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Cournot versus Walras in Dynamic Oligopolies with Memory

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Abstract

This paper explores the impact of memory in Cournot oligopolies where firms learn through imitation of success (as suggested in Alchian (1950) and modeled in Vega-Redondo (1997)). As long as memory includes at least one period, the long-run outcomes correspond to all the quantities in the interval between the Cournot quantity and the Walras one. There is a conceptual tension between the evolutionary stability associated to the walrasian outcome, which relies on inter-firm comparisons of simultaneous profits, and the stability of the Cournot-Nash equilibrium, derived from intertemporal comparisons of profits.
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1 Introduction

The stream of the learning literature initiated by Kandori, Mailath, and Rob (1993) and Young (1993) has seen a growing number of applications to oligopoly theory in the last years. Starting with the analysis of a Cournot oligopoly presented in Vega-Redondo (1997) (extended in different directions in Alós-Ferrer, Ania, and Vega-Redondo (1999), Tanaka (1999) and K.R.Schenk-Hoppé (2000)), further applications have been studied for Bertrand oligopolies (Alós-Ferrer, Ania, and Schenk-Hoppé (2000) and Hehenkamp (2000)), differentiated-goods oligopolies (Tanaka (2000)), signaling (Nöldeke and Samuelson (1997)), and insurance markets (Ania, Tröger, and Wambach (2001)).

All these models are based on the ideas pointed out by Alchian (1950). In summary, firms adapt via imitation and trial and error. Imitation is based on observed success, where "success" is defined in terms of achieving the highest profits. Trial and error is modeled through an experimentation (also called "mutation") parameter, which gives firms a (small) probability of trying something new.

The success of this approach in oligopoly models is due both to conceptual and technical reasons. Conceptually, real oligopolies are extremely complex situations where agents (firms) are bound to to use simple rules of thumb (relative to the complexity of the environment) to save decision costs. Technically, the long-run outcome of the process as the experimentation parameter goes to zero (i.e. for small perturbations) can be characterized through the techniques developed in Freidlin and Wentzell (1998), which were not available when Alchian wrote his seminal paper, but nowadays are already contained in several leading books on the subject of learning (see e.g. Fudenberg and Levine (1998), Samuelson (1997), or Young (1998)).

The general idea is analogous to the one underlying evolutionary models, where agents obtaining higher payoffs thrive at the expense of others. In oligopoly applications, the strategies (whether quantities, prices, or insurance contracts) leading to higher profits are promptly imitated and come to dominate the population.

In an oligopoly, this idea has interesting implications. In Vega-Redondo (1997) and Alós-Ferrer, Ania, and Vega-Redondo (1999), the Cournot-Nash equilibrium is quickly discarded in favor of the "walrasian" one (where firms set price equal to marginal cost) due to the effects of *spite*, which show that, with imitation of highest profits only *relative* payoffs matter. On the one hand, even if firms deviating from the walrasian equilibrium earn higher profits than before, those not deviating are left with even higher profits; thus, the deviant finds himself in a bad relative position. On the other hand, firms deviating from the Cournot-Nash equilibrium to produce the walrasian quantity achieve lower profits than before, but those not deviating are left with even lower profits; thus, the deviant finds himself in a good relative position.

As pointed out in Alós-Ferrer (2000), there is nothing to object to the referred idea in a biological framework, where agents live one period and are replaced by their offspring. Doubts arise, though, when the same arguments are applied to learning models in economics, in particular to firms. Since firms do not die, but presumably try to learn, the effects of spite rely on the fact that previous outcomes (e.g. the profits of the Cournot-Nash equilibrium) are immediately forgotten and only the current comparison of profits matters. Besides, if the results of even the most recent period are forgotten, it is difficult to interpret the experimentation process as trial and error, since an error might only be perceived as such when compared with previous results. Hence, it might be worth investigating how the models mentioned above models are affected once firms are allowed to use the information gained in, at least, the most recent periods of play.

Alós-Ferrer (2000) poses this question for general learning models and shows that the introduction of (bounded) memory can have quite strong implications.[1] For instance, the results obtained in Kandori, Mailath, and Rob (1993) for coordination games can be reversed. More important, it is shown that, even in models based on imitation and experimentation, where best response plays no role, there is still a certain significance to the concept of Nash equilibrium, since the intertemporal comparison of payoffs after a unilateral deviation from a Nash equilibrium will always, by definition, allow the agents to perceive such deviation as an error and correct it by imitation of past (before-deviation) strategies.

This paper explores the impact of adding finite memory to the models of Cournot Oligopoly mentioned above. It is shown that, as long as memory includes at least one period, the walrasian outcome is not anymore the only

[1] Young (1993) studies a different model where agents, sampled from two different populations, obtain a sample of records from past interactions before engaging in an asymmetric game. Due to the addition of sampling and fundamental differences in the modeling of agent interaction, Young's memory can not be directly compared to the one in Alós-Ferrer (2000).

long-run outcome as in Vega-Redondo (1997). On the contrary, there is a clear tension between the walrasian outcome and the Cournot equilibrium, which stabilizes the whole range of quantities between them.

It is important to stress that this result obtains already with a *single* period of memory, for an arbitrary number of firms. This shows that the result in Vega-Redondo (1997), which is based on the relative payoff considerations mentioned above, is not robust with respect to the absolute payoff considerations introduced by memory. It is also worth noticing that, as in Vega-Redondo (1997), firms are pure imitators (i.e. they are neither supposed to know their profit functions nor to be able to compute a best reply). Hence, the difference in the results is due exclusively to the introduction of memory.

2 The Model

2.1 Preliminaries

Vega-Redondo (1997) studies a model where boundedly rational firms compete in quantities following a dynamics of imitation and experimentation. The striking conclusion is that the Cournot-Nash equilibrium does not play any role in the long-run, but rather the quantity corresponding to a walrasian equilibrium is selected, i.e. a non-Nash outcome of the one-shot game.

In this paper, we examine the effects of the introduction of memory in such a framework. For the sake of tractability, we will work with a well-behaved Cournot Oligopoly, as reflected by the following technical assumptions. A more general analysis would add nothing to the understanding of the effects of memory.

There are $N \geq 2$ firms producing a homogeneous good whose demand is given by the inverse-demand function $P : \mathbb{R}_+ \mapsto \mathbb{R}_+$. We assume this function to be twice-differentiable in a closed interval $[0, Q_{max}]$, downward-sloping (in the differentiable sense: $P'(\cdot) < 0$) and concave ($P''(\cdot) \leq 0$). Furthermore, we assume that $P(0) = P_{max} > 0$ and $P(x) = 0$ for all $x \geq Q_{max}$. All firms have an identical cost function $C : \mathbb{R}_+ \mapsto \mathbb{R}_+$, assumed upward-sloping ($C'(x) > 0 \; \forall \; x > 0$) and strictly convex ($C''(\cdot) > 0$).

Definition 2.1. The Walras quantity x^W is such that $P(Nx^W)x^W - C(x^W) \geq P(Nx^W)x - C(x)$ for all x.

Definition 2.2. The Cournot quantity x^C is such that $P(Nx^C)x^C - C(x^C) \geq P((N-1)x^C + x)x - C(x)$ for all x.

We directly assume the existence of a unique, strictly positive Cournot quantity x^C and a unique, strictly positive Walras quantity x^W.[2]

Time is discrete. Each period, firms are assumed to choose their output from a common finite grid $\Gamma = \{0, \delta, ..., v\delta\}$, with arbitrary $\delta > 0$, thought to be small, and $v \in \mathbb{N}$. Those quantities of interest will be assumed to be in the grid (e.g. Walras and Cournot quantities).

[2]Existence of a positive Walras quantity is guaranteed under mild additional assumptions (e.g. $c'(0) < P_{max}$; otherwise the market is not meaningful), and then the assumptions above guarantee its uniqueness. Those assumptions also guarantee the existence of a unique Cournot quantity, and similarly mild conditions ensure it to be be non-zero.

In the dynamics studied by Vega-Redondo (1997), firms observe the quantities produced and the profits realized by all competitors, and then imitate the quantities that led to the highest profits. Independently across time and firms, with a (small) probability $\varepsilon > 0$, a firm will instead experiment with a randomly chosen new quantity (all quantities in the grid having positive probability of being chosen). This event is called "a mutation." The dynamics without mutations is called "unperturbed."

Formally, the model described is a finite Markov chain (for each fixed ε) which can be studied using the by-now standard techniques brought into economics by Kandori, Mailath, and Rob (1993) and Young (1993) (see Kandori and Rob (1995) or Ellison (2000) for other self-contained summaries). The objective is to find the *stochastically stable states*, which are those in the support of the limit invariant distribution of the process as the experimentation probability ε goes to zero. The interpretation is that, for small ε, in the long run the process will spend most of the time on the stochastically stable states.

Given the quantities $x_1, ..., x_N$ produced in a given period, the modeler (but not the firms) can deduce the profits obtained by each firm through

$$\Pi_i(x_i, x_{-i}) = P\left(\sum_{j=1}^{N} x_j\right) x_i - C(x_i)$$

and hence the state space of the chain can be summarized by Γ^N, in spite of the fact that firms observe quantities and realized profits.

Given a quantity $x \in \Gamma$, denote $\text{mon}(x) = (x, ..., x) \in \Gamma^N$. Such states are called *monomorphic*. The first trivial observation is that the absorbing sets of the unperturbed dynamics correspond to singletons formed by monomorphic states (unsurprisingly, since the only force at work is imitation). It is then a standard result that only these states might be stochastically stable. Vega-Redondo (1997) shows that, in the dynamics described above, the only stochastically stable state is $\text{mon}(x^W)$, i.e. the walrasian equilibrium. The key to this result is the strong relative advantage that a firm producing x^W gets. Specifically:

Lemma 2.3. *For all $x \neq x^W, 1 \leq m \leq N$,*

$$P((N-m)x + mx^W)x^W - C(x^W) > P((N-m)x + mx^W)x - C(x)$$

Proof. See Vega-Redondo (1997, p.381). ■

Taking $m = N - 1$, we see that the Walras quantity features *spite* (see e.g. Schaffer (1988) or Crawford (1991)): if a mutant deviates from the state $\text{mon}(x^W)$, even to a best-response, the payoffs of the non-mutants, still choosing x^W, will rise even more than the mutant's payoff.

Taking $m = 1$, we see that the Walras quantity also features what we could call *negative spite* (see Alós-Ferrer (2000)): if a mutant firm deviates to x^W from any monomorphic state, $\text{mon}(x)$, with $x \neq x^W$, even if its payoffs after deviation have decreased, they will still be higher than those of the non-mutants. For instance, suppose the economy is at the Cournot equilibrium, i.e. the state $\text{mon}(x^C)$. If a mutant firm deviates to x^W, its profits will decrease, but the profits of the other firms will decrease even more. Hence, in absence of memory, the mutation is successful.

These two properties are enough to guarantee that $\text{mon}(x^W)$ is the only stochastically stable state of the dynamics without memory. Intuitively, it is much more probable to reach this state from any other than to leave it. In the terms of Kandori and Rob (1995), the minimum-cost ω-tree is constructed connecting all states to $\text{mon}(x^W)$ with a single mutation (negative spite). Since it is impossible to leave $\text{mon}(x^W)$ with a single mutation (spite), all the ω-trees of other states have strictly larger costs. In the terms of Ellison (2000), $\text{mon}(x^W)$ has Coradius one by negative spite, and Radius larger than or equal to two by spite.

2.2 Introducing memory

We want to investigate the effects of the introduction of memory in this framework. Assume that firms remember the quantities produced and the profits realized in the last $K > 0$ periods in addition to the present one ($K = 0$ is the model without memory). Formally, the state space is enlarged to $\Gamma^{N\cdot(K+1)}$, and the imitation rule simply specifies to copy one of the quantities that has lead to highest payoffs in the last $K + 1$ periods (including the present one).

The intuition is that, if firms remember past profits, destabilizing Cournot will not be such an easy task. After a single mutation from $\text{mon}(x^C)$, the mutant may earn more than the non-mutants, but the largest payoff remembered will still be that of the Cournot equilibrium, and hence the mutant will "correct the mistake," even in absence of any strategic considerations. This allows us to interpret the experimentation process as "trial-and-error".[3]

Given a quantity $x \in \Gamma$, denote $\text{mon}(x, K) = ((x, ..., x), ..., (x, ..., x))$, i.e. the state where all firms have produced x for the last $K + 1$ periods. In the model with memory, we call these states *monomorphic*.

Obviously, the recurrent classes of the unperturbed process are the singletons formed by monomorphic states. In the absence of mutation, the imitation process can not lead away from a monomorphic state, and, given any non-monomorphic state, there is always strictly positive probability that all firms imitate the same quantity.[4] This guarantees that the analysis can be restricted to the monomorphic states. See e.g. Young (1993) or Ellison (2000).

3 Analysis

3.1 Relative advantage vs. absolute success

Two comparisons are of importance to consider the relative advantages of any given quantity. The first is the difference in payoffs between a mutant and the non-mutants after a single mutation from a monomorphic state. This is given

[3] Vega-Redondo (1997) further assumes the presence of exogenous inertia, which has no effect there. This assumption, though, would effectively prevent the analysis of memory by enabling transitions where agents are "frozen" until the relevant payoffs are forgotten. See Alós-Ferrer (2000) for a discussion.

[4] With memory, the argument is slightly more complex. Each of the following $K + 1$ periods, there is positive probability that all firms imitate the same quantity as other firms (though not necessarily the same as in other periods). This leads to a state of the form $((x_1, ..., x_1), (x_2, ..., x_2), ..., (x_{K+1}, ..., x_{K+1}))$. For the next $K + 1$ all of them imitate the (remembered) quantity that yields higher profits when all firms produce it.

by:

$$D(x,y) = P((N-1)x + y)(y-x) + C(x) - C(y)$$

i.e. the difference, after mutation, between the payoffs of a mutant producing y and the non-mutants, all of them still producing x. If $D(x,y) > 0$, then, in a model without memory, a single mutation is enough for a transition from mon(x) to mon(y). The spite of x^W results from the fact that $D(x^W, y) < 0$ for all $y \neq x^W$, and the negative spite amounts to $D(x, x^W) > 0$ for all $x \neq x^W$. Both facts, which follow from Lemma 2.3, are illustrated in Figure 1.

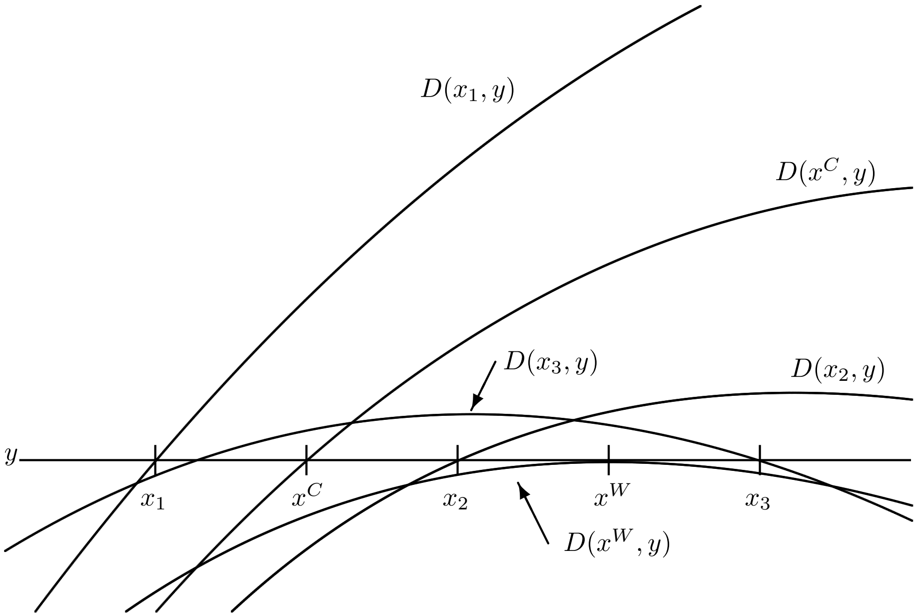

Figure 1: $D(x,y)$ is the payoff difference between a mutant playing y and the non-mutants playing x. If it is positive, the mutant fares better. The horizontal axis is the second variable of $D(\cdot,\cdot)$, i.e. the mutation y.

The second important comparison is the difference between the payoff of a mutant, producing y after mutation (from a monomorphic state where all firms produce x) and the payoff that the same mutant had before mutation, that is, the absolute gain or loss due to the mutation:

$$M(x,y) = P((N-1)x + y)y - C(y) - P(N \cdot x)x + C(x)$$

If $M(x,y) > 0$, then the mutant obtains, after deviation, higher profits than before. The fact that the Cournot outcome is a (strict) Nash equilibrium translates to $M(x^C, x) < 0$ for all $x \neq x^C$.

It is shown in Lemma A.3 (see Appendix A) that, under our assumptions, $M(x, x^C) > 0$ for all $x \neq x^C$, which shows a certain analogy between the Walras and Cournot quantities. The Walras quantity is such that deviations to or from it always render a situation where relative payoffs are higher for the firms producing x^W after mutation. The Cournot quantity is such that single deviations to or from it always render a situation where an "intertemporal" comparison between pre- and post-mutation payoffs for the mutant always favors x^C. This is illustrated in Figure 2.

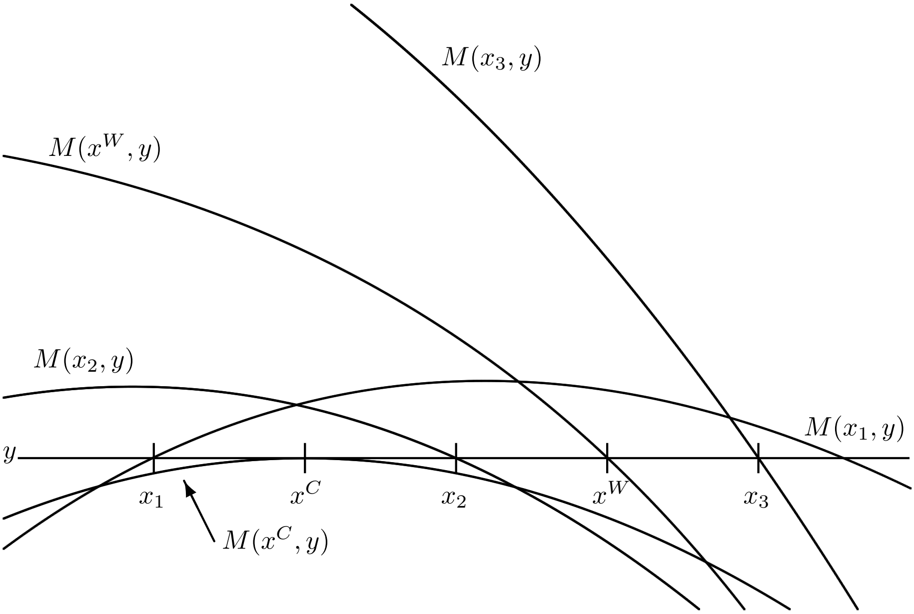

Figure 2: $M(x, y)$ is the payoff difference experienced by a mutant deviating from x to y, when all other firms remain playing x. If it is positive, the mutant is better off. The horizontal axis is the second variable of $M(\cdot, \cdot)$, i.e. the mutation y.

The quantities $D(x, 1)$ and $M(x, y)$ are not unrelated. In Lemma A.1, it is shown that a deviation to a lower quantity ($y < x$) is always better in absolute than in relative terms (i.e. $M(x, y) > D(x, y)$), whereas exactly the opposite is true for deviations to higher quantities.

In a model with memory, both absolute and relative payoff comparisons are crucial. In the remaining analysis we see how this observation yields *both* the Walras and the Cournot quantities as stable outcomes in the model detailed above.

3.2 The stable set: quantities between Cournot and Walras

As a first approximation, the following theorem shows that both quantities above x^W and below x^C can be quickly discarded.

Theorem 3.1. *For any $K \geq 1$, the set of stochastically stable states is contained in*

$$\{\text{mon}(x, K)/x^C \leq x \leq x^W, x \in \Gamma\}$$

i.e. it is a subset of the set of monomorphic states corresponding to quantities between (and including) the Cournot and the Walras ones. Moreover, $\text{mon}(x^W)$ *is always stochastically stable.*

Essentially, the proof (which is relegated to Appendix A) proceeds as follows. From a monomorphic state, single mutations leave the mutant with higher after-mutation profits than those of the non-mutants only if the mutation happens in the direction of x^W (see Figure 1). Such mutations are thus successful in relative terms, but not necessarily in absolute terms. Analogously, single mutations leave the mutant with higher profits than before only if the mutation happens in the direction of x^C (see Figure 2). Such mutations are thus successful in absolute terms, but not necessarily in relative ones.

This is illustrated in Figure 3, which depicts $M(x, x^W)$, $M(x, x^C)$, $D(x, x^W)$, and $D(x, x^C)$. Whereas the walrasian quantity is guaranteed to represent an advantage in relative, spiteful terms (due to Lemma 2.3), the Cournot quantity is guaranteed to represent an advantage in absolute, intertemporal terms.

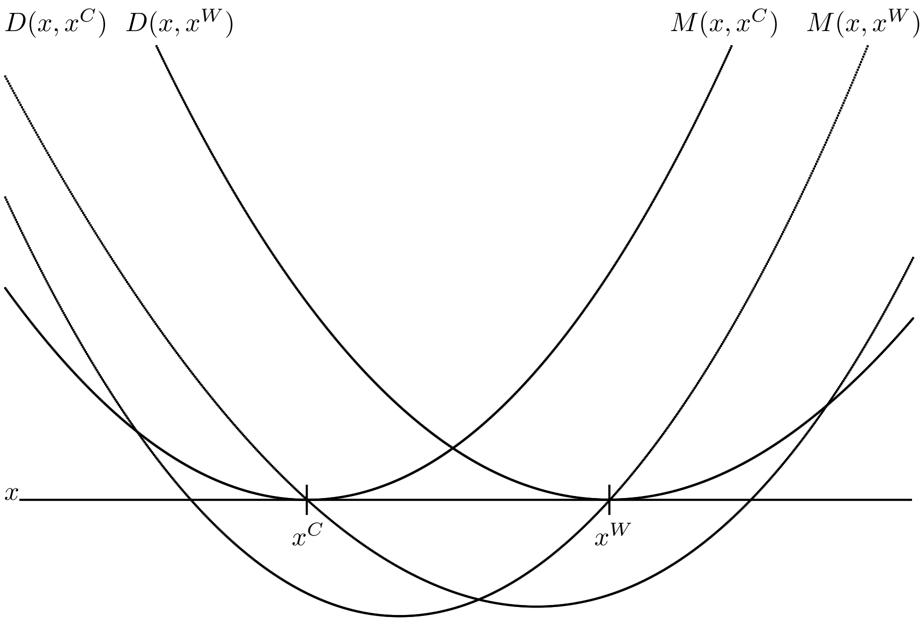

Figure 3: Deviations to Cournot and Walras. A mutant from x to $y = x^C$ or x^W gains in relative terms if and only if $D(x, y) > 0$. He gains in absolute terms if and only if $M(x, y) > 0$. The horizontal axis is here the first variable of the functions $D(\cdot, \cdot)$ and $M(\cdot, \cdot)$.

Quantities below x^C are unstable, because a mutation to a higher quantity (e.g. x^C) can go simultaneously in the direction of x^C and x^W. A single mutant, hence, will earn more payoffs than the non-mutants both after and before mutation, and will be imitated. The same holds for quantities above x^W are unstable, because a mutation to a lower quantity (e.g. x^W) can go simultaneously in the direction of x^W and x^C.

It turns out, though, that quantities in $[x^C, x^W]$ are more stable. More precisely, no single mutation form the corresponding monomorphic state can be successful. This fact is established in Lemma A.6. The argument (which is partially reflected in Figure 3), is as follows.

Consider any quantity $x \in [x^C, x^W]$. A mutation from $\text{mon}(x, K)$ "upwards," towards a higher quantity (e.g. x^W) might leave the mutant with higher after-mutation profits than the non-mutants (e.g., $D(x, x^W) > 0$, see Figure 3). Still, before-mutation profits are the highest profits remembered (e.g., $M(x, x^W) < 0$. Hence, the mutation would not be successful. A mutation "downwards," towards a lower quantity (e.g. x^C) might leave the mutant with higher payoffs than before mutation (e.g., $M(x, x^C) > 0$). The largest payoff observed, though, would be that of the non-mutants (after mutation) (e.g., $D(x, x^C) < 0$). Hence, such a mutation would also be unsuccessful. In summary, a single mutation from a state where all firms produce $x \in [x^C, x^W]$ will never be successful (this is proven in Lemma A.6), whereas a single mutation is enough to destabilize all other states (Lemmata A.4 and A.5). This reflects a different level of stability and is enough to drive the result: all stochastically stable states must be monomorphic states corresponding to quantities between the Cournot and the Walras ones.

Remark 3.2. The arguments above imply that, in the terminology of Ellison (2000), the set $\{\text{mon}(x, K)/x^C \leq x \leq x^W, x \in \Gamma\}$ has Radius strictly larger than one, but Coradius 1. Theorem 1 in Ellison (2000) then yields an alternative proof of the result, and additionally implies that the estimated time of first arrival into the mentioned set is of order ε^{-1}, that is, convergence is as fast as it can be, and independent of population size.

Theorem 3.1 also states that the walrasian quantity is always stochastically stable. The essence of the proof is the following. Starting from a quantity $x \in [x^C, x^W)$, it is always possible to move upwards from $\text{mon}(x, K)$ to $\text{mon}(x+\delta, K)$, where $x < x + \delta \leq x^W$, with two simultaneous mutations. For example, from $\text{mon}(x, K)$, consider two simultaneous mutations to quantities $x - \delta$ and $x + \delta$. This leaves the total produced quantity, and hence the market price $p = P(N \cdot X)$, unchanged. With a fixed price, though, the profit function $p \cdot x - C(x)$ is concave and attains its maximum above x^W (see Lemma A.7). Hence, the mutant producing $x + \delta$ earns larger profits than the non-mutants (both before and after mutation), and than the mutant producing $x - \delta$. Thus, $x + \delta$ is imitated.

Repeating this argument, we can construct a chain that connects $\text{mon}(x, K)$ with $x \in [x^C, x^W)$ to $\text{mon}(x^W, K)$, where each transition is done with exactly two mutations. In the terminology of Kandori, Mailath, and Rob (1993), this is enough to build the $\text{mon}(x^W, K)$-tree of minimal cost.

3.3 The duopoly case

Theorem 3.1 shows that the Cournot quantity and the Walras one are respectively the lower and upper bound of the quantities that can be observed in the market in the long run. Quantities that are too low (respectively too high) are promptly destabilized because deviations to higher (respectively lower) quantities are made both in the direction of the Cournot quantity and the Walras one. Hence, the result makes use of two powerful forces. The first force is the one associated to Nash equilibria, reinterpreted as an intertemporal comparison of own payoffs. The second, the already familiar effects of spite and negative spite.

Between x^C and x^W, these two forces clash. We see now a first case where the outcome is "undecided" and their clash stabilizes the whole interval of quantities: the duopoly.

Proposition 3.3. *If $N = 2$ (a duopoly), and for any $K \geq 1$, the set of stochastically stable states is identically equal to*

$$\{\mathrm{mon}(x, K)/x^C \leq x \leq x^W, x \in \Gamma\}$$

i.e. it is formed by all the monomorphic states corresponding to quantities between (and including) the Cournot and the Walras ones.

The proof is relegated to Appendix A. The result relies on the fact that, with a population of just two firms, after two simultaneous mutations to the same quantity there are no relative-payoff considerations, because there are no non-mutants left. If only absolute-payoff considerations are present, mutations in the direction of x^C are again successful. This allows to construct transitions "downwards" from monomorphic states corresponding to quantities $x \in (x^C, x^W]$ with two mutations. This implies that, within $[x^C, x^W]$, it is just as costly (in terms of mutations) to move up or downwards. Upwards transitions are as mentioned above for the $\mathrm{mon}(x^W, K)$-tree of minimal cost. Downwards transitions use two simultaneous mutations to the same (smaller) quantity.

Thus, we know that, although the walrasian quantity is always contained in the prediction, it will in general not be the only quantity there, as in Vega-Redondo (1997). Moreover, the Cournot quantity might be in the prediction.

3.4 The general case with more than two firms

We concentrate now on $N \geq 3$ (since the case $N = 2$ is solved by Proposition 3.3). We will conclude that the whole interval $[x^C, x^W]$ is also stable in this case, although the required construction to prove this result is not as simple as in the duopoly case.

We know from Theorem 3.1 that the walrasian outcome is stochastically stable, and that the only other candidates for stochastic stability are the states $\mathrm{mon}(x, K)$ with $x \in [x^C, x^W]$. The key to establish that all these states are indeed stable is that it is possible to destabilize Walras with the same cost (in terms of mutations) as any other quantity in $[x^C, x^W]$.

We need to consider transitions leaving $\mathrm{mon}(x^W, K)$. This cannot be accomplished with one single mutation, but it is easy to see that two mutations would suffice. To get an intuition of how, we consider an extreme example. Let, for instance, $x^W_{N-1} > x^W$ be the Walras quantity when there are only $N-1$ firms in the market. Suppose two mutant firms produce respectively 0 and x^W_{N-1}. In all practical respects, the situation is as if there were only $N-1$ firms in the market, and the one that now produces x^W_{N-1} earns larger profits than the rest by Lemma 2.3. But these firms are earning more than before mutation, since their costs remain the same and the price has risen (x^W_{N-1} is easily seen to be lower than $2 \cdot x^W$). In summary, the firm that has deviated to x^W_{N-1} is better off both in relative and absolute terms, and hence the transition to $\mathrm{mon}(x^W_{N-1}, K)$ follows.

Once a transition to a larger quantity has been achieved with two mutations, the system is in the unstable region out of $[x^C, x^W]$. The new monomorphic state can be destabilized by a single mutation to a lower quantity, and it can be computed that quantities strictly lower than x^W can be reached now in such a way. Consider one such quantity, x' with $x^C \leq x' < x^W$. This shows that $\text{mon}(x', K)$ must be stochastically stable. Consider the $\text{mon}(x^W, K)$-tree which shows stability of Walras. Add the two transitions discussed above, with a total cost of three. In exchange, the previous transitions leaving $\text{mon}(x^W_{N-1})$ (at cost one) and $\text{mon}(x', K)$ (at cost two) can be deleted, yielding for the new tree (with vertex $\text{mon}(x', K)$) the same cost as that of the original $\text{mon}(x^W, K)$-tree.

It turns out that transitions such as these can be used to show stability of the whole spectrum of quantities in $[x^C, x^W]$, as the next theorem states. The detailed proof is relegated to Appendix B. It is noteworthy that, in this case, it is enough to add a single period of memory to alter the conclusions of the model without memory.

Theorem 3.4. *For any $K \geq 1$, $N > 2$, and δ small enough, the set of stochastically stable states is*

$$\{\text{mon}(x, K)/x^C \leq x \leq x^W, x \in \Gamma\}$$

i.e. all the monomorphic states corresponding to quantities between the Cournot and the Walras ones.

An example of the procedure used to show stability of non-walrasian quantities is illustrated in Figure 4 (based on quadratic costs and linear demand). At the state $\text{mon}(x^W, K)$, consider two mutations to quantities 0 and x with $x^W < x < 2 \cdot x$. Let $H(x)$ be the after-mutation payoff difference between the $x-$mutant and the incumbents. As long as $H(x)$ is positive, the $x-$mutant is successful. The function $H(x)$ is easily seen to be strictly concave, which yields a range of monomorphic states, corresponding to quantities above x^W, which can be reached from $mon(x^W, K)$ with just two mutations. But from each of these states, a single mutation to lower quantities y is successful as long as $D(x, y) \geq 0$. If it happens, for instance, that $D(x, x^C)$ is positive, the stability of $\text{mon}(x^C, K)$ follows.

4 Discussion

The presence of memory allows agents to behave as if they were able to "experiment conditionally." When a new strategy is tried out, memory allows to compare its success with that of the previous strategy. If the mutation brings payoffs down, the mutant will be able to "correct" the mistake and go back to the previous action. This observation, which is of intertemporal nature in an explicitly dynamic framework, naturally reintroduces better-response considerations into models of bounded rationality without explicitly assuming that the agents compute any best reply. However, the example analyzed in this paper shows that this is not the only consideration. It is the interplay between better response (to do better than yesterday) and relative success (to do better than the others) which creates a rich dynamic in which two properties arise as determining the long-run outcomes. The first property is the one associated to Nash equilibria, reinterpreted as an intertemporal comparison of own payoffs. The second, the already familiar effects of spite and negative spite.

In the case of a Cournot Oligopoly, we see the two forces clash, giving us an economically meaningful example where two focal points are selected: the

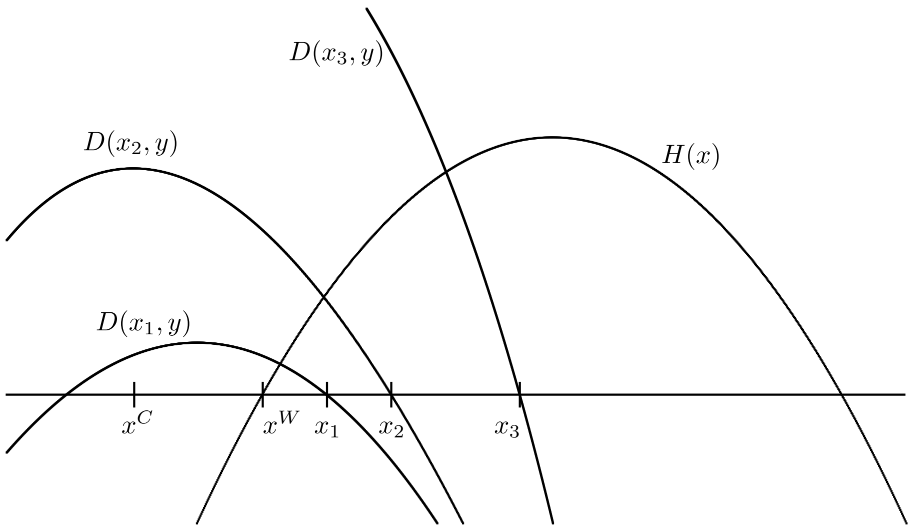

Figure 4: How to leave Walras. A transition from $\text{mon}(x^W, K)$ to $\text{mon}(x, K)$, $x > x^W$ can be achieved with two mutations if and only if $H(x) \geq 0$. A transition from $\text{mon}(x_i, K)$ to $\text{mon}(y, K)$ (e.g. $y = x^C$) can be achieved with one mutation if $D(x_i, y) \geq 0$. The horizontal axis is the second variable of $D(\cdot, \cdot)$ and the argument of $H(\cdot)$.

first thanks to its Nash-equilibrium condition, the second thanks to its spiteful properties. As these two "forces" compete, the whole interval between those two outcomes is stabilized, creating an apparently blunt prediction. In actual examples, the prediction reduces to a small interval of quantities (necessarily shrinking for large number of firms), not unlike the interval of prices found in Alós-Ferrer, Ania, and Schenk-Hoppé (2000) for the Bertrand model. However, the fact that it is still an interval appropriately illustrates both the stability properties of the Cournot-Nash equilibrium and of the spite-driven walrasian outcome.

From the point of view of Industrial Organization, this paper shows the lack of stability of quantities above the Walras one, and below the Cournot one. It shows that both of them play a clear role, limiting the interval of stable quantities, but neither of them is to be expected as a unique solution.

Since the stochastic analysis relies on identifying the less costly, i.e. more probable transitions, it is tempting to re-interpret the arguments in the proofs of the main result as pointing at high-probability paths. Under such an (admittedly risky) interpretation, it is clear that we would rarely observe quantities outside the interval limited by the Cournot and Walras quantities. We would rather observe a dynamic, almost-cyclical behavior. Once the market settles in a given quantity, slowly some firms will deviate from it, both to lower and higher quantities, but only the ones with higher quantities will be successful, due to spite-driven considerations. This process will keep raising the market quantity until some firm will raise it too much, above the walrasian one, enjoying a short-lived prosperity which will be quickly undermined by other firms switching to much lower quantities. From these lower quantities, the "cycle" will start again.

A Proof of Theorem 3.1

The proof proceeds through a series of lemmata, some of which will also be useful for the proof of Theorem 3.4.

First, recall that the recurrent classes of the unperturbed process are the monomorphic states. Thus, we can concentrate the analysis on them (see e.g. Young (1993) or Ellison (2000)).

Lemma A.1. $M(x,y) > D(x,y)\ \forall\ y < x; M(x,y) < D(x,y)\ \forall\ y > x$

Proof. Notice that $M(x,y) = D(x,y) + x \cdot [P((N-1)x+y) - P(N \cdot x)]$. If $y < x$, then $(N-1)x + y < N \cdot x$. Since P is strictly decreasing, this implies that $P((N-1)x+y) - P(N \cdot x) > 0$, proving the claim. If $y > x$, the proof proceeds analogously. ■

Lemma A.2. *For any x, $M(x,\cdot)$ is a strictly concave function. Define $y^M(x) = argmax\{M(x,y)/0 \le y \le Q_{max}\}$. Then, $y^M(x)$ is continuous and decreasing. Moreover, there exists $Q^M > x^C$ such that $y^M(x)$ is differentiable and strictly decreasing in $(0,Q^M)$ and $y^M(x) = 0$ for all $x \in [Q^M, Q_{max}]$.*

Proof. For a given, fixed x, the function $M(x,y)$ is strictly concave in y:

$$\frac{\partial^2 M(x,y)}{\partial y^2} = P''((N-1)x+y)y + 2P'((N-1)x+y) - C''(y) < 0$$

Hence, the first order condition

$$\frac{\partial M(x,y)}{\partial y} = P'((N-1)x+y)y + P((N-1)x+y) - C'(y) = 0$$

is necessary and sufficient for an interior global maximum.

The function y^M is continuous by the Maximum Theorem. For all x, we have that $M(x,Q_{max}) = -C(Q_{max}) - P(N \cdot x)x + C(x) < -C(0) - P(N \cdot x)x + C(x) = M(x,0)$ for all x. It follows that $y^M(x) < Q_{max}$. That is, either $y^M(x) = 0$ or it is an interior solution ($0 < y^M(x) < Q_{max}$).

In the sub-domain where it is strictly positive, $y^M(x)$ is implicitly defined by the first order condition above. By the Implicit Function Theorem, this function is differentiable in this set and has first derivative

$$\frac{dy^M}{dx} = -\left(\frac{\partial^2 M(x,y)}{\partial y^2}\right)^{-1} [(N-1)P''((N-1)x+y)y + (N-1)P'((N-1)x+y)] < 0$$

Hence, $y^M(x)$ is strictly decreasing in this set, and equal to zero outside it. Continuity of $y^M(x)$ in its whole domain and the fact that $y^M(x^C) = x^C > 0$ complete the claim. ■

Lemma A.3. $M(x,x^C) > 0$ *for all* $x \neq x^C$

Proof. Notice that $y^M(x^C) = x^C$. From Lemma A.2, it follows that, for all $x < x^C$, $y^M(x) > x^C > x$. Analogously, for all $x > x^C$, $y^M(x) < x^C < x$. In both cases, since $M(x,x) = 0$, $M(x,\cdot)$ is strictly concave, and $y^M(x)$ is its maximum, we have that $M(x,x^C) > 0$. ■

Lemma A.4. *For any $x < x^C$, a single mutation suffices for the transition from* $\mathrm{mon}(x,K)$ *to* $\mathrm{mon}(x^C,K)$.

Proof. Starting at $\text{mon}(x, K)$, suppose one mutant firm deviates to x^C. By Lemma A.3, $M(x, x^C) > 0$; i.e. the mutant attains higher payoffs than all firms before mutation, when all produced x. By Lemma A.1, $D(x, x^C) > M(x, x^C) > 0$; i.e. the mutant also attains higher payoffs than the non-mutants after mutation. Since the mutant's is the highest observed payoff, the mutation is successful and the transition to $\text{mon}(x^C, K)$ is completed after K consecutive periods. ■

Lemma A.5. *For any $x > x^W$, a single mutation suffices for the transition from* $\text{mon}(x, K)$ *to* $\text{mon}(x^W, K)$.

Proof. Starting at $\text{mon}(x, K)$, suppose one mutant firm deviates to x^W. By Lemma 2.3, $D(x, x^W) > 0$; i.e. the mutant attains higher payoffs than the non-mutants after mutation. By Lemma A.1, $M(x, x^W) > D(x, x^W) > 0$; i.e. the mutant also attains higher payoffs than the all firms before mutation, when all produced x. Since the mutant's is the highest observed payoff, the mutation is successful and the transition to $\text{mon}(x^C, K)$ is completed after K consecutive periods. ■

Lemma A.6. *No state* $\text{mon}(x, K)$ *with $x^C \leq x \leq x^W$ can be destabilized with only one mutation.*

Proof. Consider a mutation to $y > x$. By Lemma A.2, $M(x, \cdot)$ is strictly concave. Moreover, since $x \geq x^C$, it follows that $y^M(x) \leq y^M(x^C) = x^C \leq x$, and hence $M(x, \cdot)$ is decreasing in $[x, Q_{max}]$. In particular, $M(x, y) < M(x, x) = 0$, i.e., after a single mutation to $y > x$, the mutant will obtain smaller profits than before. It follows that the process will simply go back to $\text{mon}(x, K)$ in absence of further mutations.

Consider now a mutation to $y < x$. Analogously, suppose that $D(x, \cdot)$ is increasing in $[0, x]$. Then, $D(x, y) < D(x, x) = 0$ for $y < x$, i.e. the mutant obtains smaller profits than the non-mutants after a single mutation to $y < x$. It follows that the process will go back to $\text{mon}(x, K)$ in absence of further mutations.

It remains to show that, for $x^C \leq x \leq x^W$, $D(x, \cdot)$ is increasing in $[0, x]$. The first derivative is

$$\frac{\partial D}{\partial y} = P'((N-1)x + y)(y - x) + P((N-1)x + y) - C'(y)$$

and we will show that it is strictly positive in $[0, x)$, therefore proving the claim. Since $P'(\cdot) < 0$ and $y < x$, the first part of the expression is strictly positive. For the second part, notice that the function $P(N \cdot z) - C'(z)$ is strictly decreasing in z and has a zero at $z = x^W$.[5] Then, since $x \leq x^W$, we have that $P(N \cdot x) \geq C'(x)$. Since $y < x$, it follows that $P((N-1)x + y) > P(N \cdot x) \geq C'(x) > C'(y)$, completing the proof. ■

Lemma A.7. *For $p \in [0, P_{max}]$, the function $g_p(z) = p \cdot z - C(z)$ is strictly concave in z, and it attains a unique maximum $z(p)$. The function $z(p)$ is increasing and $z(P(N \cdot x^W)) = x^W$.*

[5]The first derivative is $N \cdot P'(N \cdot z) - C''(z) < 0$. The walrasian quantity x^W is a zero of this function by definition.

Proof. The second derivative of g_p is simply $-C''(z) < 0$, proving strict concavity and the existence of a unique maximum in the compact set $[0, Q_{max}]$. Moreover, $z(p)$ is continuous by the Maximum Theorem, and, for every p, either $z(p)$ is an interior maximum, or $z(p) \in \{0, Q_{max}\}$. The first order condition for an interior maximum is $p = C'(z)$, i.e. $z(p) = [C']^{-1}(p)$. This function is strictly increasing because C' is strictly increasing. Hence, the function z must be increasing by continuity. Finally, $z(P(N \cdot x^W)) = x^W$ follows by definition. ■

Lemma A.8. *For all $x^C \leq x \leq x^W - \delta$, two mutations suffice for the transition from* $\text{mon}(x, K)$ *to* $\text{mon}(x + \delta, K)$ *to occur.*

Proof. Consider the situation $\text{mon}(x, K)$, where all firms are producing a quantity x and selling it at a price $P = P(N \cdot x)$. Consider two simultaneous mutations of the following type. One firm deviates to $x + \delta$, and another firm deviates to $x - \delta$. The market price in the new situation, $(x + \delta, x - \delta, x, ..., x)$, is $P(x + \delta + x - \delta + (N - 2) \cdot x) = P(N \cdot x) = P$.

Since $z(p)$ is increasing by Lemma A.7, $P = P(N \cdot x) > P(N \cdot x^W)$ implies that $z(P) \geq z(P(N \cdot x^W)) = x^W$. Since the function $g_P(z)$ is strictly concave and has a maximum at $z(P)$, this implies that it is strictly increasing on $[0, x^W]$, and, in particular, $P \cdot (x + \delta) - C(x + \delta) > P \cdot (x) - C(x) > P \cdot (x - \delta) - C(x - \delta)$. Hence, after the two mutations described above, the mutant producing $x + \delta$ not only earns more profits than the non-mutants, but also higher ones than any profits in the previous periods, implying that the transition to $\text{mon}(x + \delta, K)$ will be successfully completed. ■

Proof of Theorem 3.1. We have to show that the monomorphic state $\text{mon}(x^W, K)$ is stochastically stable, and that no monomorphic state $\text{mon}(x, K)$ with $x \notin [x^C, x^W]$ can be stochastically stable.

Call n_1 the number of quantities in $[x^C, x^W] \bigcap \Gamma$, and n_2 the number of quantities in $\Gamma \backslash [x^C, x^W]$.

Consider any ω-tree. Stochastically stable states are those with minimal-cost ω-trees (see e.g. Kandori, Mailath, and Rob (1993) or Ellison (2000) . In any such tree, the cost necessary to leave any state but the vertex is two for monomorphic states corresponding to quantities in $[x^C, x^W] \bigcap \Gamma$ (by Lemmata A.6 and A.8), and at least one for all others (because monomorphic states can not be destabilized without mutation). If we are able to construct an ω-tree such that

1. The vertex ω corresponds to a state $\text{mon}(z, K)$ with $z \in [x^C, x^W] \bigcap \Gamma$
2. For all other $x \in [x^C, x^W] \bigcap \Gamma$, the state $\text{mon}(x, K)$ is connected at cost two.
3. For all $x \in \Gamma \backslash [x^C, x^W]$, the state $\text{mon}(x, K)$ is connected at cost one.

then this would show that the state ω is stochastically stable. The cost of this tree would be $2 \cdot (n_1 - 1) + n_2 = 2 \cdot n_1 - 2 + n_2$. Moreover, it follows that the stochastically stable states are exactly those having ω-trees of this cost.

We construct now such a tree with vertex $\text{mon}(x^W, K)$.

For all $x < x^C, x \in \Gamma$, the state $\text{mon}(x, K)$ is connected at cost one to $\text{mon}(x^C, K)$, by Lemma A.4.

For all $x > x^W, x \in \Gamma$, the state $\text{mon}(x, K)$ is connected at cost one to $\text{mon}(x^W, K)$, by Lemma A.5.

For all $x \in [x^C, x^W) \bigcap \Gamma$, the state $\text{mon}(x, K)$ is connected at cost two to $\text{mon}(x + \delta, K)$.

This completes a $\text{mon}(x^W, K)$-tree of the desired cost. Hence, $\text{mon}(x^W, K)$ is stochastically stable.

Consider any quantity $x \in \Gamma \backslash [x^C, x^W]$. The minimum cost that could be attained by any $\text{mon}(x, K)$-tree is at least $2 \cdot n_1 + n_2 - 1$ by Lemma A.6, which is larger than required. This proves that $\text{mon}(x, K)$ is not stochastically stable. ■

Proof of Proposition 3.3. If $N = 2$, then the state $\text{mon}(x, K)$ can be connected to the state $\text{mon}(x - \delta, K)$ with two mutations (both to $x - \delta$) for all $x > x^C, x \in \Gamma$. This follows from the fact that the function $P(2 \cdot x) \cdot x - C(x)$ is strictly concave and its maximum, that is, the collusion output, is lower than x^C.

For any $x \in [x^C, x^W) \bigcap \Gamma$, consider the $\text{mon}(x^W, K)$-tree constructed in the proof of the Theorem 3.1. Consider all the arrows leaving states $\text{mon}(x', K)$ for $x' \in [x, x^W) \bigcap \Gamma$. These arrows, which connected $\text{mon}(x', K)$ to $\text{mon}(x' + \delta, K)$ with two mutations, can be reversed so that they connect now $\text{mon}(x' + \delta, K)$ to $\text{mon}(x', K)$, also with two mutations. This procedure yields a $\text{mon}(x, K)$-tree of minimal cost, which, together with Theorem 3.1, proves the result. ■

B Proof of Theorem 3.4

In this section, let $N \geq 3$. The proof of the main Theorem consists of a series of interrelated, often technical lemmata.

We already know that the stochastically stable states must be contained in the set of monomorphic states corresponding to quantities in $[x^C, x^W]$, and that $\text{mon}(x^W, K)$ is stochastically stable. To prove Theorem 3.4, we have to build $\text{mon}(x, K)$-trees (with $x \in [x^C, x^W)$) with the same cost (in terms of mutations) as the $\text{mon}(x^W, K)$-tree built in the proof of Theorem 3.1. We will see that, for this purpose, it is enough to modify the mentioned $\text{mon}(x^W, K)$-tree, by deleting certain arrows and adding transitions from higher to lower quantities. This will show stochastic stability of the whole interval. The first Lemma, though, establishes that this cannot be done with "direct" transitions.

Lemma B.1. *Let $x \in [x^C, x^W]$. No transition from* $\text{mon}(x, K)$ *to* $\text{mon}(x', K)$ *with $x' < x$ is possible with two mutations or less.*

Proof. Transitions with one mutation are precluded by Lemma A.6. It suffices, hence, to consider two simultaneous mutations to x', x'' with $x' < x$.

Recall that, by Lemma A.7, the profit function $g_p(z) = p \cdot z - C(z)$ is strictly concave and its unique maximum $z(p)$ is increasing in p, with $z(P(N \cdot x^W)) = x^W$.

Let $Q = (N - 2)x + x' + x''$ be the total quantity produced after mutation, and let $p = P(Q)$ be the corresponding market price.

If $Q < Nx^W$, then $p > P(Nx^W)$ and hence $z(p) \geq x^W$. It follows that $g_p(x') < g_p(x)$, i.e. the x'-mutant earns less than the non-mutants *after* mutation (remember that $N \geq 3$), and hence can not be successful.

If $Q > Nx^W$, then $p < P(Nx^W) < P(Nx)$ and $p \cdot x' - C(x') < P(Nx)x' - C(x')$. Since $z(P(Nx)) \geq x^W$, it follows that $P(Nx)x' - C(x') < P(Nx)x - C(x)$. Combining both facts, we get that $p \cdot x' - C(x') < P(Nx)x - C(x)$, i.e. the x'-mutant earns less than the non-mutants *before* mutation and hence can not be successful. ■

Since transitions to lower quantities can not be achieved directly with two mutations, we must take a detour. The basic idea is that quantities in $[x^C, x^W]$ can be first destabilized "upwards," to high quantities above x^W from where we can add further transitions reaching lower quantities than the original ones. The next two lemmata concentrate on the first step in such transitions, giving us a range of higher quantities that we can reach from a given one (actually, from the corresponding monomorphic state).

Lemma B.2. *Let $x \in [x^C, x^W]$.*[6] *A transition from* $\text{mon}(x, K)$ *to* $\text{mon}(y, K)$, *with $x < y \leq 2 \cdot x$ can be achieved with two mutations if $H(x, y) \geq 0$, where*

$$H(x, y) = P(y + 0 + (N - 2) \cdot x)(y - x) - C(y) + C(x).$$

Proof. Consider the situation $\text{mon}(x, K)$, where all firms are producing the quantity x. Consider two simultaneous mutations to quantities $x_1 = y, x_2 = 0$, with $x < y \leq 2 \cdot x$. Call then $Q = y + (N - 2) \cdot x \leq N \cdot x$.

Note that $P(Q)x - C(x) \geq P(N \cdot x)x - C(x)$. It follows that the considered mutation can be successful if and only if $P(Q)y - C(y) \geq P(Q)x - C(x)$, i.e. if and only if

$$H(x, y) = P(y + 0 + (N - 2) \cdot x)(y - x) - C(y) + C(x)$$

is positive.[7] ■

Fix x and denote $H_x(y) = H(x, y)$. Note that $H_x(x) = 0$ and that $H'_x(x) = P((N - 1) \cdot x) - C'(x) > P(N \cdot x) - C'(x) \geq 0$ (since $x \leq x^W$). This proves that for $y > x$ close enough to x, such a transition is possible. The problem is whether the transition is also possible for $y > x$ "far away enough" for our purposes.

Lemma B.3. *For $x \in [x^C, x^W]$, define*

$$h(x) = max\{y \in [x, 2 \cdot x] | H(x, y) \geq 0\}$$

Then, h is continuous, and either $h(x) = 2x$, or $h(x) \in (x, 2 \cdot x)$. In the latter case, $h(x)$ is implicitly defined by $H(x, h(x)) = 0$, and it is strictly decreasing.[8]

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[6] This Lemma holds actually for a larger range of quantities.

[7] The considered mutations (involving a mutant to quantity zero) might seem arbitrary. Actually, a transition from $\text{mon}(x, K)$ to $\text{mon}(y, K)$ through two mutations to $x_1 = y, x_2 \geq 0$ (such that $x_1 > x, x_1 + x_2 \leq 2 \cdot x$) is possible if and only if the same transition can be achieved with two mutations to x_1 and $x'_2 = 0$.

To prove this, first note that if the transition is possible with the mutations to x_1, x_2, then $P(x_1 + x_2 + (N - 2) \cdot x)(x_1 - x) - C(x_1) + C(x) \geq 0$ which, since P is decreasing, implies that $P(x_1 + 0 + (N - 2) \cdot x)(x_1 - x) - C(x_1) + C(x) \geq 0$ which in turn implies the claim.

The reverse implication is trivial. It suffices to set $x_2 = 0$.

[8] Hence, h is either decreasing or "tent-shaped," because $2x$ is increasing and the implicit function defined by $H(x, h(x)) = 0$ is decreasing.

Proof. Fix x and denote $H_x(y) = H(x, y)$. Then,

$$\begin{aligned} H'_x(y) &= P'(y + (N-2)x)(y - x) + P(y + (N-2)x) - C'(y) \\ H''_x(y) &= P''(y + (N-2)x)(y - x) + 2P'(y + (N-2)x) - C''(x) < 0 \end{aligned}$$

and hence H_x is strictly concave. It follows that the set

$$K_x = \{y \in [x, 2 \cdot x] | H(x, y) \geq 0\}$$

is convex, and the correspondence $x \mapsto K_x$ is continuous and compact-valued, so $h(x)$ is continuous (e.g. by an application of the Maximum Theorem to the identity function $y \mapsto y$), and $K_x = [x, h(x)]$ (by concavity). Moreover,

$$H'_x(x) = P((N-1)x) - C'(x) > P(N \cdot x) - C'(x) \geq 0$$

since $x \leq x^W$, and hence $h(x) > x$. Hence, either $h(x)$ is in the interior of the interval (and then, by continuity, $H(x, h(x)) = 0$), or $h(x) = 2x$.

It remains to show that $h(x)$ is strictly decreasing whenever $H(x, h(x)) = 0$. By the Implicit Function Theorem, in this case the function $h(x)$ is differentiable and has first derivative

$$h'(x) = -\left(\frac{\partial H(x, h(x))}{\partial x}\right) \frac{1}{H'_x(h(x))}.$$

Since H_x is strictly concave and $H_x(x) = H_x(h(x)) = 0$, with $h(x) > x$, it follows that $H'_x(h(x)) < 0$. By direct computation,

$$\begin{aligned} &\frac{\partial H(x, y)}{\partial x} = (N-2)P'((N-2)x + y)(y - x) - P((N-2)x + y) + C'(x) \\ &\frac{\partial H(x, h(x))}{\partial x} = \\ &\quad = (N-2)P'((N-2)x + h(x))(h(x) - x) - P((N-2)x + h(x)) + C'(x) \end{aligned}$$

We claim this last quantity to be negative. Notice that $P'(\cdot) < 0$ and $(h(x) - x) > 0$, hence the first part is negative. For the second part, notice that from $H(x, h(x)) = 0$ follows that

$$P((N-2)x + h(x)) = \frac{C(h(x)) - C(x)}{h(x) - x}$$

and this quantity is strictly larger than $C'(x)$ by strict convexity of C.[9] Hence, the second part is also negative. In summary, $h'(x) < 0$. ∎

The next lemma paves the way for the second step in the transitions explained above. Once we have reached a monomorphic state with a certain high quantity, we need to determine how far "down" we can go back. We are only interested in reaching quantities in $[x^C, x^W]$. Intuitively, we can concentrate on the relative advantages of mutants because they adjust their quantity in the direction of x^C and are hence better off in absolute terms. More rigorously, Lemma A.1 implies that a mutation from a higher to a lower quantity is always beneficial in absolute terms when it is so in relative terms. Therefore, we concentrate on the function $D(x, \cdot)$.

[9] Taking a Taylor expansion, $C(h(x)) = C(x) + C'(x)(h(x) - x) + \frac{1}{2}C''(\theta_x)(h(x) - x)^2$ and it follows that $\frac{C(h(x)) - C(x)}{h(x) - x} > C'(x)$ since $C''(\cdot) > 0$.

Lemma B.4. *For $x > x^W$, define*

$$f(x) = min\{y \in [0, x] | D(x, y) \geq 0\}.$$

Then, f is continuous, and either $f(x) = 0$ or $f(x) \in (0, x^W)$. In the latter case, f is implicitly defined by $D(x, f(x)) = 0$, and it is strictly decreasing. Moreover, $D(x, y) > 0$ for all $y \in (f(x), x)$.

Proof. Consider $D_x(y) = D(x, y) = P((N-1)x+y)(y-x)+C(x)-C(y)$. Note that $D_x(x) = 0$ and

$$D'_x(y) = P'((N-1)x + y)(y - x) + P((N-1)x + y) - C'(y)$$

and hence $D'_x(x) = P(Nx) - C'(x) < 0$ for $x > x^W$. Hence, $D_x(y) > 0$ for $y < x$ close enough to x.

If $D_x(y) \geq 0$ for all $y \in (0, x)$ then the claim is true with $f(x) = 0$. If not, then there exists $x' \in (0, x)$ such that $D(x, x') = 0$.

It is enough to show that, if $D(x, x') = 0$, then $D(x, y) > 0$ for all $y \in (x', x)$. This implies that there cannot exist two different quantities $x', x'' \in (0, x)$ with $D(x, x') = 0$ and $D(x, x'') = 0$ and, by continuity of D, proves the claim.

Suppose, then, $D(x, x') = 0$. Call $P' = P((N-1)x + x')$. We have that

$$P' \cdot x' - C(x') = P' \cdot x - C(x)$$

Since the function given by $g_{P'}(z) = P' \cdot z - C(z)$ is strictly concave by Lemma A.7, it follows that $P' \cdot y - C(y) > P' \cdot x - C(x)$ for all $y \in (x', x)$.

Consider any $y \in (x', x)$. Since $y > x'$, we have that $P((N-1)x + y) < P((N-1)x + x') = P'$. Since $y < x$, it follows that $P((N-1)x + y)(y - x) > P' \cdot (y - x)$ and hence

$$D(x, y) = P((N-1)x+y)(y-x)-C(y)+C(x) > P' \cdot (y-x)-C(y)+C(x) > 0$$

as needed.

Note that, by Lemma 2.3, $D(x, x^W) > 0$, which implies that $f(x) < x^W$.

It remains to show that f is continuous for $x > x^W$. This follows from the Maximum Theorem since $f(x)$ is the argmin of the identity function in the set $K'_x = \{y/D(x, y) \geq 0\}$. We have shown that this set is a closed interval (compact and convex) for all $x > x^W$ (and we can define $K'_{x^W} = \{x^W\}$), and we know that $D(\cdot, \cdot)$ is a continuous function, hence continuity of the correspondence $x \mapsto K'_x$ follows.

In the sub-domain where it is strictly positive, $f(x)$ is implicitly defined by $D(x, f(x)) = 0$. By the Implicit Function Theorem, this function is differentiable in this set and has first derivative

$$f'(x) = -\left(\frac{\partial D(x, f(x))}{\partial x}\right) \frac{1}{D'_x(f(x))}.$$

By definition of $f(x)$ and continuity, it follows that $D'_x(f(x)) > 0$. By direct computation,

$$\frac{\partial D(x, y)}{\partial x} = (N-1)P'((N-1)x + y)(y - x) - P((N-1)x + y) + C'(x)$$

$$\frac{\partial D(x, f(x))}{\partial x} =$$

$$= (N-1)P'((N-1)x + f(x))(f(x) - x) - P((N-1)x + f(x)) + C'(x)$$

We claim this last quantity to be positive. Notice that $P'(\cdot) < 0$ and $(f(x) - x) < 0$, hence the first part is positive. For the second part, notice that from $D(x, f(x)) = 0$ follows that

$$P((N-1)x + f(x)) = \frac{C(x) - C(f(x))}{x - f(x)}$$

and this quantity is strictly lower than $C'(x)$ by strict convexity of C.[10] This implies the second part to be also positive.

In summary, f is a strictly decreasing function in the set where it is implicitly defined by $D(x, f(x)) = 0$, and zero outside it. By continuity, f is decreasing in its whole domain. ■

For high quantities (above x^W), mutations to lower ones are beneficial in relative terms. Last lemma shows that, as we consider higher and higher quantities, the range of such beneficial deviations increases in size. In the next result we tackle a complementary consideration. Fixing a deviation y, we show that there exists a quantity $\phi(y)$ such that the deviation y is beneficial for all monomorphic states corresponding to quantities above $\phi(y)$.

Lemma B.5. *The function $D(x, y)$ is strictly convex in x for $x > y$. For all $y < x^W$ there exists a unique $\phi(y) > x^W$ such that $D(\phi(y), y) = 0$ and $D(x, y) > 0 \ \forall\ x > \phi(y)$. Taking $\phi(x^W) = x^W$, the function ϕ is continuous in $(0, x^W]$.*[11]

In particular, if $x_2 > x_1 > y$ and $D(x_1, y) \geq 0$, then $D(x_2, y) > 0$.

Proof. Fix y. Consider $d_y(x) = D(x, y) = P((N-1)x + y)(y - x) + C(x) - C(y)$. Note that $d'_y(x) = \frac{\partial D(x,y)}{\partial x}$ computed in the proof of Lemma B.4, and

$$d''_y(x) = (N-1)^2 P''((N-1)x + y)(y - x) - 2(N-1)P'((N-1)x + y) + C''(x) > 0$$

i.e. $D(x, y)$ is strictly convex in x for $x > y$.

Let $y < x^W$. Note that $d'_y(y) = -P(N \cdot y) + C'(y)$. Since the function $P(N \cdot z) - C'(z)$ is strictly decreasing in z and has a zero at x^W (see footnote 5 in Lemma A.6), it follows that $d'_y(y) < 0$. Noting that $d_y(Q_{max}) = C(Q_{max}) - C(y) > 0$, it follows that there exists $\phi(y) \in (y, Q_{max})$ such that $d_y(\phi(y)) = 0$ and $d_y(x) > 0 \ \forall\ x > \phi(y)$. Since $D(x^W, y) < 0$ by Lemma 2.3, necessarily $\phi(y) > x^W$.

Continuity of ϕ follows from convexity of $D(\cdot, y)$ analogously to Lemmata B.3 and B.4.

In particular, consider $x_2 > x_1 > y$ such that $d_y(x_1) = D(x_1, y) \geq 0$. Since $d_y(y) = D(y, y) = 0$, it follows from convexity that $d_y(x)$ is increasing for $x > x_1$ (since, if d_y has a minimum, it must be lower than x_1). Hence, $D(x_2, y) \geq D(x_1, y) \geq 0$. ■

Suppose that a monomorphic state $\text{mon}(x, K)$ with $x \in [x^C, x^W]$ has been destabilized with two mutations and a new state $\text{mon}(x', K)$ has been reached.

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[10] Taking a Taylor expansion, $C(f(x)) = C(x) + C'(x)(f(x) - x) + \frac{1}{2}C''(\theta_x)(f(x) - x)^2$ and it follows that $\frac{C(f(x)) - C(x)}{f(x) - x} < C'(x)$ since $C''(\cdot) > 0$ and $(f(x) - x) < 0$.

[11] It is easy to see that the functions f and ϕ are partial inverses, i.e. $f(\phi(x)) = x$ for all $x \in [0, x^W]$, but $\phi(f(x)) \neq x$ in general (f is not invertible).

Lemma B.3 guarantees that this is possible for all $x' \in (x, h(x)]$. The next Lemma implies that, if x' is high enough, it must be the case that $D(x', x)$ is strictly positive, and hence, from the new state $\text{mon}(x', K)$, a single mutation back to x or, by continuity, to an even lower quantity, will be beneficial in relative terms. This will be the core of the argument to show how to "go back" and construct transitions towards lower quantities in $[x^C, x^W]$ (recall that we already know that $\text{mon}(x^W, K)$ is stochastically stable).

Lemma B.6. *Let $x \in [x^C, x^W]$. Then, $D(h(x), x) > 0$. Moreover, $h(x) > x^W$.*

Proof. By Lemma B.3, either $h(x) \in (x, 2x)$, or $h(x) = 2x$. We distinguish both cases. If $h(x) \in (x, 2x)$, again by Lemma B.3, we have that $H(x, h(x)) = 0$, i.e.

$$P(h(x) + (N-2)x)(h(x) - x) = C(h(x)) - C(x).$$

Then,

$$\begin{aligned} D(h(x), x) = P((N-1)h(x) + x)(x - h(x)) - C(x) + C(h(x)) = \\ [P(h(x) + (N-2)x) - P((N-1)h(x) + x)] \cdot (h(x) - x) \end{aligned}$$

Since $h(x) > x$, it follows that $h(x) + (N-2)x < (N-1)h(x) + x$, thus

$$P(h(x) + (N-2)x) > P((N-1)h(x) + x)$$

implying $D(h(x), x) > 0$.

Suppose now that $h(x) = 2x$. Then,

$$D(2x, x) = P((N-1)2x + x)(x - 2x) - C(x) + C(2x)$$

Let $\overline{p} = P((N-1)2x + x) = P((2N-1)x)$. Since $N > 2$, $\overline{p} < P((N+1)x)$. Taking a Taylor expansion,

$$P((N+1)x) = P(Nx) + P'(Nx)x + \frac{1}{2}P''(\xi_x)x^2 \leq P(Nx) + P'(Nx)x \leq C'(x)$$

where the first inequality follows from $P''(\cdot) \leq 0$, and the second from the fact that $x \geq x^C$.[12] Hence, $\overline{p} < C'(x)$.

Recall that, by Lemma A.7, $g_p(z) = p \cdot z - C(z)$ is strictly concave and attains a maximum at $z(p)$. Since $\overline{p} < C'(x)$, it follows that $g'_{\overline{p}}(x) < 0$ and $z(\overline{p}) \leq x$, implying that $g_{\overline{p}}$ is strictly decreasing for $z \geq x$. In particular,

$$D(2x, x) = \overline{p}(x - 2x) - C(x) + C(2x) = g_{\overline{p}}(x) - g_{\overline{p}}(2x) > 0$$

which proves the claim.

It remains to prove that $h(x) > x^W$. Suppose otherwise. Then, by Lemma B.5, $D(h(x), x) > 0$ implies that $D(x^W, x) > 0$, a contradiction with Lemma 2.3. ■

The next lemma implies that, given any quantity $x \in [x^C, x^W]$, there exists a range of higher quantities y such that the corresponding monomorphic states can be reached from $\text{mon}(x, K)$ with two mutations as in Lemma B.2, but new monomorphic states $\text{mon}(x', K)$ with $x' < x$ can be reached from $\text{mon}(y, K)$ with a single mutation (see Figure 5).

[12] The function $P(Nx) + P'(Nx)x - C'(x)$ is strictly decreasing and has a unique zero at $x = x^C$.

Lemma B.7. *For all* $x \in [x^C, x^W]$, $\phi(x) < h(x)$.

Proof. Suppose $\phi(x) \geq h(x)$. Then, let $z \in [h(x), \phi(x)]$. Since $z \leq \phi(x)$ and $z \geq h(x) > x$ (by Lemma B.3), it follows from Lemma B.5 that $D(z, x) \leq 0$ (recall that $D(\cdot, x)$ is strictly convex with $D(x, x) = D(\phi(x), x) = 0$). By Lemma B.6, $D(h(x), x) > 0$. Since $z \geq h(x)$, Lemma B.5 implies that $D(z, x) > 0$, a contradiction. ■

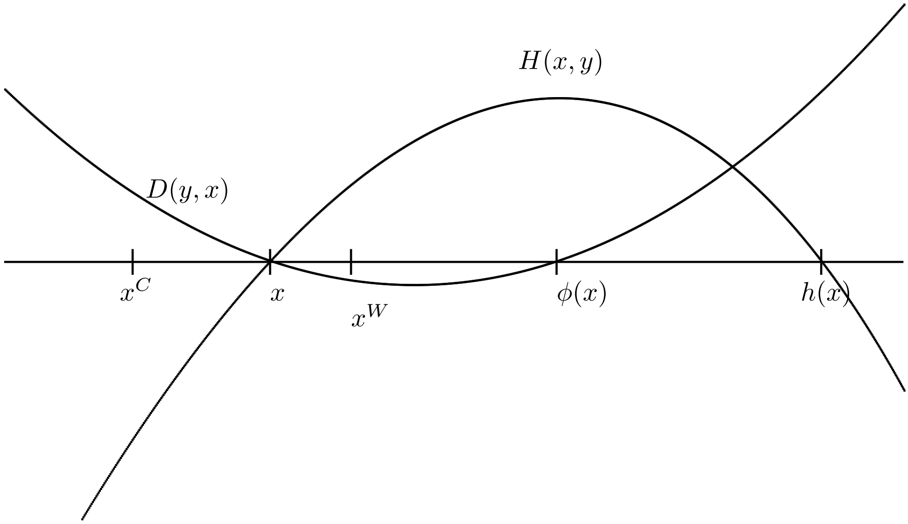

Figure 5: Two-step transitions. In a first step, a transition from mon(x, K) to mon(y, K), $y > x$ can be achieved with two mutations if $H(x, y) \geq 0$. In a second step, a transition from mon(y, K) to mon(x, K) and to mon(x', K) with $x' < x$ can be achieved with one mutation if $D(y, x) > 0$. The horizontal axis is the first variable of $D(\cdot, \cdot)$ and the second of $H(\cdot, \cdot)$. The fact that $\phi(x) < h(x)$ makes possible a net "downward" transition.

Now we are ready to prove the main theorem. The remaining proof shows that, for any $x \in [x^C, x^W] \bigcap \Gamma$, we can construct a mon$(x, K)$-tree of the same cost (i.e. number of involved mutations) as that of the mon(x^W, K)-tree constructed in the proof of Theorem 3.1. The proof is complicated by considerations regarding the discretization of the strategy space.

Proof of Theorem 3.4. Consider the mon(x^W, K)-tree of minimal cost constructed in the proof of Theorem 3.1. Theorem 3.4 will be proven if we can show that we can modify this tree into a mon(x, K) of the same cost for all $x \in [x^C, x^W] \bigcap \Gamma$. We distinguish two cases.

Case 1: For all $x \in [x^C, x^W]$, $h(x) \in (x, 2x)$.

In this case, $h(x) > x$ is a strictly decreasing function, implicitly defined by $H(x, h(x)) = 0$ (see Lemma B.3). Moreover, $h(x) \geq h(x^W) > x^W$ for all $x \in [x^C, x^W]$.

Let $\delta_h = min\{h(x) - \phi(x) \mid x \in [x^C, x^W]\}$. Since h and ϕ are continuous by Lemmata B.3 and B.5 and $h(x) > \phi(x)$ for all $x \in [x^C, x^W]$ by Lemma B.7, we know that $\delta_h > 0$.

We can define a continuous function ψ by

$$\psi(x) = \frac{1}{2}\left(h(x) + \phi(x)\right) \in (\phi(x), h(x))$$

Let $x \in [x^C, x^W)$. By Lemma B.5, $\phi(x) > x^W > x$ with $D(\phi(x), x) = 0$. By Lemma B.4, since $D(\phi(x), x) = 0$ it must be that $f(\phi(x)) = x$ and f is strictly decreasing at $\phi(x)$. Since $\phi(x) < \psi(x) < h(x)$ and $f(\phi(x)) = x > 0$, it follows from Lemma B.4 that $x = f(\phi(x)) > f(\psi(x)) \geq f(h(x))$. For $x = x^W$, $\phi(x^W) = x^W$ and hence $f(\phi(x^W)) = x^W > f(\psi(x^W))$. In summary, $f(\psi(x)) < x$ for all $x \in [x^C, x^W]$.

Let $\delta_f = min\{x - f(\psi(x)) \mid x \in [x^C, x^W]\}$. Since f and ψ are continuous by Lemmata B.3, B.4, and B.5 and $x > f(\psi(x))$ for all $x \in [x^C, x^W]$, we know that $\delta_f > 0$.

Let $\delta < \frac{1}{3}\min\{\delta_h, \delta_f\}$.

Consider $x_0 = x^W$. Let $y_0 \in (\psi(x_0), h(x_0)) \bigcap \Gamma$ (it exists because $\delta < \frac{1}{3}\delta_h$). Since $H_x(y) = H(x, y)$ is strictly concave in y by Lemma B.3, we know that $H(x_0, y_0) > 0$, i.e. a transition from $\text{mon}(x_0, K)$ to $\text{mon}(y_0, K)$ with two mutations is possible.

Let $x_1 \in (f(\psi(x_0)), x_0) \bigcap \Gamma$. It exists because $\delta < \delta_f$.

Since f is decreasing, $f(y_0) \leq f(\psi(x)) \leq x_1$, which implies that $D(y_0, x_1) \geq 0$, i.e. the transition from $\text{mon}(y_0, K)$ to $\text{mon}(x_1, K)$ is possible with one mutation.

Consider the $\text{mon}(x^W, K)$-tree of minimal cost constructed in the proof of Theorem 3.1. Delete the arrows leaving $\text{mon}(x_1, K)$ (which has cost 2) and $\text{mon}(y_0, K)$ (cost 1). Replace them with the arrow from $\text{mon}(x_0, K)$ to $\text{mon}(y_0, K)$ (cost 2) and the arrow from $\text{mon}(y_0, K)$ to $\text{mon}(x_1, K)$ (cost 1). This yields a $\text{mon}(x_1, K)$-tree with the same cost as the original one and hence proves that $\text{mon}(x_1, K)$ is stochastically stable. Moreover, by Lemma A.8 all states $\text{mon}(x, K)$ with x in $[x_1, x^W] \bigcap \Gamma$ are also stochastically stable.

Notice that the grid is a priori fixed and $x_1 < x_0$, i.e. $x_1 \leq x_0 - \delta$. This procedure has added at least one stochastically stable state. Repeating the argument from x_1, we obtain a new stochastically stable state $\text{mon}(x_2, K)$ with $x_2 \leq x_1 - \delta$. After a finite number of iterations, we obtain that $\text{mon}(x^C, K)$ is stochastically stable, and hence all monomorphic states with quantities in $[x^C, x^W]$ are stochastically stable.

Case 2: For some $x \in [x^C, x^W]$, $h(x) = 2x$.

Apply the same procedure as in Case 1. Let x_i be the first quantity in the sequence constructed there such that $h(x_i) = 2x_i$. It follows that $h(x) = 2x > x^W$ for all $x \in [x^C, x_i])$ (because h is continuous by Lemma B.3, strictly decreasing whenever $h(x) \in (x, 2x)$, and $2x$ is strictly increasing).

By Lemma B.4 f is decreasing. Since $2x_i \geq 2x^C$, it follows that $f(2x_i) \leq f(2x^C)$. Since $h(x^C) = 2x^C$ and $D(h(x^C), x^C) > 0$ by Lemma B.6, we know that $f(2x^C) < x^C$. Hence, $f(2x_i) < x^C$ and $D(2x_i, x^C) > 0$.

We can set now $x_{i+1} = x^C$ and complete the previous tree with an arrow from $\text{mon}(x_i, K)$ to $\text{mon}(2x_i, K)$ (at cost 2) and an arrow from $\text{mon}(2x_i, K)$ to $\text{mon}(x^C, K)$ (at cost 1). ■

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