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Nonexistence of Constrained Efficient Equilibria when Markets are Incomplete

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Nonexistence of Constrained Efficient Equilibria when Markets are Incomplete*

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Abstract

We consider economies with incomplete markets, production, and a given distribution of initial endowments. The main purpose of the paper is to present a robust example of an economy with only one firm and one good per state in which no production decision entails a constrained efficient outcome. In particular, the unique Drèze equilibrium is dominated by every other production decision.

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1 Introduction

We consider economies with incomplete markets, production, and private ownership of initial endowments and restrict ourselves to one good per state. There are two time periods, $t = 0$ and $t = 1$. Production transforms good 0, the good at $t = 0$, into state dependent outputs at $t = 1$. At $t = 0$ consumers face risks that depend on how their initial endowments vary across future states of nature. The existence of assets including equity contracts gives rise to the possibility of insuring against risk. However, since markets are incomplete, insurance remains partial and the production plan of a firm can very well affect the asset span, that is to say, the insurance possibilities. Thus, firms not only provide profits to their owners but also ways to transfer wealth across time and states.

In general, shareholders have conflicting opinions and the firm faces a social choice problem [cf. Arrow (1983), p. 2].[1] Drèze (1974) proposed a way of resolving the conflict among shareholders. The framework originally considered by Drèze (1974) differs from the present one. Drèze (1974) leaves the distribution of initial endowments at $t = 0$ among consumers unspecified, whereas we focus on market equilibria that are defined with respect to an a priori given distribution of initial endowments. We want to show that taking initial endowments as fixed has important consequences if markets are incomplete. For this purpose we present an example in a simple setting with constant returns to scale. Since firms do not exert market power, individual wealth is determined by initial endowments and the zero profit condition. As profits vanish and firms are assumed to serve the new owners, original ownership plays no role.

After production decisions are made, individual demand for shares is determined by utility maximization under a budget constraint. Due to the assumption of constant returns to scale and the zero profit condition, the budget sets are independent of the scale of operation. Every firm is assumed to produce on a market clearing scale. In this way, every system of production rays induces a market equilibrium allocation. Observe that the set of allocations the market can achieve, that is to say the set of all market equilibrium allocations, is smaller than the set of allocations that are reached if total initial endowments at $t = 0$ can be freely distributed among consumers before the economy begins to operate.

A Drèze equilibrium specifies production plans, consumption plans and shareholdings as follows. Consumers take the production plans as given. Every consumer chooses shares optimally and a market equilibrium with respect to the production decisions results. Furthermore, in the determination of the production plans, individual shareholdings are taken as given. The production plans must pass the following test: It must be impossible for the shareholders of a firm to achieve a Pareto improvement by adopting another production plan and by

[1]Headnote to Arrow (1950) in Arrow (1983). We are grateful to M. Hellwig for drawing our attention to this headnote.

making sidepayments in terms of good 0 that enable the winners of the change to compensate the losers while all shares are kept fixed.

The test of the production plans is based on the following idea. The (new) shareholders of firm j meet at $t = 0$, after a system of production plans has been proposed and after they have chosen their shares optimally. Unanimity is required to change a production plan. To obtain every shareholder's consent only sidepayments in terms of good 0 are permitted. In a Drèze equilibrium there is no firm whose shareholders are able to overrule the production plan proposed by their firm and sidepayments never take place.[2]

The equilibrium concept introduced by Drèze is geared towards constrained efficiency. Constrained efficiency can be described as follows. Consider a hypothetical planner who can simultaneously choose all production plans and portfolios and who can, in addition, allocate consumption at time $t = 0$ under the condition that total consumption at $t = 0$ equals total initial endowments at $t = 0$ minus total inputs. The set of all allocations the planner can implement is called the constrained feasible set. Constrained efficiency means Pareto efficiency within the constrained feasible set. Under the usual conditions on production sets and preferences, constrained efficient allocations exist.

The set of constrained feasible allocations depends on the aggregate endowment at $t = 0$ but not on its distribution. Thus, a major difference between the planner and the market lies in the fact that the planner is not constrained by the original distribution of initial endowments at $t = 0$. The set of market equilibria with respect to fixed production decisions is contained in the set of constrained feasible allocations, but it is considerably smaller. We show that this fact has serious consequences.

It is well known that a Drèze equilibrium satisfies the first order conditions for constrained efficiency [see Magill and Quinzii (1996), p. 369]. However, since the constrained feasible set is nonconvex as pointed out by Drèze (1974), the first order conditions can be satisfied at a constrained feasible, but constrained inefficient allocation. In the definition of a Drèze equilibrium, the fact that shares are taken as fixed when production decisions are made has no first order effect according to the envelope theorem. However, effects of higher order are important due to the nonconvexity of the constrained feasible set. Accordingly, one would expect that both, efficient and inefficient Drèze equilibria, exist simultaneously. However, this intuition is false.

The purpose of this note is to present an example of an economy with privately owned endowments such that no market equilibrium with respect to fixed production decisions is constrained efficient. That is to say, all allocations which can be achieved by the market can be Pareto dominated by a planner who is not

[2]For an extensive treatment of Drèze equilibria in a setting with private ownership of initial endowments, the reader is referred to Magill and Quinzii (1996), chapter 6.

constrained by the given distribution of initial endowments at $t = 0$. In particular, all Drèze equilibria are constrained inefficient. Drèze equilibria are the only candidates for constrained efficient market equilibria, since a constrained efficient market equilibrium satisfies the definition of a Drèze equilibrium.

To clarify the nature of our example we look at some well known sources of constrained inefficiency of a Drèze equilibrium. First, price effects on the spot markets are known to create inefficiency [cf. Geanakoplos et al. (1990)]. However, they are ruled out here because we have only one good per state.

Furthermore, Drèze (1974) gives examples with two firms in which there is a constrained efficient as well as a constrained inefficient Drèze equilibrium. The constrained inefficient equilibrium is Pareto dominated by the constrained efficient one. To move from the inefficient to the efficient equilibrium, the planner has to simultaneously change the two production plans and the ownership of the firms. In order to rule out that the inefficiency of a Drèze equilibrium results from such a coordination failure, we construct our example with only one firm.

Another well known reason for inefficiency of Drèze equilibria results from the following fact. Whenever the firm proposes a production decision, its approval or rejection depends only on the group of consumers with positive shareholdings. If the firm had made another decision, other consumers could have benefited. In our example this potential source of inefficiency is avoided because the group of shareholders always coincides with the set of all consumers so that no consumer's welfare is ever neglected.

For the above reasons, we present a robust example of an economy with private ownership of initial endowments, one good per state, and one firm with constant returns to scale in which, for every production decision, every consumer has positive shareholdings. In the example, no market equilibrium with respect to some production decision is constrained efficient. In our example there is a unique Drèze equilibrium and this equilibrium is constrained inefficient.

In the examples of constrained inefficient equilibria given by Drèze (1974) each of the two firms has only one owner and this owner chooses a utility maximizing production plan. Therefore, the social choice problem typically faced by a firm's shareholders does not arise. Our example illustrates a difficulty in defining the goal of a firm with several shareholders when markets are incomplete.

In a Drèze equilibrium the behavior of a firm can also be characterized as follows: Each firm maximizes profits with respect to the weighted average of the normalized utility gradients of its shareholders, where the weights are the shares, and the marginal utilities with respect to good 0 are normalized to 1 [cf. Magill and Quinzii (1996), p. 364]. If a firm does not maximize profits as specified above, its shareholders can find another production plan and a system of sidepayments in good 0 only, such that they all become better off. For this reason, profit maximization with respect to the weighted average of the utility gradients is considered to be an adequate social compromise. Moreover, if markets are complete and all utility gradients coincide with the price system, the traditional concept of profit maximization results and the goal of a firm that is embodied in the concept of a Drèze equilibrium is seen as a natural extension of the classical goal of profit maximization.

In our example there is a unique Drèze equilibrium. However, a Pareto improvement can be obtained by adopting any production plan in which profits are not maximized with respect to the weighted average of the utility gradients, and by making sidepayments in good 0 as well as in shares. Hence, it remains unclear why profit maximization as it occurs in Drèze equilibria has attractive welfare properties in the example. One might argue that the firm should implement a production plan that cannot be unanimously rejected by the shareholders at their meeting, if sidepayments in terms of good 0 as well as shares are made. In our example no such production plan exists.

The example is driven by the existence of a consumer whose demand for shares depends on the distribution of initial endowments at $t = 0$. Consider a redistribution of good 0 making this consumer richer. The consumer reacts by buying fewer shares. Thus, the consumer's weight in the firm's objective is lowered. The consumers whose endowments are reduced are rewarded by obtaining a more favorable production plan. This mechanism gives rise to allocations that Pareto dominate the Drèze equilibrium. Contrary to the planner, the market cannot implement such a redistribution of endowments.

The example is built upon a severe conflict between the issues of distribution and efficiency, which is absent from the traditional theory of perfect competition with complete markets. As shown by Drèze (1974), all constrained efficient allocations can be obtained as stock market equilibria defined for economies without private ownership of initial endowments. The set of Drèze equilibria is defined here with respect to a given distribution of initial endowments and is much smaller than the set of stock market equilibria. The difference is analogous to the difference between the set of Walrasian equilibria and the set of price equilibria with transfers considered in the second welfare theorem with complete markets.

A similar phenomenon was first described in a seminal paper by Guesnerie (1975). In a framework with complete markets and a nonconvex production set, Guesnerie showed that all marginal cost pricing equilibria can be Pareto inefficient although efficiency requires that prices equal marginal costs. In his and in our setting, redistribution is required before the market can achieve efficiency. In both cases the inefficiency of all market equilibria is due to an inherent nonconvexity. In Guesnerie's framework, the production set is nonconvex, whereas in ours the set of constrained feasible consumption allocations is nonconvex. These nonconvexities entail that efficiency considerations cannot be separated from the way initial endowments are distributed.

2 The Example.

We consider an economy with two periods $t = 0, 1$, and two possible states of nature at $t = 1$. The states at time $t = 1$ are denoted $s = 1$ and $s = 2$ and the unique state at $t = 0$ is included as the state $s = 0$. There is a single good in each state.

There are three consumers. Consumers 2 and 3 are of the same type. Consumer 1 has the initial endowment $e^1 = (0.95,\ 0,\ 0)$ while consumers $i = 2, 3$ initially own $e^i = (1, 0, 0)$. In order to construct an example in which every Drèze equilibrium is constrained inefficient, income effects are needed. If all consumers have quasilinear preferences, a constrained efficient Drèze equilibrium is obtained by maximizing the social surplus of the consumers; see Dierker, Dierker, and Grodal (1999). To embody strong income effects into consumer 1's preferences, we proceed as follows. Consider the CES-indifference curve $x_0^{0.9} + x_1^{0.9} = 1$ and take its image under the linear mappings $(x_0, x_1) \mapsto (x_0, \gamma x_1)$ where $0 \leq \gamma < 1$. The resulting indifference pattern defines the preferences of consumer 1 in the part that is relevant for the example. The example is calibrated such that the consumption of consumer 1 stays well below the curve $x_0^{0.9} + x_1^{0.9} = 1$. In particular, 1's consumption of good 0 will have an upper bound of $3/4$, which is clearly below 1. Therefore, the "turning point" $(1, 0)$ associated with the above construction is outside the domain of interest in the example. There is no need to extend the preferences to the whole nonnegative orthant, although such an extension can easily be made. In the relevant range, consumer 1's preferences are given by the quasiconcave utility function

$$U^1(x_0, x_1, x_2) = \frac{x_1}{(1 - x_0^{9/10})^{10/9}}\,.$$

The two consumers of type 2 have the quasilinear and quasiconcave utility function

$$U^i(x_0, x_1, x_2) = x_0 + {x_2}^{1/2}, \quad i = 2, 3\,.$$

There is a unique firm that transforms inputs at $t = 0$ into state dependent outputs at $t = 1$. The firm has constant returns to scale and makes zero profits. Its technology is given by a family of normalized production plans $(-1, \lambda, 1 - \lambda)$. More precisely, every production plan, y, takes the form $y = \alpha(-1, \lambda, 1 - \lambda)$ for some $\alpha \geq 0$ and $\lambda \in [2/3,\ 0.99]$. It is assumed that shares in the firm represent the only way in which consumers can transfer wealth across time. By investing a certain amount of input at $t = 0$, consumer i is entitled to the corresponding part of the random output at $t = 1$. The right endpoint 0.99 has been chosen instead of 1 to make sure that both types of consumers always want to hold shares.

If the firm selects the normalized production plan $(-1, \lambda, 1 - \lambda)$, consumer i can choose within the set $\{e^i + \alpha^i(-1, \lambda, 1 - \lambda) \in \mathbb{R}^3_+ \mid \alpha^i \geq 0\}$ and will select α^i

so as to maximize utility. Let $\alpha^i(\lambda)$ denote i's optimal choice and let $\alpha(\lambda) = \sum_i \alpha^i(\lambda)$. Then total production equals $y(\lambda) = \alpha(\lambda)(-1, \lambda, 1-\lambda)$. Consumer i has a consumption of $x^i(\lambda) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$ and a share in the firm equal to $\vartheta^i(\lambda) = \alpha^i(\lambda)/\alpha(\lambda)$.[3] For any $\lambda \in [2/3, 0.99]$, the allocation $(y(\lambda), x^1(\lambda), x^2(\lambda), x^3(\lambda))$ is called a *market equilibrium with respect to* λ.

Since shares $\vartheta^i(\lambda)$ are positive for every consumer i and every $\lambda \in [2/3,\ 0.99]$, all consumers are always shareholders. Thus, no consumer is disregarded when a production plan is evaluated. Clearly, since consumer 1 is only interested in state 1 and consumers of type 2 derive no utility from that state, they have opposing views about what production ray, λ, the firm should select.

A market equilibrium $(y(\tilde{\lambda}), x^1(\tilde{\lambda}), x^2(\tilde{\lambda}), x^3(\tilde{\lambda}))$ with respect to $\tilde{\lambda}$ is a *Drèze equilibrium* if it is impossible to find a normalized production plan $(-1, \lambda, 1-\lambda)$ and a system of sidepayments τ^i at $t = 0$ with $\sum_i \tau^i = 0$ such that

$$U^i(e^i + \tau^i(1,0,0) + \alpha^i(\tilde{\lambda})(-1, \lambda, 1-\lambda)) > U^i(x^i(\tilde{\lambda}))$$

for every i. Note that the production plan $(-1, \lambda, 1-\lambda)$ on the left hand side of the above inequality is multiplied by the investment level $\alpha^i(\tilde{\lambda})$ that is optimal at the ray $\tilde{\lambda}$.[4]

An allocation (y, x^1, x^2, x^3), where i's consumption bundle is $x^i \geq 0$ and the production plan takes the form $y = \alpha(-1, \lambda, 1-\lambda)$ with $\alpha \geq 0$ and $\lambda \in [2/3, 0.99]$, is *constrained feasible* if there exist individual investments $\alpha^i \geq 0$ and individual consumptions $x_0^i \geq 0$ at $t = 0$ with $\sum \alpha^i = \alpha$ and $\sum x_0^i = \sum e_0^i - \alpha$ such that $x^i = (x_0^i, \alpha^i\lambda, \alpha^i(1-\lambda))$ for every i. An allocation is *constrained efficient* if there is no strictly Pareto superior constrained feasible allocation.

The set of constrained feasible allocations does not depend on how the aggregate initial endowment at date 0, $e_0 = \sum e_0^i$, is distributed among consumers, whereas market equilibria depend on how much of e_0 each consumer i owns. Therefore, the set of constrained feasible allocations is much larger than the set of market equilibrium allocations the firm can induce by choosing any $\lambda \in [2/3,\ 0.99]$. That is to say, there are constrained feasible allocations which cannot be implemented as market equilibria with respect to any λ given the distribution of initial endowments.

Moreover, if a market equilibrium $(y(\tilde{\lambda}), x^1(\tilde{\lambda}), x^2(\tilde{\lambda}), x^3(\tilde{\lambda}))$ with respect to some $\tilde{\lambda}$ is constrained efficient, then it is a Drèze equilibrium by definition. As a

[3] Since the production set has constant returns to scale, no market clearing condition for shares needs to be imposed.

[4] In the usual definition of a Drèze equilibrium, shares ϑ^i, and not the investment levels α^i, are taken as fixed when a production plan is evaluated. The two definitions are equivalent; cf. Dierker, Dierker, and Grodal (1999).

consequence, if all Drèze equilibria are constrained inefficient, then all production decisions entail constrained inefficiency.

A numerical computation shows that there is a unique Drèze equilibrium at $\tilde{\lambda} \approx 0.7$ in our example. The production ray λ is said to dominate the ray $\tilde{\lambda}$ if a planner can find $x_0^i \geq 0$ and $\alpha^i \geq 0$ with $\sum x_0^i = \sum e_0^i - \sum \alpha^i$ such that

$$U^i(x_0^i, \alpha^i(\lambda, 1-\lambda)) > U^i(x^i(\tilde{\lambda}))$$

for every i. If there exists a ray λ dominating the equilibrium $\tilde{\lambda} \approx 0.7$, the unique Drèze equilibrium is constrained inefficient. Clearly, the ray λ dominates $\tilde{\lambda}$ iff the planner can find a system of sidepayments τ^i at $t = 0$ with $\sum_i \tau^i = 0$ such that

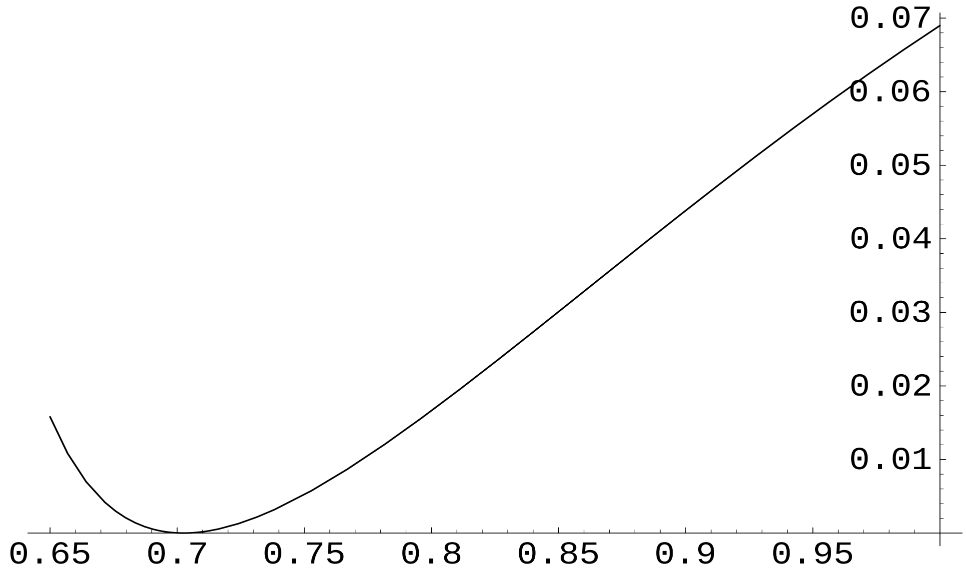
$$\max_{\alpha^i \geq 0} U^i(e^i + \tau^i(1,0,0) + \alpha^i(-1, \lambda, 1-\lambda)) > U^i(x^i(\tilde{\lambda}))$$

for every i.

We show that the ray $\tilde{\lambda}$ corresponding to the unique Drèze equilibrium in the example is dominated by every ray $\lambda \neq \tilde{\lambda}$. To explain how the example works we first analyze consumer 1. Assume that the consumer gets a sidepayment τ^1 at $t = 0$ and therefore possesses the amount $\bar{e}_0^1 = 0.95 + \tau^1 < 1$ at $t = 0$. Consumer 1 then invests $\bar{e}_0^1 - (\bar{e}_0^1)^{10}$ into production independently of the proposed ray λ, and therefore, will consume $(\bar{e}_0^1)^{10}$ at $t = 0$. In the Drèze equilibrium consumer 1 invests $0.95 - 0.95^{10} \approx 0.35$ so that approximately 0.6 units of the consumer's endowment remain for consumption at $t = 0$. Consumer 1's investment pattern is such that, if made richer, the consumer invests less in production.

We now briefly discuss the consumers of type 2. A simple computation shows that the consumers' equilibrium utility level corresponding to λ equals $1 + (1 - \lambda)/4$. Thus, their marginal willingness to pay for a change of λ is identically equal to $-1/4$. Hence, the sidepayment that is just sufficient to compensate a consumer of type 2 for a change, $\Delta\lambda$, in production is of the linear form $\Delta\lambda/4$.

In order to show that the Drèze equilibrium is dominated by every other $\lambda \in [2/3, 0.99]$, we fix the utility level of both type 2 consumers at its equilibrium value. Thus, put $\Delta\lambda = \lambda - \tilde{\lambda}$ and let each consumer of type 2 get the sidepayment $\Delta\lambda/4$. Accordingly, consumer 1 gets $-\Delta\lambda/2$. For $\Delta\lambda > 0$, consumer 1 gets a negative sidepayment. As a consequence, consumer 1 consumes less at $t = 0$, but invests more in production. A computation shows that 1's utility is monotonically increasing in $\Delta\lambda > 0$. Similarly, for $\Delta\lambda < 0$, the sidepayment to be given to 1 becomes positive. Hence, the wealth and also the consumption at $t = 0$ of consumer 1 increase. However, due to the restriction $\lambda \geq 2/3$, consumer 1's consumption never exceeds 3/4. That is to say, over the whole range of rays under consideration, 1's consumption at $t = 0$ stays well below the critical value 1. Again, the utility of consumer 1 increases if λ moves further away from its equilibrium value $\tilde{\lambda} \approx 0.7$. Thus, consumer 1 prefers to move away from the equilibrium ray $\tilde{\lambda}$, if the sidepayment given to both consumers of type 2 is just sufficient to make them indifferent to their position in the Drèze equilibrium. In particular, the Drèze equilibrium is dominated by every $\lambda \neq \tilde{\lambda}$. Consumer 1's corresponding utility increase is shown in the figure.



Consumer 1's utility increase after the sidepayment

In the setting of this example, total initial endowments at time $t = 0$ can be redistributed in a way that the phenomenon just described disappears. This fact illustrates the importance of the distribution of initial endowments for the welfare properties of an economy, which was first pointed out by Guesnerie (1975) in another context. For instance, take away 0.2 units from consumer 1's endowment at $t = 0$ and distribute this amount equally among the quasilinear consumers of type 2. The resulting economy has a unique Drèze equilibrium at consumer 1's favorite ray $\lambda = 0.99$ and this equilibrium is constrained efficient.

Moreover, the Drèze equilibrium in the new economy Pareto dominates the one in the original economy before redistribution. Observe, however, that many other Pareto improvements can be achieved in a similar way. In particular, one can also move the unique Drèze equilibrium in the original economy to type 2's favorite production ray, $\lambda = 2/3$, by a redistribution in the other direction and all agents are again better off than at the original Drèze equilibrium.

It can be shown that examples with a unique, totally dominated Drèze equilibrium also exist, if all consumers have von Neumann-Morgenstern utility functions; see Dierker, Dierker, and Grodal (1999).

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