to show that $\lambda(x)$ is uniformly bounded away from zero. Intuitively, this means that for any current state x and any direction of motion z in $\mathbf{R}_0^n$, the amount of randomness in the motion of the process X_t^N in the direction z is nonnegligible.

To establish the bound on $\lambda(x)$, we again let

$$m \equiv \min_{x\in\Delta} \min_{i\in S} \tilde{B}_i(x) > 0.$$

Lemma 10: *The minimum eigenvalue* $\lambda(x)$ *of* Σ^x *satisfies* $\lambda(x) \geq m$.

Proof: Fix an arbitrary unit length vector $\theta \in \mathbf{R}^n_0$; it is enough to show that $\theta \cdot \Sigma^x \theta \geq m$. A calculation reveals that

$$\Sigma^x_{ij} = \begin{cases} -x_i x_j - \tilde{B}_i(x)\tilde{B}_j(x) & \text{if } i \neq j; \\ x_i(1-x_i) + \tilde{B}_i(x)(1-\tilde{B}_i(x)) & \text{if } i = j. \end{cases}$$

Since x and $\tilde{B}(x)$ lie in the simplex, and since $\sum_{i=1}^n \theta_i = 0$ and $\sum_{i=1}^n \theta_i^2 = 1$, we find that

$$\begin{aligned} \theta \cdot \Sigma^x \theta &= \theta \cdot diag(x)\theta - \theta' x x' \theta + \theta \cdot diag(\tilde{B}(x))\theta - \theta' \tilde{B}(x)\tilde{B}(x)' \theta \\ &= \sum_i \theta_i^2 x_i - \left(\sum_j \theta_j x_j\right)^2 + \sum_i \theta_i^2 \tilde{B}_i(x) - \left(\sum_j \theta_j \tilde{B}_j(x)\right)^2 \\ &= \sum_i \left(\theta_i - \sum_j \theta_j x_j\right)^2 x_i + \sum_i \left(\theta_i - \sum_j \theta_j \tilde{B}_j(x)\right)^2 \tilde{B}_i(x) \\ &\geq m \sum_i \left(\theta_i - \sum_j \theta_j \tilde{B}_j(x)\right)^2 \\ &= m\left(\sum_i \theta_i^2 - 2\left(\sum_i \theta_i\right)\left(\sum_j \theta_j \tilde{B}_j(x)\right) + n\left(\sum_j \theta_j \tilde{B}_j(x)\right)^2\right) \\ &\geq m. \quad ❑ \end{aligned}$$

To complete the proof, we must check the conditions which support BH Theorem 1.5. Proposition 3.7 and Theorem 3.8 show that (after a linear transformation), the dynamics (P) form a cooperative, irreducible dynamical system on Δ, so BH Hypothesis 1.2 is satisfied. Since the increments are uniformly bounded above, and since $\lambda(x)$ is uniformly bounded below by Lemma 10, BH Proposition 2.3 implies that BH Hypothesis 1.4 holds. Finally, since each X_t^N is takes values in the compact set Δ, the tightness assumption in BH Theorem 1.5 is satisfied. Therefore, BH Theorem 1.5 implies that $\lim_{N\to\infty} \mu^N(Q) = 1$ for any open set Q containing the Lyapunov stable rest points of (P). ■

The proof of Theorem 7.1 requires the following lemma.

Lemma 11: *Let* $R \subset RP$*, and let* $\{\sigma_t\}$ *be a solution to* (B) *with* $\omega\{E\sigma_t\} \subset R$*. Then* $\omega\{\sigma_t\} \subset \mathbf{B}(R)$.

The proof of this lemma for the case where R is a singleton is contained in the proof of Theorem 6.4 of Ely and Sandholm (2000) (henceforth ES); the proof for general R requires only a minor modification.

The Proof of Theorem 7.1(*ii*)

Suppose that $\{\sigma_t\}$ is a solution to (B). Then ES Theorem 5.1 implies that $\{E\sigma_t\}$ is a solution to (AB) = (P). Since A is a potential game, Theorem 3.3 implies that $\omega\{E\sigma_t\} \subset RP$. Thus, Lemma 11 allows us to conclude that $\omega\{\sigma_t\} \subset \mathbf{B}(RP)$. The remaining claims follow immediately from Lemma 11 and ES Theorem 6.4. ∎

The Proof of Theorem 7.1(*iii*)

Since A is a supermodular game, Theorem 3.9 (*ii*) shows that there is an open dense set $O \subset \Delta$ such that solutions to (P) from every initial condition $x_0 \in O$ converge to some point in RP. We will show that $E^{-1}(O)$ is an open dense subset of Σ, and that solutions to (B) from every initial condition $\sigma_0 \in E^{-1}(O)$ converge to some equilibrium strategy profile in $\mathbf{B}(RP)$.

Since E is continuous, $E^{-1}(O)$ is open. To establish that $E^{-1}(O)$ is dense, we must show that for every $\sigma \in \Sigma$ and every $\varepsilon > 0$, the set $E^{-1}(O)$ and the open ε-neighborhood of σ are not disjoint. Let $N_\varepsilon^\Sigma(\sigma) \subset \Sigma$ be the open ε-neighborhood of σ, and for $x \in \Delta$ let $N_\varepsilon^\Delta(x) \subset \Delta$ be the open ε-neighborhood of x. Since O is dense in Δ, there exists a $y \in \Delta$ such that $y \in N_\varepsilon^\Delta(E\sigma) \cap O$. By ES Lemma 6.1, there is a $\rho \in \Sigma$ such that $\rho \in N_\varepsilon^\Sigma(\sigma)$ and $E\rho = y$. Thus, $\rho \in N_\varepsilon^\Sigma(\sigma) \cap E^{-1}(O)$, and so $E^{-1}(O)$ is dense in Σ.

Now suppose that $\{\sigma_t\}$ is a solution to (B) with $\sigma_0 \in E^{-1}(O)$. Since $E\sigma_0 \in O$, the solution to (AB) starting from $E\sigma_0$ converges to some $x^* \in RP$. ES Theorem 5.2 therefore implies that $\{E\sigma_t\}$ converges to x^*, and so Lemma 11 implies that $\{\sigma_t\}$ converges to $\mathbf{B}(x^*)$. This completes the proof of the theorem. ∎

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